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Cosmological constant in the Bianchi type-I-modified Brans–Dicke cosmology

A K AZAD and J N ISLAM
Research Centre for Mathematical and Physical Sciences, Chittagong University, Bangladesh

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Abstract. In 1961, Brans and Dicke [1] provided an interesting alternative to general relativity based on Mach's principle. To understand the reasons leading to their field equations, we first consider homogeneous and isotropic cosmological models in the Brans–Dicke theory. Accordingly we start with the Robertson–Walker line element and the energy tensor of a perfect fluid. The scalar field ϕ is now a function of the cosmic time only.

Then we consider spatially homogeneous and anisotropic Bianchi type-I-cosmological solutions of modified Brans–Dicke theory containing barotropic fluid. These have been obtained by imposing a condition on the cosmological parameter $\Lambda(\phi)$. Again we try to focus the meaning of this cosmological term and to relate it to the time coordinate which gives us a collapse singularity or the initial singularity. On the other hand, our solution is a generalization of the solution found by Singh and Singh [2]. As far as we are aware, such solution has not been given earlier.

Keywords. Cosmology; cosmological constant; modified Brans–Dicke cosmology.

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1. Introduction

As is well known, Einstein originally introduced the cosmological constant in order to obtain a static model of the Universe, because the large scale recession of the galaxies, i.e., the expansion of the Universe had not been discovered at the time; this was discovered later by Hubble in the twenties and the thirties. It was later (than Einstein's static model) in 1922 that Friedmann found his dynamic solutions. As is also well known, Einstein later regretted his introduction of the cosmological constant to obtain a static model, for if he had relied on his original equations, he might have been able to predict the expansion of the Universe, or at least that the Universe is in a dynamic state (see Islam [3] second edition, 2002).

Ever since Einstein introduced it, and more particularly in recent years, however, the cosmological constant has turned up in various forms and for various reasons. For example, Zeldovich [4] tried to derive it from considerations of particle physics and quantum fluctuation phenomena. The inflationary models of Guth [5], Linde [6] and Albrecht and Steinhardt [7] also have connections with the cosmological constant. The behavior of Einstein's equations when one introduced the Higgs field that is supposed to trigger off a phase transition in the early Universe is somewhat akin to the behavior in the presence of the cosmological constant. Further, Linde has argued that the cosmological term arises from spontaneous symmetry breaking and suggested that the term is not a constant but a function of temperature. Also Dreitlein [8] connects the mass of Higg's scalar boson with both the cosmological term and gravitational constant. In cosmology, the term may be understood by incorporation with Mach's principle which suggests the acceptance of the Brans–Dicke Lagrangian as a realistic case.

Another motivation for the term arises in the work of Hawking and Penrose [9] (cited in Islam [10]), in which one has to know about the anti-de-Sitter space (the de-Sitter and anti-de-Sitter spaces arise for opposite non-zero values of Λ). Hawking shows, for example, in $N = 8$ supergravity theory (see, e.g. Freund [11]) that a phase transition occurs at a certain critical value of the coupling constant; below the critical value of the coupling constant, the ground state happens to be an anti-de-Sitter space (Islam [3]).

In this work we will discuss the extension of some work by Singh and Singh [2] in connection with a form of Brans–Dicke cosmology, in which the cosmological constant is regarded as a function of the scalar field ϕ (see also Bergmann [12], Wagoner [13], Endo and Fukui [14]). In the field equations a certain arbitrary function $f(\phi)$ of the scalar field ϕ occurs. In fact, the cosmological 'constant' $\Lambda(\phi)$ is related to the scalar field as follows:

$$\Lambda - \phi \frac{\mathrm{d}\Lambda}{\mathrm{d}\phi} = A\phi^{-1}\Box\phi, \tag{1}$$

where A is a constant and $\Box\phi = g^{\mu\nu}\phi_{;\mu\nu}$.

From (1) it follows that $\Box\phi = f(\phi)$, a function of ϕ. Singh and Singh assume perfect fluid form for the energy momentum tensor $T_{\mu\nu}$ (we use ρ here instead of ε to conform to their notation).

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - pg_{\mu\nu}, \tag{2}$$

where p and ρ are the pressure and energy density, and u_μ and u_ν are the components of fluid four-velocity vector. We try to discuss the application of Brans–Dicke cosmology in homogeneous and isotropic cosmological models of R–W model. Singh and Singh further assume the following form for $f(\phi)$:

$$f(\phi) = m\phi^n, \tag{3}$$

where m and n are constants, and they are able to get explicit solutions for homogeneous but anisotropic cosmologies of the form

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{e}^{2r}\mathrm{d}x^2 - \mathrm{e}^{2\theta}\mathrm{d}y^2 - \mathrm{e}^{2\psi}\mathrm{d}z^2, \tag{4}$$

where r, θ, ψ are functions of t only. This metric is called the cosmological models of Bianchi type-I. For the barotropic fluid case the equation of state can be written as

$$p = (\lambda - 1)\rho, \quad 1 \leq \lambda \leq 2. \tag{5}$$

Singh and Singh get explicit solutions for $\lambda = 1$ (zero pressure) and $\lambda = 2$ (for stiff fluid). For $\lambda = 4/3$ (pure radiation) no solutions exist because one gets the vacuum solution. In general, it is difficult to specify the function $f(\phi)$ so that one can get explicit solutions. In this paper we will give a formulation of the equations which will facilitate the search for such explicit solutions and we shall indicate how to get one single explicit solution (distinct to that obtained by Singh and Singh). As far as we are aware, such a formulation has not been given earlier. We shall confine ourselves to the relevant equations here, more details can be obtained from the paper by Singh and Singh [2].

2. Application of Brans–Dicke modified theory to Bianchi-type-I cosmology

As mentioned earlier (see eq. (1)), the cosmological 'constant' in this case is a function of $\phi : \Lambda = \Lambda(\phi)$. The modified Brans–Dicke equations are (eq. (1) of Singh and Singh [2])

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = (K/\phi)T_{\mu\nu} - (\omega/\phi^2)(\phi_{,\mu}\phi_{,\nu} - \tfrac{1}{2}g_{\mu\nu}\phi_{,\alpha}\phi^{,\alpha}) - (1/\phi)(\phi_{;\mu\nu} - g_{\mu\nu}\Box\phi). \quad (6)$$

In addition to (6), other relevant equations are

$$\Box\phi = KT, \quad (7)$$

where $T = g^{\mu\nu}T_{\mu\nu} = (\rho - 3p)$, with K a constant. From the conservation equation

$$T^{\mu\nu}_{;\nu} = 0,$$

we get (conforming to Singh and Singh)

$$\rho' = (\rho + p)(r' + \theta' + \psi') : \text{ [where } \varepsilon = \rho \text{ and } c = 1], \quad (8)$$

where a prime denotes differentiation with respect to t. With the barotropic assumption (5), eq. (8) reduces to the following one:

$$\rho' = -\lambda\rho(r' + \theta' + \psi'). \quad (9)$$

The condition $\Box\phi = f(\phi)$ gives the following equation:

$$\Box\phi = \phi'' - \phi'(r' + \theta' + \psi') = f(\phi). \quad (10)$$

This equation has not been considered by Singh and Singh, who rely on (3), and $\Box\phi = f(\phi)$ to get their explicit solution. From (9) and (10) we get

$$\phi'' + (\rho'\phi'/\lambda\rho) = f(\phi). \quad (11)$$

From (5) and (7) we get (recalling that $T = (\rho - 3p) = (4 - 3\lambda)\rho$)

$$\Box\phi = f(\phi) = KT = K(4 - 3\lambda)\rho = \hat{k}\rho. \quad (12)$$

Thus

$$\hat{k}\rho' = \frac{\mathrm{d}f}{\mathrm{d}\phi}\phi' \equiv f_\phi\phi'; \quad \rho'/\rho = f_\phi\phi'/\phi, \quad (13)$$

when (11) reduces to

$$\phi'' + (f_\phi \phi'^2/\lambda f) = f(\phi). \tag{14}$$

Multiply (14) by f on both sides, we get

$$f\phi'' + \lambda^{-1} f_\phi \phi'^2 = f^2. \tag{15}$$

Let $\xi = \phi'$, so that

$$\phi'' = \frac{\mathrm{d}}{\mathrm{d}t}(\phi') = \frac{\mathrm{d}}{\mathrm{d}\phi}(\phi')\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{\mathrm{d}\xi}{\mathrm{d}\phi}\phi' = \xi\frac{\mathrm{d}\xi}{\mathrm{d}\phi} = \xi\xi_\phi,$$

when eq. (15) becomes

$$f\xi\xi_\phi + \frac{1}{\lambda} f_\phi \xi^2 = f^2. \tag{16}$$

Multiply eq. (16) by $2f^{2/\lambda-1}$, we get

$$2f^{2/\lambda}\xi\xi_\phi + (2/\lambda) f^{2/\lambda-1} f_\phi \xi^2 = 2f^{2/\lambda+1}. \tag{17}$$

This can be written as

$$\frac{\mathrm{d}}{\mathrm{d}\phi}(f^{2/\lambda}\xi^2) = 2f^{2/\lambda+1}. \tag{18}$$

Integrating with respect to ϕ, we get

$$f^{2/\lambda}\xi^2 = f^{2/\lambda}\phi'^2 = 2\int^\phi (f^{2/\lambda+1})\mathrm{d}\phi. \tag{19}$$

This gives

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = f^{-1/\lambda}\left\{2\int^\phi (f^{2/\lambda+1})\mathrm{d}\phi\right\}^{1/2}, \tag{20}$$

when we get by quadratures

$$t = \int^\phi f^{1/\lambda}\left\{2\int^\phi (f^{2/\lambda+1})\mathrm{d}\phi\right\}^{-1/2}\mathrm{d}\phi. \tag{21}$$

Thus for any given function $f(\phi)$, ϕ can be determined in principle as a function of t, for any value of λ. Let us try

$$f(\phi) = \beta\mathrm{e}^{\alpha\phi}, \tag{22}$$

where β and α are constants. Note that this function is distinct from (3) assumed by Singh and Singh. Then, as is readily verified (21) yields

$$t \sim \mathrm{e}^{-1/2\alpha\phi}, \tag{23}$$

if we set an arbitrary constant equal to zero in the inner integral in (21). With this constant non-zero one gets a more complicated relation between t and ϕ with the use of (23) and (1). Λ can be worked out as an explicit function of t.

3. Solution of the field equations

Let us try again $f = \beta \mathrm{e}^{\alpha\phi}$.

$$f^{2/\lambda+1} = \hat{\beta}\mathrm{e}^{\hat{\alpha}\phi}. \tag{24}$$

$$\hat{\beta} = \beta^{2/\lambda+1}, \quad \hat{\alpha} = (2/\lambda+1)\alpha. \tag{25}$$

$$t = k\int^{\phi} \frac{\mathrm{e}^{\alpha\phi/\lambda}}{(K'\mathrm{e}^{\alpha\phi}+C)^{1/2}}\mathrm{d}\phi; \quad \bar{K'} = \frac{\beta}{\alpha}, \quad k = \frac{\beta^{1/\lambda}}{\sqrt{2}}. \tag{26}$$

The integral in (26) is somewhat intractable for $C \neq 0$. For example, for $\lambda = 2$, i.e., for 'stiff matter' we get

$$t = k\int^{\phi} \frac{\mathrm{e}^{(\alpha/2)\phi}}{(K'\mathrm{e}^{2\alpha\phi}+C)}\mathrm{d}\phi, \tag{27}$$

which with the substitution $y = \mathrm{e}^{\alpha\phi/2}$, yields

$$t = (2k/\alpha)\int \frac{\mathrm{d}y}{(K'y^4+C)^{1/2}} \tag{28}$$

which for $C \neq 0$ is in general an elliptic integral. The resulting exact solution f can be expressed in terms of elliptic integrals but we will not pursue this here, but take $C = 0$, for which an exact solution can be obtained explicitly for all values of λ.

Putting $C = 0$ in (28) and integrating, we get, after some simplification

$$b\exp[(1/2)\tilde{\alpha}\phi]; \quad b = (2/\tilde{\alpha})[\alpha_0(1/\lambda+\tfrac{1}{2})]^{1/2} > 0;$$

α, β are both < 0, with $\tilde{\alpha} = -\alpha > 0$, $(\alpha/\beta) = \alpha_0 > 0$.

$$\rho = \hat{k}^{-1}f = \hat{k}^{-1}b^2(t-t_0)^{-2}. \tag{29}$$

If $t = t_0$ then the density of the Universe is infinitely high like the beginning of the Universe. This is called the initial singularity at which the time coordinate $t = t_0$.

Again

$$\phi = +2\tilde{\alpha}^{-1}\log[b^{-1}(t-t_0)] \tag{30}$$

where t_0 is an arbitrary constant and b is defined in (29).

The function $\Lambda(\phi)$ satisfies the following equation (see (1) and (10)):

$$\Lambda - \phi\frac{\mathrm{d}\Lambda}{\mathrm{d}\phi} = A\phi^{-1}f(\phi) \tag{31}$$

which can be transformed as follows:

$$\Lambda - \phi(t)\frac{\mathrm{d}\Lambda}{\mathrm{d}t}\cdot\frac{\mathrm{d}t}{\mathrm{d}\phi} = A\phi^{-1}\beta\mathrm{e}^{-\tilde{\alpha}\phi}. \tag{32}$$

From (30) we get

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{2\tilde{\alpha}^{-1}}{(t-t_0)}, \tag{32a}$$

so that eq. (32) becomes

$$\Lambda + \log[b^{-1}(t-t_0)] \cdot (t-t_0)\Lambda_t = -\frac{A}{2}\beta b^2 \alpha (t-t_0)^{-2}\{\log[b^{-1}(t-t_0)]\}^{-1}$$

whence we get

$$\frac{\Lambda}{(t-t_0)\{\log[b^{-1}(t-t_0)]\}^2} + \frac{\Lambda_t}{\{\log[b^{-1}(t-t_0)]\}} = \frac{A\beta b^2 \hat{\alpha}}{2(t-t_0)^3\{\log[b^{-1}(t-t_0)]\}^3}. \tag{33}$$

Thus Λ as a function of t is given as follows:

$$\Lambda(t) = \{\log[b^{-1}(t-t_0)]\}F(t) \tag{34}$$

where $F(t)$ is given by quadratures

$$F(t) = (A/2)(\beta b^2 \hat{\alpha}) \int^t \frac{\mathrm{d}t}{(t-t_0)^3\{\log[b^{-1}(t-t_0)]\}^3}. \tag{35}$$

By making suitable transformation or interpretation of the coordinate time t, the singularity in $\Lambda(t)$ given by (34) can probably be associated with the initial singularity (see in (29)). The ϕ dependence of Λ can be obtained from (30), (32a), (34) and (35), as follows:

$$\Lambda(\phi) = \phi G(\phi) \tag{36}$$

where $G(\phi)$ is obtained from (32a), (35) as the following integral:

$$G(\phi) = -B\int^{\phi} \mathrm{e}^{-\hat{\alpha}\phi}\phi^{-3}\mathrm{d}\phi \tag{37}$$

with $B = 1/2A\beta\alpha^{-1}$.

4. Discussion

Singh and Singh obtained exact explicit solutions for $p = 0$ and $p = \rho$. Here we obtain solutions for all values of $\lambda, p = (\lambda - 1)\rho$, which can, in principle provide a 'bridge' between the values $p = 0$ and $p = \frac{1}{3}\rho$ (for the latter the solution becomes vacuous) and between $p = \frac{1}{3}\rho$ and $p = \rho$. Under their assumptions, the case $p = \frac{1}{3}\rho$ leads to $p = \rho = 0$. It is in this sense that this case is vacuous. In this paper we have been concerned mainly with finding Λ as a function of ϕ and ϕ, ρ as function of t. The functions r, θ, ψ can be worked out in principle, but the field equations are somewhat complicated, and we do not for the present purpose need the explicit values for these functions. Indeed Singh and Singh also are content to find expressions for ϕ, ρ in terms of t. However, the fact that we have solutions for all values of λ and that the solutions display singularity for a particular value of t, is of some interest.

The cosmological 'constant' continues to be of interest and a puzzle, both theoretically and observationally. Various aspects of it have been discussed by Carrol and Press [15] (see also Weinberg [16]). The solutions found in this discussion may turn out to be useful in some contexts.

The ever-interesting function ϕ given by (30) may be of interest in connection with the recent observational evidence of an accelerating Universe [17–20].

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