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Derivation of relativistic wave equation from the Poisson process

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Abstract. A Poisson process is one of the fundamental descriptions for relativistic particles: both fermions and bosons. A generalized linear photon wave equation in dispersive and homogeneous medium with dissipation is derived using the formulation of the Poisson process. This formulation provides a possible interpretation of the passage time of a photon moving in the medium, which never exceeds the speed of light in vacuum.

Keywords. Poisson process; linear photon wave equation; passage time.

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The motion of relativistic particles is represented by the Poisson process. Kac proposed the continuous version of Poisson process in 1956 [1]. Gaveau *et al* derived the Dirac equation in 1984 using the formulation of Poisson process [2]. McKeon and Old constructed a modified version of the Poisson process in 1992 [3], where the relativistic particles can move both forward and backward in time. These Poisson processes were extended in 2001 [4] to more general frameworks of three-dimensional Dirac equation including an external field. Bialynicki–Birula invented a linear photon wave equation in 1994 [5] as a linearized version of the d'Alembert wave equation. Sipe verified in 1995 [6] that this wave equation identifies the ordinary canonical quantization.

We treat the case that a time-dependent external potential $V(t)$ exists [4]. Let an absorption probability in a time interval δ be $I(t)\delta$, where the path can be absorbed at most once in a time interval δ by the influence of the potential at time t. The conditional expectation is represented as

$$f_{\pm}(x,t) = (1-I\,\delta)\times[\,(1-a\,\delta)\,E(\mathrm{e}^{s(t)L}\varphi_{\pm}(x)|\,(-1)^{N_a(t)} = (-1)^{N_a(t-\delta)} = \pm 1)$$
$$+a\,\delta\,E(\mathrm{e}^{s(t)L}\varphi_{\pm}(x)|-(-1)^{N_a(t)} = (-1)^{N_a(t-\delta)} = \mp 1)\,] + O(\delta^2), \qquad (1)$$

where $1-I\,\delta$ corresponds to a time shift generator in a time interval δ, and the one-dimensional space-shift generator is $L = -c\partial/\partial x$. The one-dimensional space-shift generator has to be generalized to a three-dimensional generator with minimal coupling, $L = -c\,\boldsymbol{\sigma}\cdot(\boldsymbol{\nabla} - ie\mathbf{A}(\mathbf{x})/\hbar)$ in the case that vector field $\mathbf{A}$ exists, where $\boldsymbol{\sigma}$ gives the Pauli matrix [2]. It should be noted that the evolution operator is given by a simple form, such as $\exp(s(t)L)$. Since L is a 2×2 matrix, $\varphi_\pm(x)$, a characteristic function of the interval between x and $x\mp v\delta$, is replaced by a two-component spinor function $\boldsymbol{\varphi}_\pm(\mathbf{x})$, and $f_\pm(x,t)$ is replaced by a two-component spinor $\mathbf{f}_\pm(\mathbf{x},t)$. In the continuous limit of time, (1) becomes the three-dimensional master equation. If we set $a=-imc^2/\hbar, I=iV/\hbar$ and $\psi^T=\mathrm{e}^{at}(\mathbf{f}_+^T,\mathbf{f}_-^T)$, the three-dimensional master equation becomes the three-dimensional Dirac equation in an external field

$$i\hbar\frac{\partial\psi}{\partial t}=mc^2\sigma_1\otimes\mathbb{1}\,\psi-ic\hbar\sigma_3\otimes\boldsymbol{\sigma}\cdot\left(\boldsymbol{\nabla}-i\frac{e}{\hbar}\mathbf{A}(\mathbf{x})\right)\psi+V(t)\,\psi. \tag{2}$$

Now we want to derive a three-dimensional linear bosonic wave equation. Then the generator should be changed to spin-1 space-shift generator $L=-c\,\mathbf{S}\cdot\boldsymbol{\nabla}$. Here $\mathbf{S}$ is a 3×3 $SO(3)$ matrix, which is represented by $(S_i)_{jk}=-i\varepsilon_{ijk}$. We consider linear photon wave equation in dispersive medium with the time dependent material parameters, a dielectric constant $\varepsilon(t)$, a permeability $\mu(t)$ and an electrical conductivity $\sigma(t)$. We assume that they are homogeneous in space and the velocity of light is well represented by $c=1/\sqrt{\varepsilon(t)\mu(t)}$. In this case there occurs not only a time depending dispersive term $\sigma(t)$, but also additional absorption terms proportional to $\partial\varepsilon(t)/\partial t$ and $\partial\mu(t)/\partial t$. We have to incorporate this effect into the Poisson process. In the dispersive medium, the local structure of medium, $K(t)-K(t-\delta)$, determines the absorption, $I(t)\delta=K(t)-K(t-\delta)$, where the function, $K(t)$, corresponds to the absorption till the time t. Then the evolution of system at a time t has to be satisfied as follows:

$$\begin{aligned}\mathbf{F}_\pm(\mathbf{x},t)&=(1-I(t)\delta)\times[(1-a\,\delta)E(\mathrm{e}^{s(t)L}\boldsymbol{\varphi}_\pm(\mathbf{x})|(-1)^{N_a(t)}=(-1)^{N_a(t-\delta)}=\pm1)\\&\quad+a\,\delta\,E(\mathrm{e}^{s(t)L}\boldsymbol{\varphi}_\pm(\mathbf{x})|-(-1)^{N_a(t)}=(-1)^{N_a(t-\delta)}=\mp1)+O(\delta^2)],\end{aligned} \tag{3}$$

where $\boldsymbol{\varphi}_\pm(\mathbf{x})$ and $\mathbf{F}_\pm(\mathbf{x},t)$ are the three-component vector functions. The evolution of system of $\mathbf{F}=\mathbf{F}_++\mathbf{F}_-^*$ in (3) is invariant under the gauge transformation at time t and time $t-\delta$. The constructive operator $K(t)$ should be determined by the material parameters at the time t,

$$K(t)=\left(\ln\sqrt{\varepsilon(\mathrm{t})\mu(\mathrm{t})}+\rho_1\ln\sqrt{\mu(\mathrm{t})/\varepsilon(\mathrm{t})}\right)/2.$$

Here ρ_1 exchanges upper and lower components ($\rho_1\mathbf{F}_\pm=\mathbf{F}_\mp$). The continuous limit of time in (3) gives the master equation. We combine $\mathbf{F}_\pm$ into a six-component vector, $\mathscr{F}^T=(\mathbf{F}_+^T,\mathbf{F}_-^T)$ and set $a=0.5\,\sigma(t)/\varepsilon(t)$ [1/s]. Later we will show that the velocity of photons never exceeds the light velocity in vacuum because a is real and no additional phase factor is needed. The master equation becomes

$$i\hbar\left(\frac{\partial}{\partial t}+\frac{\partial K}{\partial t}\right)\mathscr{F}=c\frac{\hbar}{i}\sigma_3\otimes\mathbf{S}\cdot\boldsymbol{\nabla}\mathscr{F}+\frac{\hbar\sigma}{2i\varepsilon}(1-\sigma_1\otimes\mathbb{1})\mathscr{F}. \tag{4}$$

This is a generalized version with dissipation of the linear photon wave equation in the dispersive and homogeneous medium without dissipation (83) in [5]. The constitutive relations are $\mathbf{D}(\mathbf{x},t)=\varepsilon(t)\mathbf{E}(\mathbf{x},t)$ and $\mathbf{B}(\mathbf{x},t)=\mu(t)\mathbf{H}(\mathbf{x},t)$, and the ohmic law is $\mathbf{i}(\mathbf{x},t)=\sigma(t)\mathbf{E}(\mathbf{x},t)$ in the dispersive and homogeneous medium. The continuity equation, $\partial\rho/(\partial t)+\mathrm{div}\,\mathbf{i}=0$, holds at the arbitrary time in the medium. We set

$\mathbf{F} = \mathbf{B}/\sqrt{2\mu(t)} - i\mathbf{D}/\sqrt{2\varepsilon(t)}$. The divergence of the continuous limit (master equation) of the evolution of the system (3) becomes

$$\left(\frac{\partial}{\partial t} + \frac{\partial K}{\partial t}\right) \operatorname{div} \mathbf{F} = -\frac{i}{\sqrt{2\varepsilon}} \frac{\partial}{\partial t} \rho, \tag{5}$$

using the identity, $(\mathbf{S} \cdot \mathbf{a})\mathbf{F} = i\mathbf{a} \times \mathbf{F}$, and the continuity equation. Note that the real part of the left-hand side vanishes and this gives one of the ordinary Maxwell equations, $\operatorname{div} \mathbf{B}(\mathbf{x}, t) = 0$. This relation (5) from the evolution of the system (3) corresponds to Gauss' law at an arbitrary time t by initially setting the law in the dispersive case. The photon wave equation (4) and the relation (5) are theoretically based on the correct Maxwell equation in medium with source and current.

Next we determine the passage time of photons through the nondispersive and homogeneous medium, where ε, μ and σ are constant. The distribution function $g(t, \tau)$ of random variable $s(t)$ was given by DeWitte–Morette and Foong [7]. Mugnai *et al* proposed to define a passage time in the dissipative wave propagation [8]. One can get the mean of the randomized time, $\langle s(t) \rangle = \int_{-\infty}^{\infty} \tau g(t, \tau)\, \mathrm{d}\tau = (\varepsilon/\sigma)(1 - \mathrm{e}^{-\sigma t/\varepsilon})$. Note that $\langle s(t) \rangle$ becomes the real time t in the limit of no dissipation ($\sigma/\varepsilon \to 0$). This limit corresponds to the case of the zero probability of reversal in the Poisson process. Thus the randomized time, $\langle s(t) \rangle$ multiplied by the light velocity c in the medium is the photon flight range $l = c\langle s(t) \rangle$ in the limiting case of no dissipation. On these observations, the passage time through the range l is generally determined by a relation, $l = c\langle s(t) \rangle$. Then we propose the passage time t of the photons moving in the nondispersive and homogeneous medium of the range l,

$$t = \frac{l}{c} \ln\left(1 - \frac{\sigma l}{\varepsilon c}\right)^{-\varepsilon c/(\sigma l)} \geq \frac{l}{c} \geq \frac{l}{c_0}. \tag{6}$$

The photons in medium move slower than light velocity in vacuum.

As a result, the linear relativistic wave equations interacting the external field (not only Dirac equation but also linear photon wave equation) can be derived by the formulation of Poisson process. This formulation is useful for a generalized linear photon wave equation in dispersive and homogeneous medium with dissipation. The notion of randomized time is useful to obtain a possible interpretation of the passage time of the photon moving in the medium, and the passage time of a distance l may be estimated by (6).

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