

## The crossover from classical to quantum behavior in Duffing oscillator based on quantum state diffusion

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**Abstract.** The classical Duffing oscillator is a dissipative chaotic system, and so there is not a definite Hamiltonian. We quantize the Duffing oscillator on the basis of quantum state diffusion (QSD) which is a formulation for open quantum systems and a useful tool for analyzing nonlinear problems and classical limits. We can define a ‘Lyapunov exponent’, which corresponds to the classical one in the proper limit, and investigate the behavior of the system by varying the Planck constant effectively. We show that there exists a critical stage, where the behavior of the system crosses over from classical to quantum one.

**Keywords.** Chaos; quantum state diffusion; crossover; Lyapunov exponent.

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### 1. Introduction

It is a very interesting problem to see how a certain classical system is related to its corresponding quantum system. In this study we analyze a dissipative chaotic system, the Duffing oscillator, and discuss the crossover from classical to quantum behavior based on quantum state diffusion (QSD) [1,2], which is a formulation of open quantum system approaches and equivalent to the Lindblad master equation [3,4]. While the latter describes the time evolution of the density operator, QSD describes that of the pure state vector according to a stochastic differential equation. We can easily define a quantum analogue of the ‘Lyapunov exponent’ and investigate it. From our numerical calculations we find explicit crossover behavior, so the chaotic behavior gets completely lost in the quantum region.

### 2. The quantized Duffing oscillator

As the classical Duffing oscillator [5] is a dissipative system, there is not a definite Hamiltonian. We quantize it using QSD. Let us assume the dynamics to be Markovian, and the Hamiltonian  $\hat{H}$  and the Lindblad operator  $\hat{L}$  are defined phenomenologically as

$$\hat{H} = \frac{\hat{P}^2}{2} + \frac{\hat{Q}^4}{4} - \frac{\hat{Q}^2}{2} + \frac{\Gamma}{2}(\hat{Q}\hat{P} + \hat{P}\hat{Q}) - g \cos(\Omega t)\hat{Q},$$

$$\hat{L} = \sqrt{2\Gamma} \hat{a}.$$

Calculating the constant phase map numerically, we find that the chaotic behavior appears in the limit that the characteristic action for the system is larger than the Planck constant  $\hbar$ . We actually perform the simulation with small  $\beta$ , where  $\beta$  is the ratio of  $\hbar$  to system's action. Figure 1 shows this result. It agrees with [2].

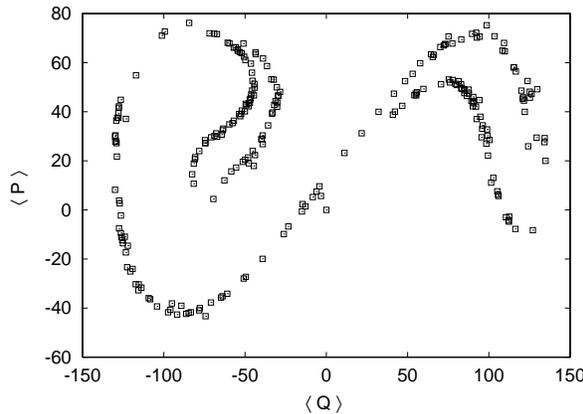
### 3. The crossover from classical to quantum behavior

But it is inadequate for the following problems. At what stage is the chaotic behavior lost? Is there such a stage? We investigate a quantum analogue of the 'Lyapunov exponent' which is the quantity characterizing chaotic systems in classical mechanics.

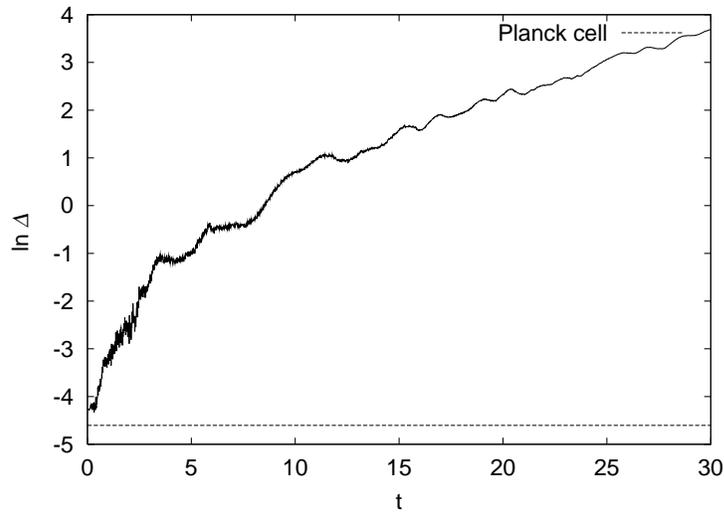
Note that any two extremely neighboring points in phase space (strictly speaking,  $\langle \hat{Q} \rangle - \langle \hat{P} \rangle$  plane) are indistinguishable due to the Heisenberg's uncertainly principle. We call the region in phase space where any two points are indistinguishable by Planck cell. The size of Planck cell is of the order of  $\beta$  in our model. Therefore, we should prepare two initial pure states  $|\psi_1\rangle, |\psi_2\rangle$  satisfying the condition for proper initial separation

$$\Delta_{12}(t=0) = \sqrt{(\bar{q}_1 - \bar{q}_2)^2 + (\bar{p}_1 - \bar{p}_2)^2} = \varepsilon \sim 0.01,$$

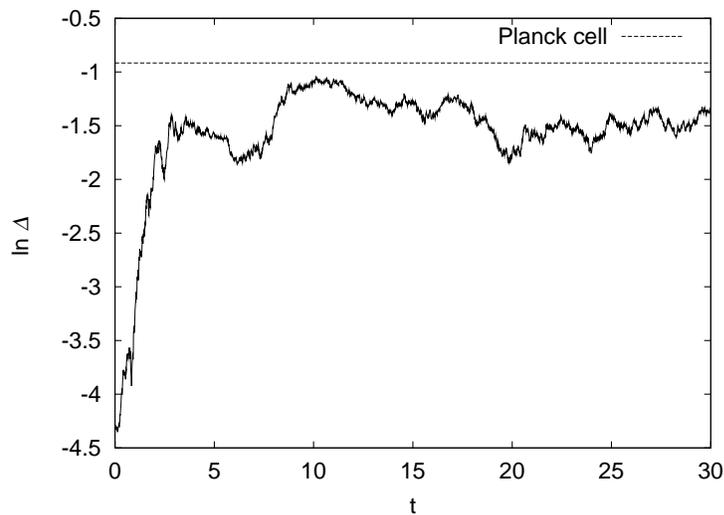
where  $\bar{q}_i = \langle \psi_i | \hat{Q} | \psi_i \rangle, \bar{p}_i = \langle \psi_i | \hat{P} | \psi_i \rangle$  ( $i = 1, 2$ ). These two points are distinguishable in the classical region ( $\beta = 0.01$ ). We calculate time-dependent expectation values corresponding to each initial condition. And we obtain the time dependence of the separation. We average  $\Delta_{12}(t)$  over any proper initial separations in the fixed  $\varepsilon$ . Then we obtain the time dependent distance of 'trajectory'  $\Delta(t)$  under the proper initial separations. See figures 2 and 3. Figure 2 shows that in the classical region  $\Delta(t)$  represents exponentially



**Figure 1.** The constant phase map obtained by plotting value in phase space by  $2\pi/\Omega$  in  $\beta = 0.01$ .  $(\Gamma, g, \Omega) = (0.125, 0.3, 1.00)$ .



**Figure 2.**  $\Delta(t)$  in  $\beta = 0.01$ .  $(\Gamma, g, \Omega) = (0.125, 0.3, 1.00)$ . This is obtained by averaging over 20 initial conditions.



**Figure 3.**  $\Delta(t)$  in  $\beta = 0.40$ .  $(\Gamma, g, \Omega) = (0.125, 0.3, 1.00)$ . This is obtained by averaging over 10 initial conditions and averaging over an ensemble of 100 stochastic processes in QSD for each initial condition.

blowing-up behavior, corresponding to the maximal Lyapunov exponent being positive. Moreover, figure 3 shows that at some stage (about  $\beta = 0.40$ )  $\Delta(t)$  never leaves the Planck cell, so at this stage the chaotic dynamics is completely lost.

#### 4. Summary

We have analyzed the crossover from quantum to classical behavior for the Duffing oscillator based on QSD. We have shown that we can discuss the crossover phenomena through the investigation of the 'Lyapunov exponent', changing the Planck constant effectively and that there exists a stage at which the classical dynamical property is completely lost. The point is that we can define the proper separation, considering the quantum effect that any two points in phase space never leaves the Planck cell. It should be very interesting to see whether there is such a crossover in a general dissipative system. It should also be important to investigate what effect it produces using other methods, for example, the analysis of the Wigner function and the von Neuman entropy [6].

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