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The geometry of Larmor nutation

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Abstract. The nutation accompanying the well-known Larmor precession discussed with respect to an inertial frame in an earlier paper is now considered in the rotating Larmor frame where the precession is absent and the nutation is therefore fully emphasized. It is shown that, in this frame, the nutating vector generates, in general, what may be called *interpenetrating partial cones*, the two parts of which merge into a single cone traced twice over, when the orbit of the charged particle changes to a circle – giving an immediate explanation of the discontinuous jump in the nutation frequency to twice its value as the orbit changes continuously from an ellipse to a circle.

Keywords. Larmor precession; angular momentum; rotating frame.

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1. Introduction

In an earlier paper [1] it was shown that, the angular momentum vector $\vec{L}$ of a charged particle in the combined central electric field and a uniform magnetic field executes, in addition to the well-known Larmor precession [2–4], a high frequency nutation in an inertial frame. The expression for $\cos\theta_L$ (eq. (3.6) of [1]), where θ_L is the angle between the angular momentum vector and the magnetic field, turned out to be somewhat involved reflecting the fact that the motion of $\vec{L}$ is the combined motion of a precession and a nutation. Graphs depicting the motion for selected values of parameters were also given. We now show that in the Larmor frame, rotating with the Larmor frequency ω, the precessional part $\vec{L}^*$ of $\vec{L}$ is obviously frozen and the nutational part stands out in relief. We have, in this frame, as vivid a geometrical picture of nutation, as we have of precession in the inertial frame. We shall see that in the general case of an elliptic orbit of the charged particle, the nutational part $\vec{\lambda}$ of the angular momentum, which we shall call the nutation vector, actually generates the lateral surface of a double cone, the two parts of which are on the same side of a common vertex, and which, in general, are irregular and interpenetrating, depending upon the position of the elliptic orbit of the charged particle. The two parts of the double cone approach each other as the eccentricity e of the elliptical orbit tends to zero and actually merge into a single cone when $e = 0$. Its lateral surface is then traced twice over in one period of the charge explaining, at once, the discontinuous jump of the nutation frequency to twice its value as the orbit changes from an ellipse into a circle.

For ready reference, we collect here, the main results of our earlier paper [1]. The equation of motion of a charged particle of charge $-q$ $(q > 0)$ moving in the combined central electric field and a uniform magnetic field is

$$\frac{\mathrm{d}(m\vec{v})}{\mathrm{d}t} = f(r)\hat{r} - \frac{q}{c}\,\vec{v}\times\vec{B}. \tag{1.1}$$

On taking the cross product with $\vec{r}$, we obtain

$$\frac{\mathrm{d}\vec{L}}{\mathrm{d}t} = -2m\vec{r}\times(\vec{v}\times\vec{\omega}), \quad \vec{L} = \vec{r}\times m\vec{v}, \quad \vec{\omega} = \frac{q\vec{B}}{2mc} \tag{1.2}$$

where $\vec{L}$ is the angular momentum of the charge and $\omega = |\vec{\omega}| = q|\vec{B}|/2mc$ is the Larmor frequency. Writing the right hand side of eq. (1.2) as a sum of two equal halves and expressing one of them using Jacobi identity [5] as

$$-m\vec{r}\times(\vec{v}\times\vec{\omega}) = m\vec{v}\times(\vec{\omega}\times\vec{r}) + m\vec{\omega}\times(\vec{r}\times\vec{v}) \tag{1.3}$$

we obtain

$$\frac{\mathrm{d}\vec{L}}{\mathrm{d}t} = \vec{\omega}\times\vec{L} + \frac{\mathrm{d}\vec{\lambda}}{\mathrm{d}t} \tag{1.4}$$

where

$$\vec{\lambda} = m\vec{r}\times(\vec{\omega}\times\vec{r}). \tag{1.5}$$

Defining

$$\vec{L}^{*} = \vec{L} - \vec{\lambda} \tag{1.6}$$

eq. (1.4) takes the form

$$\frac{\mathrm{d}\vec{L}^{*}}{\mathrm{d}t} = \vec{\omega}\times\vec{L}^{*} + \vec{\omega}\times\vec{\lambda}. \tag{1.7}$$

Since $\vec{\omega}\times\vec{r}$ is the *velocity of transport* [6] of the charged particle, $\vec{\lambda} = m\vec{r}\times(\vec{\omega}\times\vec{r})$ would be the *angular momentum of transport* and since $\vec{L}$ is the absolute angular momentum, i.e., the angular momentum with respect to the inertial frame, we see that $\vec{L}^{*}$ is the *relative angular momentum*, i.e., the angular momentum with respect to the rotating Larmor frame. Denoting the time derivative with respect to the rotating frame by $\mathrm{d}^{*}/\mathrm{d}t$ and noting that

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}^{*}}{\mathrm{d}t} + \vec{\omega}\times \tag{1.8}$$

eq. (1.7) would assume the form

$$\frac{\mathrm{d}^{*}\vec{L}^{*}}{\mathrm{d}t} = \vec{\omega}\times\vec{\lambda}. \tag{1.9}$$

We now note that $\vec{\omega}\times\vec{\lambda}=\vec{\omega}\times m\,(\vec{r}\times(\vec{\omega}\times\vec{r}))$ is of the order $1/c^2$ and is quadratic in $\vec{B}$. Neglecting this term for the case of weak magnetic fields, eqs (1.7) and (1.9) respectively assume the form

$$\frac{\mathrm{d}\vec{L}^*}{\mathrm{d}t}\approx\vec{\omega}\times\vec{L}^* \tag{1.10}$$

and

$$\frac{\mathrm{d}^*\vec{L}^*}{\mathrm{d}t}\approx 0. \tag{1.11}$$

Equation (1.10) shows that it is $\vec{L}^*$ (and not $\vec{L}$) which is precessing around the field $\vec{B}$ and eq. (1.11) shows that the relative angular momentum $\vec{L}^*$ is a constant vector in the rotating Larmor frame in the approximation where $\vec{\omega}\times\vec{\lambda}$ is neglected. In other words, the dynamical behavior of the angular momentum $\vec{L}=\vec{L}^*+\vec{\lambda}$, in the Larmor frame is entirely that of the angular momentum of transport $\vec{\lambda}$.

2. The geometry of nutation

The fact that $\vec{L}^*$ is constant in the Larmor frame implies that the orbit of the charged particle lies in a plane perpendicular to $\vec{L}^*$. Taking the z-axis of the inertial frame along the direction of the field $\vec{B}$ (or $\vec{\omega}$) and the Z-axis of the Larmor frame along $\vec{L}^*$, we choose the Y-axis of the rotating frame in the Z–z plane so that the vector $\vec{\omega}$ has, in this frame, the components

$$\vec{\omega}=(0,\,\omega\sin\theta,\,\omega\cos\theta) \tag{2.1}$$

where θ is the angle between the Z- and z-axes. The X-axis is evidently perpendicular to $\vec{\omega}$. If $\vec{r}$ is the position vector of the charged particle, we clearly have, in the Larmor approximation

$$\vec{r}\equiv(X,\,Y,\,0) \tag{2.2}$$

and the nutation vector $\vec{\lambda}=m\vec{r}\times(\vec{\omega}\times\vec{r})$ with components $\vec{\lambda}\equiv(\lambda_X,\,\lambda_Y,\,\lambda_Z)$, which may be calculated by the method of successive approximation, on using the X, Y of eq. (2.2) obtained in the previous Larmor approximation. We have

$$\lambda_X=-m\omega\sin\theta XY \tag{2.3}$$

$$\lambda_Y=m\omega\sin\theta X^2 \tag{2.4}$$

$$\lambda_Z=m\omega\cos\theta(X^2+Y^2). \tag{2.5}$$

We note that the orbit of the charged particle is, in general, an ellipse in the XY plane and we assume that the major axis of the ellipse is inclined at an angle ψ to the X-axis (see figure 1).

Introducing polar coordinates (r,ϕ) defining the position of the charge where the polar angle ϕ is measured from the initial line taken along the major axis of the ellipse, the equation of the orbit is evidently

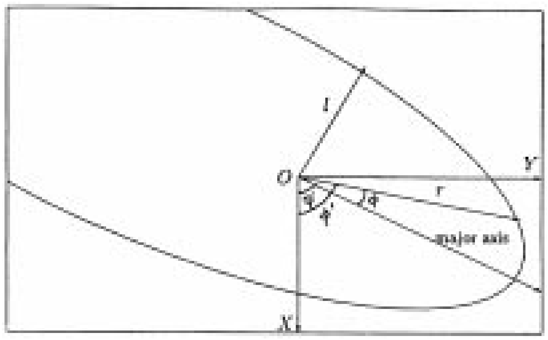

Figure 1. Elliptic orbit of the charged particle in the Larmor frame.

$$\frac{l}{r} = 1 + e\ \cos\phi \tag{2.6}$$

where e is the eccentricity of the orbit and l, the semilatus rectum:

$$l = a(1 - e^2), \tag{2.7}$$

a, being the semi-major axis of the ellipse. Thus the Cartesian coordinates X, Y of the charge are given by

$$X = r\cos\phi' \tag{2.8}$$

$$Y = r\sin\phi' \tag{2.9}$$

$\phi' = \phi + \psi$ and $r = l/(1 + e\ \cos(\phi' - \psi))$. Using eqs (2.6), (2.8) and (2.9) in eqs (2.3)–(2.5) we obtain

$$\lambda_X(\phi') = -\frac{m\omega l^2 \sin\theta\ \cos\phi'\ \sin\phi'}{(1 + e\ \cos(\phi' - \psi))^2} \tag{2.10}$$

$$\lambda_Y(\phi') = \frac{m\omega l^2 \sin\theta\ \cos^2\phi'}{(1 + e\ \cos(\phi' - \psi))^2} \tag{2.11}$$

$$\lambda_Z(\phi') = \frac{m\omega l^2 \cos\theta}{(1 + e\ \cos(\phi' - \psi))^2}. \tag{2.12}$$

Note that $X\lambda_X + Y\lambda_Y = 0$ showing that the vector $\vec{\lambda}_P \equiv (\lambda_X,\ \lambda_Y,\ 0)$ is always orthogonal to $\vec{r}$.

Alternatively, the orientation of the vector $\vec{\lambda}$ relative to the Larmor frame is fully specified if we know the angular separation α from the Z-axis (or from $\vec{L}^*$) and the angle β between the X-axis and the projection $\vec{\lambda}_P$ of $\vec{\lambda}$ on the X–Y plane. From eqs (2.10)–(2.12) we have

$$\lambda_P \equiv |\vec{\lambda}_P| = \sqrt{\lambda_X^2 + \lambda_Y^2} = \frac{m\omega l^2 \sin\theta\,|\cos\phi'|}{(1 + e\ \cos(\phi' - \psi))^2} \tag{2.13}$$

and therefore

$$\tan\alpha = \frac{\lambda_P}{\lambda_Z} = \tan\theta\ |\cos\phi'|, \tag{2.14}$$

$$\tan\beta = \frac{\lambda_Y}{\lambda_X} = -\cot\phi' \Rightarrow \beta = \phi' \pm \pi/2. \tag{2.15}$$

Since λ_Y is always positive, β can only lie in the range $0 \le \beta \le \pi$ and therefore we must have

$$\beta = \begin{cases} \phi' + \pi/2 & \text{when } 0 \le \phi' \le \pi/2, \\ \phi' - \pi/2 & \text{when } \pi/2 \le \phi' \le 3\pi/2, \\ \phi' + \pi/2 & \text{when } 3\pi/2 \le \phi' \le 2\pi. \end{cases} \tag{2.16}$$

We see from eq. (2.14) that α is maximum when $\phi' = 0$ and $\phi' = \pi$. In each case, we have, $\beta = \pi/2$ and $\vec{\lambda}_P$ is along the positive direction of the Y-axis while $\alpha_{\max} = \theta$, the angle between Z- and z-axes.

In order to study the manner in which the nutation vector $\vec{\lambda}$ varies as the charge moves in its orbit, it is convenient to consider first the projection curve traced by the tip of the vector $\vec{\lambda}_P$ in the X–Y plane and this information, together with the way λ_Z varies, gives the complete picture of the variation of $\vec{\lambda}$ as ϕ' changes from 0 to 2π. Since the projection curve in the X–Y plane in the general case of $\psi \neq 0$ retains most of the characteristics of the case with $\psi = 0$, as we shall see presently, it is desirable first to discuss this special case $\psi = 0$ as it shows some illuminating symmetry properties and its analysis is simple and elegant. In the general case of $\psi \neq 0$, we need only examine the portion of the curve in the neighborhood of the origin O of the coordinate system, where there is a significant difference between the two cases.

In the first case where $\psi = 0$, we note that when ϕ' is replaced by $-\phi'$ (or $(2\pi - \phi')$), $\lambda_Y(\phi')$ remains unaltered while $\lambda_X(\phi')$ changes sign. This shows that the projection curve is symmetric about the Y-axis. Secondly, we see that as ϕ' steadily increases from 0 to 2π, λ_X *becomes zero* at $\phi' = 0,\ \pi/2,\ \pi,\ 3\pi/2,\ 2\pi$ and since

$$\frac{\mathrm{d}\lambda_Y}{\mathrm{d}\phi'} = \frac{2\ \lambda_X}{(1 + e\ \cos\phi')}\ , \tag{2.17}$$

these are precisely the values of ϕ' for which λ_Y assumes its extremal values given by

$$\begin{aligned} \lambda_Y(0) &= \lambda_Y(2\pi) = \frac{m\omega l^2\ \sin\theta}{(1+e)^2}\ (\text{ relative maximum}), \\ \lambda_Y(\pi/2) &= \lambda_Y(3\pi/2) = 0\ (\text{ minimum}), \\ \lambda_Y(\pi) &= \frac{m\omega l^2\ \sin\theta}{(1-e)^2}\ (\text{ maximum}). \end{aligned} \tag{2.18}$$

We shall now sketch the form of the projection curve traced by the tip of the vector $\vec{\lambda}_P \equiv (\lambda_X,\ \lambda_Y)$, which we shall see is a loop within a loop.

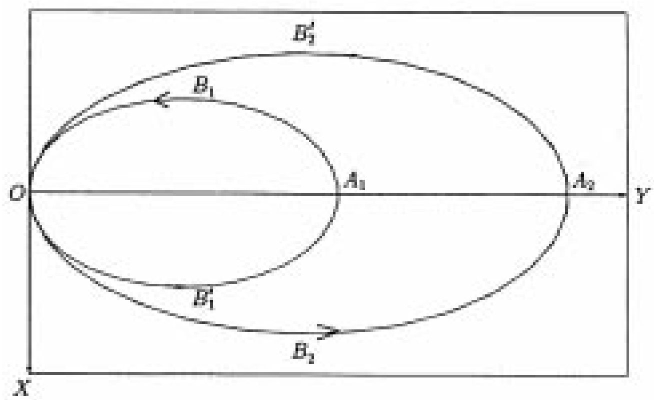

Figure 2. Projection curve traced by the tip of the nutation vector $\vec{\lambda}$, projected on to the *X*–*Y* plane (i.e., the plane to which the motion of the charged particle is confined) of the Larmor frame, when $\psi = 0$.

If $\phi' = 0$ at $t = 0$ we start with the point A_1 of figure 2: $OA_1 = \lambda_Y(0) = (m\omega l^2 \sin\theta)/(1+e)^2$ and $\lambda_X = 0$. As ϕ' increases the representative point moves towards the negative direction of the X-axis as λ_X is negative for such ϕ'. The next zero of λ_X occurs at $\phi' = \pi/2$, when λ_Y is also equal to zero and we arrive at the origin O of the coordinate system. This accounts for the part A_1B_1O of the curve for ϕ' in the range $0 \le \phi' \le \pi/2$.

As observed before, the mirror reflection $A_1B_1'O$ of A_1B_1O is also a part of the curve and this is traced when ϕ' is replaced by $-\phi'$ or equivalently by $2\pi - \phi'$ so that the part $A_1B_1'O$ is obtained when $3\pi/2 \le \phi' \le 2\pi$. Thus when ϕ' is increased from $\pi/2$, the representative point cannot move along $OB_1'A_1$ but along the part OB_2A_2 for the range $\pi/2 \le \phi' \le \pi$ and reaches the maximum value $OA_2 = \lambda_Y(\pi) = m\omega l^2 \sin\theta/(1-e)^2$. The mirror reflection $A_2B_2'O$ is now traced for ϕ' in the range $\pi \le \phi' \le 3\pi/2$. Finally, the representative point touches A_1 again after tracing the part $OB_1'A_1$ for $3\pi/2 \le \phi' \le 2\pi$. Thus, in one complete period of the charged particle, this double loop is traced by the tip of the two-dimensional vector $\vec{\lambda}_P$ which has components (λ_X, λ_Y). For a circular orbit of the charge, $e = 0$, the inner loop of the projection curve swells up and the outer one shrinks to a common circle of diameter $m\omega l^2 \sin\theta$, and the circle is traced twice over in one period of the charge, which immediately explains the discontinuous jump in the nutation frequency to twice its value as the orbit of the particle changes from an ellipse to a circle. More explicitly, when $e = 0$, we have

$$\lambda_X^2 + \left(\lambda_Y - \frac{1}{2} m\omega l^2 \sin\theta\right)^2 = \left(\frac{1}{2} m\omega l^2 \sin\theta\right)^2$$

which is the equation of a circle with radius $\frac{1}{2} m\omega l^2 \sin\theta$ and centre $\left(0,\ \frac{1}{2} m\omega l^2 \sin\theta\right)$.

When $\psi \neq 0$, we note that

$$\lambda_X(\pi/2 + \psi) = m\omega \sin\theta\, l^2\ \cos\psi\ \sin\psi = \lambda_X(3\pi/2 + \psi)$$
$$\lambda_Y(\pi/2 + \psi) = m\omega \sin\theta\, l^2 \sin^2\psi = \lambda_Y(3\pi/2 + \psi) \tag{2.19}$$

i.e., the two segments of the projection curve intersect at a point. Since the corresponding β (see eq. (2.16)) values are $\beta = \phi' - \pi/2 = \pi/2 + \psi - \pi/2 = \psi$ and $\beta = \phi' + \pi/2 =$

$3\pi/2+\psi+\pi/2=\psi$, it is clear that the vectors $\vec{\lambda}_P(\pi/2+\psi)$ and $\vec{\lambda}_P(3\pi/2+\psi)$ are coincident along the major axis of the orbit. The projection curves corresponding to the cases $\psi=\pi/4$ and $\pi/2$ are given in figures 3 and 4.

Since the general form of the projection curve traced by the tip of $\vec{\lambda}_P$ is now available, the form of the space curve traced by the tip of $\vec{\lambda}$ is readily obtained by considering the variation of λ_Z with ϕ'.

Observe that λ_Z is positive when $0\le\theta<\pi/2$, while it is negative when $\pi/2<\theta\le\pi$. In the foregoing discussion, we will restrict ourselves to the case $0\le\theta<\pi/2$ and note that when $\pi/2<\theta\le\pi$, the resulting space curve is merely a mirror reflection (about the X–Y plane) of that obtained for $0\le\theta\le\pi/2$.

We first note that

$$\frac{\mathrm{d}\lambda_Z(\phi')}{\mathrm{d}\phi'}=\frac{2m\omega l^2 e\cos\theta\ \sin(\phi'-\psi)}{[1+e\ \cos(\phi'-\psi)]^3} \tag{2.20}$$

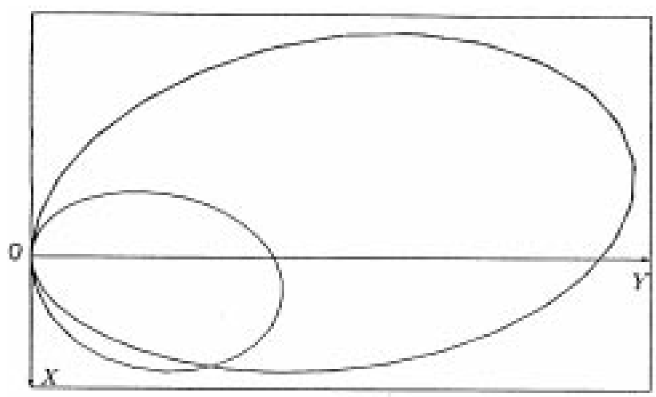

Figure 3. Projection curve for the case $\psi=\pi/4$.

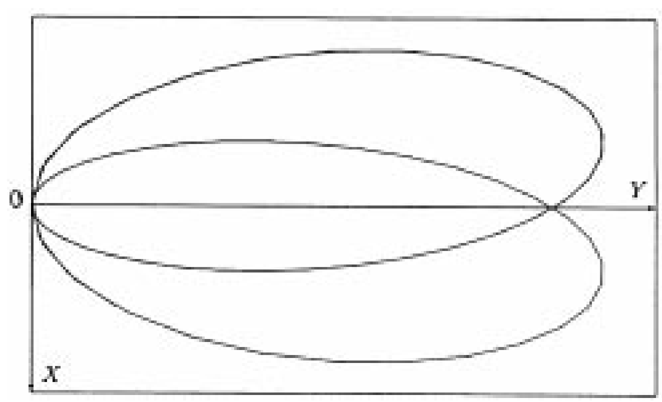

Figure 4. Projection curve for the case $\psi=\pi/2$.

is positive and therefore λ_Z is an increasing function in the range $0 < \phi' - \psi < \pi$ while it is negative so that λ_Z is a decreasing function in the range $\pi < \phi' - \psi < 2\pi$ and the derivative vanishes for $\phi' - \psi = 0,\ \pi$. It is clear from eq. (2.12) that λ_Z is minimum for $\phi' - \psi = 0$ or for $\phi = 0$ and λ_Z is maximum for $\phi' - \psi = \pi$ or for $\phi = \pi$. But for these values of ϕ', the corresponding β values which follow from eq. (2.16) are clearly $\beta = \phi' + \pi/2 = \psi + \pi/2$ and $\beta = \phi' - \pi/2 = \pi + \psi - \pi/2 = \psi + \pi/2$ again. Thus

$$\lambda_Z^{\min} = \frac{m\omega \cos\theta l^2}{(1+e)^2}$$

$$\lambda_Z^{\max} = \frac{m\omega \cos\theta l^2}{(1-e)^2} \tag{2.21}$$

both lie in the plane containing the Z-axis and the *latus rectum* of the orbit of the charged particle. Also, the relation

$$\lambda_Z(\pi/2+\psi) = m\omega l^2 \cos\theta = \lambda_Z(3\pi/2+\psi) \tag{2.22}$$

shows that the two segments of the space curve intersect at a point which lies in the plane containing the Z-axis and the *major axis* (see eq. (2.19)) of the elliptic orbit.

Taking $\psi \le \pi/2$ to be specific we now study the variation of λ_Z with ϕ', as it varies from 0 to 2π: At $\phi' = 0$, we have $\lambda_Z = m\omega l^2 \cos\theta/(1+e\,\cos\psi)^2$ and as ϕ' increases λ_Z initially decreases to its minimum value $\lambda_Z^{\min} = \lambda_Z(\psi) = m\omega \cos\theta l^2/(1+e)^2$ and thereafter increases with ϕ' when it reaches its maximum $\lambda_Z^{\max} = \lambda_Z(\pi+\psi) = m\omega \cos\theta l^2/(1-e)^2$.

Between these two extrema we have

(i) $\lambda_Z(\pi/2) = m\omega \cos\theta l^2/(1+e\,\sin\psi)^2$ at which $\phi',\ \lambda_X = 0 = \lambda_Y$ so that the representative point of the locus is on the Z-axis;

(ii) $\phi' = \pi/2+\psi$ corresponds to the intersection point where $\lambda_Z(\pi/2+\psi) = m\omega \cos\theta l^2$ and

(iii) $\phi' = \pi$, $\lambda_Z(\pi) = m\omega l^2 \cos\theta/(1-e\,\cos\psi)^2$, where the point of the locus lies in the Z–Y plane since $\lambda_X(\pi) = 0,\ \lambda_Y(\pi) = m\omega l^2 \sin\theta/(1-e\,\sin\psi)^2$.

Now, as ϕ' increases further from $\pi+\psi$ to 2π, λ_Z steadily decreases assuming the values $\lambda_Z(3\pi/2) = \lambda_Z(3\pi/2) = m\omega l^2 \cos\theta/(1-e\,\sin\psi)^2$, the point of the locus being on the Z axis; $\lambda_Z(3\pi/2+\psi) = m\omega l^2 \cos\theta$ at the intersection point and finally $\lambda_Z(2\pi) = \lambda_Z(0) = m\omega l^2 \cos\theta/(1+e\,\cos\psi)^2$.

These values of λ_Z together with the form of the projection curve discussed earlier show that the nutation vector $\vec{\lambda}$ generates – what may be called – *interpenetrating partial cones.* We observe that the Z- and z-axes both lie on the surface of this *double cone*. The intersection point of the two segments lies in the plane containing the Z-axis and the major axis, while the extrema lie in the plane defined by the Z-axis and the latus rectum of elliptic orbit.

3. Summary

In this paper, the nutational motion of the angular momentum vector of a charged particle moving under the influence of a central electric field and a uniform magnetic field has been analyzed in the rotating Larmor frame. The advantange in adapting the Larmor frame is that the well-known Larmor precession of the angular momentum is absent in this frame and therefore the nutational motion of the angular momentum can be investigated in an elegant manner. It is shown that the nutation vector generates *interpenetrating partial cones*, the two parts of which are on the same side of a common vertex. As the orbit of the charged particle changes continuously from an ellipse to a circle (i.e., when the eccentricity e of the elliptic orbit approaches zero), the two parts of the double cone generated by the nutation vector tend to merge into a single cone, the lateral surface of which is traced twice over in one period of the charged particle. This provides a natural explanation of the discontinuous jump of the nutation frequency to twice its value as the orbit of the charged particle changes gradually from an ellipse to a circle.

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References

[1] K N Srinivasa Rao and A V Gopal Rao, *Pramana – J. Phys.* **48**, 775 (1997)
[2] K R Symon, *Mechanics* (Reading, Massachusetts, Addison-Wesley, 1960) pp. 271–278
[3] L D Landau and E M Lifshitz, *Mechanics* (Pergamon Press, Oxford, 1976) pp. 32–39
[4] R P Feynman, R B Leighton and M Sands, *The Feynman lectures on physics* (Reading, Massachusetts, Addison-Wesley, 1964) pp. 34–6, 34–7
[5] The cross product of three vectors $\vec{A}$, $\vec{B}$, $\vec{C}$ satisfies the Jacobi identity: $\vec{A}\times(\vec{B}\times\vec{C})+\vec{B}\times(\vec{C}\times\vec{A})+\vec{C}\times(\vec{A}\times\vec{B})=0$
[6] For the precise definitions of absolute velocity, relative velocity, velocity of transport and their mutual relationship, see Stefan Banach, *Mechanics* (originally in Polish), English translation by E J Scott (Hafner Publishing Company, New York, 1951) pp. 56–61