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Some stochastic aspects of quantization

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Abstract. From the advent of quantum mechanics, various types of stochastic-dynamical approach to quantum mechanics have been tried. We discuss how to utilize Nelson's stochastic quantum mechanics to analyze the tunneling phenomena, how to derive relativistic field equations via the Poisson process and how to describe a quantum dynamics of open systems by the use of quantum state diffusion, or the stochastic Schrödinger equation.

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1. Introduction

It is well known that all the phenomena in classical mechanics are described by deterministic dynamical laws, even the classical chaotic phenomena. In chaos, two initially distinct points with very tiny difference in phase space separate exponentially from each other in a finite time duration. Thus we cannot actually prepare an initial state so precisely as to approach a definite final state. However, its time evolution is completely determined by classical dynamical laws.

There also exist stochastic or random dynamical processes in classical phenomena. This state of affairs, e.g., tossing a coin, rolling a die and Brownian motion of a small particle, are related to one another by the probability theory. Brownian motion of a small particle suspended in a fluid is described by a probability. A particle at (x,t) will be found at $(x+\Delta x, t+\Delta t)$ with the probability

$$\mathrm{Prob}(x+\Delta x, t+\Delta t|x,t), \tag{1}$$

and this is computable from the deterministic law for the motion of a particle in a fluid, acted on by a noise or a random force with known statistical properties which accounts for the ignorance of data of the precise values of the coordinates (x_i, p_i) of molecules of the fluid, the container and the Brownian particles themselves. However, in principle, it is possible to predict the motion.

In quantum mechanics, the situation is somewhat similar to the above. It is possible to make more refinement, by the use of a lower dimensional subspace of the total Hilbert

space. The lowest one is a subspace of 1 dimension. However, even at this level, there exists irreducible 'quantum noise'. In this talk, this idea will be explained.

The simplest statistical description may be a binomial distribution

$$W(N_1) = \frac{N!}{N_1!(N-N_1)!}\, p^{N_1}\, (1-p)^{N-N_1}, \tag{2}$$

where p is the probability with which a certain event will occur. This type of distribution has two limiting forms:

$$W(N_1) \longrightarrow \begin{cases} \frac{\langle N_1\rangle^{N_1}}{N_1!}\mathrm{e}^{-\langle N_1\rangle}, & \text{Poisson,} \quad p \ll 1, \\ \frac{1}{\sqrt{2\pi\langle N_1\rangle}}\exp\left(-\frac{(N_1-\langle N_1\rangle)^2}{2\langle N_1\rangle^2}\right), & \text{Gaussian,} \quad N \gg 1. \end{cases} \tag{3}$$

In the following, we use these two types of statistics.

From the advent of quantum mechanics, various types of stochastic-dynamical approaches to quantum mechanics have been tried. One of the important purposes of these attempts has been to find out unknown physical reasons of quantum fluctuations. Though these attempts unfortunately have not been successful and it is conceived that almost none of them can surpass the Copenhagen interpretation, the ideas of these approaches have several merits in understanding the fundamental quantum theory. As an example, we first discuss how to utilize Nelson's stochastic quantum mechanics to analyze the tunneling phenomena. Secondly, we explain how to derive relativistic field equations via the Poisson process. Finally, we discuss how to describe the quantum dynamics of an open system by the use of quantum state diffusion, or the stochastic Schrödinger equation.

2. Nelson's quantum mechanics

In Nelson's quantum mechanics, ordinary dynamical variables are treated as random variables which are subject to Ito-type Langevin equations with Gaussian noises [1]. Here one does not need a wave function, but the random variables evolve according to the Langevin equation.

In this formalism, for a specific sequence of Gaussian noises, the time development of a dynamical variable, for example, $x(t)$ is given by the Langevin equation. The ensemble of such sample paths in the configuration space gives the same prediction as ordinary quantum mechanics, including information about the phase. In a sense Nelson's quantum mechanics gives a particle picture with the quantum phase. Each sample path $x(t)$ has its own history. Thus it provides a useful method to analyse quantum phenomena which seem to have more particle nature rather than wave nature, for example, tunneling time, quantum analogue of trajectory, and so on.

Now, let me review Nelson's quantum mechanics briefly. The dynamical variable develops according to an Ito-type stochastic differential equation, the Langevin equation,

$$\mathrm{d}x(t) = b(x(t),t)\mathrm{d}t + \mathrm{d}w(t), \tag{4}$$

for forward time evolution, and

$$\mathrm{d}x(t) = b_*(x(t),t)\mathrm{d}t + \mathrm{d}w_*(t), \tag{5}$$

for backward time evolution, where $\mathrm{d}w(t)$ is a Gaussian white noise with stochastic properties:

$$\langle \mathrm{d}w(t)\rangle = 0 \quad \text{and} \quad \langle \mathrm{d}w(t)\mathrm{d}w(t)\rangle = \frac{\hbar}{m}\mathrm{d}t, \tag{6}$$

and $\langle\cdots\rangle$ denotes the average over the Gaussian noise.

The Langevin equations yield the following equivalent Fokker–Planck equations for the distribution function $P(x,t)$:

$$\frac{\partial P(x,t)}{\partial t} = \left\{ -\frac{\partial}{\partial x} b(x,t) + \frac{\hbar}{2m}\frac{\partial^2}{\partial x^2} \right\} P(x,t), \tag{7}$$

$$-\frac{\partial P(x,t)}{\partial t} = \left\{ \frac{\partial}{\partial x} b_*(x,t) + \frac{\hbar}{2m}\frac{\partial^2}{\partial x^2} \right\} P(x,t). \tag{8}$$

Addition and subtraction of these two equations give

$$u = \frac{b-b_*}{2} = \frac{\hbar}{2m}\frac{1}{P}\frac{\partial P}{\partial x}, \tag{9}$$

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x}(vP), \tag{10}$$

respectively, where u is an osmotic velocity and v is a current velocity,

$$v = \frac{b+b_*}{2}. \tag{11}$$

Elimination of P from eqs (9) and (10) leads to the kinematical equation,

$$\frac{\partial u}{\partial t} = -\frac{\hbar}{2m}\frac{\partial^2 v}{\partial x^2} - \frac{\partial}{\partial x}(uv). \tag{12}$$

Nelson introduced the 'mean forward time derivative' $Df(t)$,

$$Df(t) \equiv \lim_{\Delta t \to +0} \left\langle \frac{f(t+\Delta t) - f(t)}{\Delta t} | f(s)\,(s \le t) \ \text{ fixed} \right\rangle, \tag{13}$$

and similarly the 'mean backward time derivative' $D_* f(t)$. The 'mean balanced acceleration' is introduced as

$$\begin{aligned} a(x(t),t) &\equiv \frac{DD_* + D_*D}{2} x(t) \\ &= \frac{\hbar}{2m}\frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial x}(v^2 - u^2) + \frac{\partial v}{\partial t}. \end{aligned} \tag{14}$$

Here we have used (4) and (5). The dynamical condition is the classical Newton equation

$$ma = -\frac{\partial V}{\partial x}, \tag{15}$$

from which we get the 'Newton–Nelson equation'

$$\frac{\partial v}{\partial t} = \frac{\hbar}{2m}\frac{\partial^2 u}{\partial x^2} - v\frac{\partial v}{\partial x} + u\frac{\partial v}{\partial x} - \frac{1}{m}\frac{\partial V}{\partial x}. \tag{16}$$

Nelson's quantum mechanics has a set of two basic equations, the kinematical equation (12) and the dynamical equation (16) for two unknown functions u and v, or equivalently, for b and b_*. Then we can determine the ensemble of sample paths or the distribution function P. It is very easy to show the equivalence between this approach and the ordinary approach of the Schrödinger equation. From the combination of eq. (12) and the imaginary version of eq. (16) we obtain the equation

$$\frac{\partial}{\partial x}\left\{ i\frac{\hbar}{m}\frac{1}{\psi'}\frac{\partial \psi'}{\partial t} + \frac{1}{2}\left(\frac{\hbar}{m}\right)^2 \frac{1}{\psi'}\frac{\partial^2 \psi'}{\partial x^2} - \frac{1}{m}V \right\} = 0, \tag{17}$$

where

$$u + iv = \frac{\hbar}{m}\frac{1}{\psi'}\frac{\partial \psi'}{\partial x}. \tag{18}$$

Equation (17) shows the relationship between ψ' and the wave function ψ as the solution of the Schrödinger equation

$$i\hbar\frac{\partial \psi}{\partial t} = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V\right)\psi, \tag{19}$$

that is,

$$\psi(x,t) = \psi'(x,t)\exp\left(-\frac{im}{\hbar}\int^t \eta(s)\mathrm{d}s\right) \tag{20}$$

with an arbitrary function of t, $\eta(t)$, which has no physical relevance. It is easy to show that

$$b = \frac{\hbar}{m}(\mathrm{Im} + \mathrm{Re})\frac{\partial}{\partial x}\ln\psi, \tag{21}$$

$$b_* = \frac{\hbar}{m}(\mathrm{Im} - \mathrm{Re})\frac{\partial}{\partial x}\ln\psi, \tag{22}$$

$$P = |\psi(x,t)|^2. \tag{23}$$

This approach of a real-time stochastic process enables us to describe individual experimental runs of a quantum system in the terminology of analogs of classical trajectories, and we have obtained information on the tunneling time [2]. Furthermore, we have analyzed the effects of inelastic scattering on the tunneling time theoretically, generalizing Nelson's quantum mechanics [3], and proposed a method to measure the Büttiker–Landauer traversal time [4]. Recently, we have also analyzed the tunneling time distribution and found wave-particle duality in the distribution [5].

3. Derivation of relativistic field equations via the Poisson process

In 1956, Kac gave a nice lecture on quantization based on the Poisson process [6]. We start a particle from the origin $x = 0$ and the particle always moves with speed v. It can move either in the positive direction with a probability $1 - a\Delta t$ or in the negative direction with a probability $a\Delta t$ in each step which is of time duration Δt and covers a distance Δx. So we have $|\Delta x| = v\Delta t$. We introduce a random variable

$$\varepsilon = \begin{cases} 1 & \text{with probability } 1 - a\Delta t, \\ -1 & \text{with probability } a\Delta t, \end{cases} \tag{24}$$

and consider a sequence of such independent random variables $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1}$. The frequency of occurence of $\varepsilon = -1$ is determined by the Poisson distribution, if $a\Delta t$ is small. If we start in the positive direction from the origin, then the displacement from the origin after n steps will be

$$S_n = v\,\Delta t\,(1 + \varepsilon_1 + \varepsilon_1\varepsilon_2 + \cdots + \varepsilon_1\varepsilon_2\cdots\varepsilon_{n-1}). \tag{25}$$

If we start in the negative direction, then the displacement would have been

$$S'_n = -v\,\Delta t\,(1 + \varepsilon_1 + \varepsilon_1\varepsilon_2 + \cdots + \varepsilon_1\varepsilon_2\cdots\varepsilon_{n-1}) = -S_n. \tag{26}$$

Let us consider the following two functions

$$F_n^+(x) = \langle \phi(x + S_n)\rangle, \tag{27}$$

$$F_n^-(x) = \langle \phi(x - S_n)\rangle. \tag{28}$$

Substituting eq. (25) into eq. (27), we have

$$F^+(x) = \langle \phi(x + v\Delta t(1 + \varepsilon_1 + \varepsilon_1\varepsilon_2 + \cdots + \varepsilon_1\varepsilon_2\cdots\varepsilon_{n-1}))\rangle, \tag{29}$$

where $\langle\cdots\rangle$ means the weighted sum over all possible sequences. Thus we obtain a path integral form of the function ϕ based on the Poisson process. In order to write a recursion formula, we rewrite eq. (29) and perform the average on ε_1,

$$\begin{aligned} F_n^+(x) &= \langle \phi(x + v\Delta t + v\Delta t\varepsilon_1(1 + \varepsilon_2 + \varepsilon_2\varepsilon_3 + \cdots + \varepsilon_2\varepsilon_3\cdots\varepsilon_{n-1}))\rangle \\ &= a\Delta t\,\langle \phi(x + v\Delta t - v\Delta t(1 + \varepsilon_2 + \varepsilon_2\varepsilon_3 + \cdots + \varepsilon_2\varepsilon_3\cdots\varepsilon_{n-1}))\rangle \\ &\quad + (1 - a\Delta t)\,\langle \phi(x + v\Delta t + v\Delta t(1 + \varepsilon_2 + \varepsilon_2\varepsilon_3 + \cdots + \varepsilon_2\varepsilon_3\cdots\varepsilon_{n-1}))\rangle \\ &= a\Delta t F_{n-1}^-(x + v\Delta t) + (1 - a\Delta t)F_{n-1}^+(x + v\Delta t). \end{aligned} \tag{30}$$

Similarly, we have

$$F_n^-(x) = a\Delta t F_{n-1}^+(x + v\Delta t) + (1 - a\Delta t)F_{n-1}^-(x + v\Delta t). \tag{31}$$

Now we can rewrite eq. (30):

$$\begin{aligned} \frac{F_n^+(x) - F_{n-1}^+(x)}{\Delta t} &= \frac{F_{n-1}^+(x + v\Delta t) - F_{n-1}^+(x)}{\Delta t} \\ &\quad - aF_{n-1}^+(x + v\Delta t) + aF_{n-1}^-(x + v\Delta t). \end{aligned} \tag{32}$$

And now we can pass to the limit of $\Delta t \to 0$ to get

$$\frac{\partial F^+}{\partial t} = v\frac{\partial F^+}{\partial x} - aF^+ + aF^-. \tag{33}$$

From eq. (31), we get in a similar way

$$\frac{\partial F^-}{\partial t} = -v\frac{\partial F^-}{\partial x} + aF^+ - aF^-. \tag{34}$$

Kac showed that the function $(F^+ + F^-)/2$ satisfies telegrapher's equation. The Dirac equation is derived from this formalism [7]. This method has been extended to the more general case of a three-dimensional Dirac equation in an external field [8]. Recently we have derived a linearized photon wave equation in a dispersive medium [9].

4. Quantum state diffusion

Recent experimental and theoretical researches have developed a new realm of the quantum world. They revealed individual quantum systems which interact significantly with their environments. There exist many such systems, for example, those dealing with ultra-cold atoms at pico kelvin, few photons in a cavity, entanglement experiments, quantum cryptography, effects of noise in quantum computers, nature of quantum measurement, and so on. We can follow the behavior of a single quantum system as it evolves, influenced by the environment including the measurement apparatus itself. These facts expose the weakness in the usual interpretation of quantum mechanics. All these quantum systems are not closed and quantization of open systems is demanded. They require alternative quantum theories. In the usual approach to these problems, the closed total system is divided into a system of interest and the environment. The dynamics of the open system of interest is obtained by tracing out the environmental variables. The dissipative effects come from interaction between the two subsystems.

In 1984, Gisin proposed the idea of quantum state diffusion (QSD) which gives the theoretical foundations and a useful method for numerical simulations of dissipative systems phenomenologically [10]. QSD provides a formalism which can be used to describe the above situations. The QSD equation gives the evolution of the state vector $|\psi\rangle$ in the Hilbert space as

$$|\mathrm{d}\psi\rangle = -\frac{i}{\hbar}\hat{H}|\psi\rangle\mathrm{d}t + \left(\langle\hat{L}^\dagger\rangle\hat{L} - \frac{1}{2}\hat{L}^\dagger\hat{L} - \frac{1}{2}\langle\hat{L}^\dagger\rangle\langle\hat{L}\rangle\right)$$
$$|\psi\rangle\mathrm{d}t + (\hat{L} - \langle\hat{L}\rangle)|\psi\rangle\mathrm{d}\xi, \tag{35}$$

where L is a Lindblad operator which is a representative of the interaction between the system and the environment. Here $M(\mathrm{d}\xi) = 0, M(\mathrm{d}\xi\mathrm{d}\xi) = 0$, and $M(\mathrm{d}\xi\,{}^*\mathrm{d}\xi) = \mathrm{d}t$, $\mathrm{d}\xi$ describes the increment of a complex Wiener process, $\langle\cdots\rangle$ means $\langle\psi|\cdots|\psi\rangle$, and M is the ensemble average on the complex Wiener process. In QSD, a system is always a pure state, evolving according to eq. (35). It is easily shown that the density matrix given by eq. (35), $\rho = M(|\psi\rangle\langle\psi|)$ satisfies the Lindblad master equation

$$\dot{\rho} = -\frac{i}{\hbar}[\hat{H}, \rho] + \left(\hat{L}\rho\hat{L}^{\dagger} - \frac{1}{2}\hat{L}^{\dagger}\hat{L}\rho - \frac{1}{2}\rho\hat{L}^{\dagger}\hat{L}\right). \tag{36}$$

The advantage of QSD lies with numerical calculations. Let us consider a system of N degrees of freedom and calculate the time evolution of its density matrix. The number of basis required for the computation should be $N^2 - 1$. On the other hand, in QSD, we only have to calculate the evolution of the state vector in the Hilbert space and this requires only N basis. QSD is a useful tool for simulation of nonlinear problems and systems with many degrees of freedom.

Furthermore, in the classical limit, the trajectory in the phase space should be localized for any sequence of random noises. In such a case, only a small part of the prepared basis contributes to the time evolution of the system, and almost all the remaining do not. The situation changes with time. Then, if we can choose the most appropriate set of basis mainly which contribute to the evolution of the system at that moment, the simulation works more effectively. Brun, Percival and Schack invented the 'moving basis' to significantly reduce the number of basis needed to represent the system [11,12]. Thus it can be said that QSD is a very useful tool to investigate classical or semiclassical behavior of quantum systems.

One of the most interesting problems in quantum mechanics is how classical behavior arises in quantum mechanical systems. Recently, we have examined the problem of quantum dissipative chaos, which has been worked very little, partly because this is one of the challenging problems involved in studying dissipation in quantum systems. We examined the forced and damped Duffing oscillator [12] to illustrate the crossover from the quantum behavior to the classical behavior by changing the Planck constant $\hbar$ effectively [13]. We introduced a scaling factor $\beta = \hbar/(\text{typical action of the system})$. We prepare a set of two initial points in the phase space with a tiny difference which lie outside the effective Planck cell to be distinguishable from each other in the classical limit ($\beta = 0.01$). QSD provides the evolution of this separation, and then we can define an analogue of the Lyapunov exponent, changing Planck constant effectively. We found that while in the classical limit there exists a Lyapunov exponent, it disappears in the region $(\beta \geq 0.40)$. There seems to exist some stage where the classical dynamical property is completely lost. In order to clarify this issue, we are probing some other quantities in detail.

5. Summary and disscusion

There are various kinds of stochastic dynamical processes both in classical and quantum regions. While the former stochastic processes are deterministic, even in chaos, in the latter there remain intrinsic quantum fluctuations. In this talk, we explained such quantum fluctuations appearing in some specific aspects of quantization.

In Nelson's quantum mechanics, the distribution of noise is Gaussian type and the mean of the noise correlation is proportional to the Planck constant. Therefore, the quantum fluctuations from the classical motion mainly comes from the noise term in the Langevin equation. It provides a useful method for studying quantum phenomena which have more particle nature rather than wave nature.

On the other hand, we showed that the Dirac equation and other relativistic field equations can be derived from the path integral based on the Poisson distribution. It is an interesting problem to investigate a physical reason behind this approach.

Finally we explained the quantum state diffusion which provides a useful formalism to describe quantum measurement and systems interacting significantly with the environment. In such problems it has many advantages over the master equation approach. We showed that it works very well for studying dissipative quantum chaos.

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