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Planck scale physics of the single-particle Schrödinger equation with gravitational self-interaction

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Abstract. We consider the modification of a single-particle Schrödinger equation by the inclusion of an additional gravitational self-potential term which follows from the prescription that the 'mass-density' that enters this term is given by $m|\psi(\vec{r},t)|^2$, where $\psi(\vec{r},t)$ is the wave function and m is the mass of the particle. This leads to a nonlinear equation, the 'Newton–Schrödinger' equation, which has been found to possess stationary self-bound solutions, whose energy can be determined using an asymptotic method. We find that such a particle strongly violates the superposition principle and becomes a black hole as its mass approaches the Planck mass.

Keywords. Quantum mechanics; gravitation; state reduction.

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1. Introduction

In spite of tremendous success of quantum mechanics its interpretational aspects continue to puzzle us, particularly in regard to measurements and extrapolation to macroscopic levels. The process of measurement involves a collapse of the wave function to one of the eigenstates of the operator being measured, and is not describable within the framework of the normal unitary evolution of quantum mechanics. Further the quantum description when extrapolated to the macroscopic domain leads to paradoxical situations, which arise when we superpose macroscopically distinguishable quantum states. These two problems are related to each other, as the measurement process at some stage or the other involves a macroscopic apparatus and a result of the measurement by definition is a definite number resulting from a 'pointer position'. This precludes states which are superpositions of distinguishable pointer positions. The pointer position is not predictable, and is found at one of the possible positions whose probability is dictated by its quantum wave function. The very nature of our classical description is thus incompatible with quantum superpositions and attendent probabilistic interpretation.

There have been suggestions that gravity has a possible role in the fact that spatial superpositions of macroscopic objects are not seen. Our paper, which was stimulated by some recent suggestions by Penrose, is also an attempt in this direction. Central to this set of proposals is the idea that the wave functions that involve superpositions of spatially separated wavepackets should also have superpositions of the gravitational fields associated with the distinct mass distributions of the wavepackets. As has been lucidly explained by Penrose, there are basic difficulties in combining the covariance principle of general relativity and superposition principle of quantum mechanics. The difficulty is best seen in studying the superposition of states that are spatially apart. The gravitational field states involved in superposition require different space-times among which a point-wise identification is not possible. Under these conditions, it is impossible to define a unique time-translation operator and the very concept of stationary state. According to Penrose this leads to an uncertainity of energy, which makes such spatially superposed states inherently unstable.

At another level, heuristically, if we assume that a particle may be localized at most in a size given by its Compton wavelength, then for a particle whose mass becomes of the order of the Planck mass or more and whose size is taken to be of the order of its Compton wavelength, the particle could be a black hole. This suggests that there is perhaps a problem with the quantum mechanics of such particles.

One way to model these effects may be to include a nonlinear mass-dependent gravitational potential energy term in the standard Schrödinger equation, by postulating a 'mass density' given by $m|\psi(\vec{r},t)|^2$, where $\psi(\vec{r},t)$ is the wave function and m is the mass of the particle. This term will automatically violate the superposition principle. (We cannot give any *a priori* reason for gravity to violate the superposition principle and generate such a term in the Schrödinger equation).

Such a nonlinear modified equation, called the Newton–Schrödinger (NS) equation has a long history [1–13] in the context of works that look at the influence of gravity on superposition in quantum mechanics. This equation has been used in several many-body contexts, such as Hartree–Fock calculations in plasmas as well as in astrophysics [14–16]. (It is also known as Choquard's equation [17] and Schrödinger–Poisson equation [18].) The existence of stationary solutions has been known (see, for example [17]) for sometime.

In a preceding paper [19], we looked at the energy eigenstates of the equation and using an asymptotic method could find the exact eigenvalue [19]. We review this in §§ 1,2 and 3. In §§ 4.1 and 4.2 we take up the question of spatial superposition of such localized stationary solutions and compute the energy difference between the superposed and stationary states, which may be identified with the breakdown of superposition. We show that this is related to an energy uncertainty calculated by Penrose using the difference in the free-fall accelerations corresponding to the mass distributions of the two distinct spatial components of the superposition. This energy difference is used in analogy to Penrose's [13] calculation to get a time for state reduction for a particle of arbitrary mass. In § 4.3 we first observe that our NS equation is valid only up to a limiting mass and not beyond. This follows from the fact that the bound state energy of the stationary state goes as m^5 and thus overtakes the rest mass at a certain value of the mass causing our particle to be unstable.

For masses just below the limiting mass, using our exponentially localized wave function, we can calculate the expectation value of the radius of our bound state particle of mass m. On substituting this radius in the expression for the horizon parameter we find that the particle becomes a black hole somewhat below the limiting mass. We also find that superposition is strongly broken for such values of the mass.

2. Gravitational self-interaction

The implementation of the above set of semiclassical and nonrelativistic approximations amounts to saying that the particle experiences a self-gravitational potential, which arises from the gravitational potential energy due to the mass density given by $m|\psi(\vec{r},t)|^2$, where

$\psi(\vec{r},t)$ and m are the wave function and mass of the particle, respectively. Incorporating this interaction in the Schrödinger equation, one obtains the following equation:

$$i\hbar\frac{\partial\psi(\vec{r},t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi(\vec{r},t) + V(\vec{r})\psi(\vec{r},t) + mV_{\mathrm{G}}(\vec{r})\psi(\vec{r},t), \tag{1}$$

where $V(\vec{r})$ is the external potential acting on the particle and $V_{\mathrm{G}}(\vec{r})$ is the gravitational self-potential arising due to mass density obtained from the wave function of the particle itself. Thus $V_{\mathrm{G}}(\vec{r})$ is given by

$$V_{\mathrm{G}}(\vec{r}) = -G\int\frac{m|\psi(\vec{r}_1,t)|^2}{|\vec{r}-\vec{r}_1|}\mathrm{d}^3r_1, \tag{2}$$

and equivalently,

$$\nabla^2 V_{\mathrm{G}}(\vec{r}) = 4\pi Gm|\psi(\vec{r},t)|^2. \tag{3}$$

In order to get an idea of the magnitude of self-coupling, let us consider the external potential, V, to be the Coulomb potential, $V = -e^2/r$. We can write the equation in a dimensionless form by introducing a free length a. In terms of this length, the time is measured in units of $\tau = 2ma^2/\hbar$, and the energy is measured in units of $\varepsilon = \hbar/\tau = \hbar^2/2ma^2$. Let

$$t = \tau\tilde{t},\;\; r = a\tilde{r}, \tag{4}$$

$$E = \varepsilon\tilde{E},\;\; \psi = \tilde{\psi}/a^{3/2}. \tag{5}$$

In these units, our equations take the form

$$i\frac{\partial\tilde{\psi}(\tilde{\vec{r}},\tilde{t})}{\partial\tilde{t}} = -\nabla^2\tilde{\psi}(\tilde{\vec{r}},\tilde{t}) - \frac{-2me^2a}{\hbar^2}\frac{1}{\tilde{r}}\tilde{\psi}(\tilde{\vec{r}},\tilde{t}) + \tilde{V}_{\mathrm{G}}(\vec{r})\tilde{\psi}(\tilde{\vec{r}},\tilde{t}) \tag{6}$$

and

$$\nabla^2\tilde{V}_{\mathrm{G}}(\tilde{r}) = 4\pi C|\tilde{\psi}(\tilde{\vec{r}},\tilde{t})|^2, \tag{7}$$

where C is a dimensionless coupling constant given by

$$C = \frac{2Gm^3a}{\hbar^2}. \tag{8}$$

By choosing the length scale a to set the coefficient of the Coulomb term to be unity, that is $a = \hbar^2/2me^2$, we can immediately read off the value of the dimensionless parameter, $C = Gm^2/e^2 = (m^2/m_{pl}^2)/\alpha$, where α is the fine structure constant. This clearly shows that the gravitational term we have added is negligible compared to the Coulomb interaction unless masses get rather large, say m of the order of $m_{pl}\sqrt{\alpha}$.

It is useful to note that the above equation conserves particle number (or mass) and energy. For this we refer the reader to [5,19].

3. Stationary solutions

Solutions of this equation have been considered from a variational point of view in [5], and numerically in [20] and more recently in [21]. We now show that the NS equation admits stationary solutions, as found in [19], of the form where we have dropped the $\sim$,

$$\psi(\vec{r},t) = \mathrm{e}^{-iEt}\phi(\vec{r}). \tag{9}$$

The reason is that for solutions of this form the self-potential V_{G} becomes time-independent. $\phi(\vec{r})$ obeys the equation

$$E\phi(\vec{r}) = -\nabla^2\phi(\vec{r}) + V(\vec{r})\phi(\vec{r}) - C\int \frac{|\phi(\vec{r}_1)|^2}{|\vec{r}-\vec{r}_1|}\mathrm{d}^3r_1\phi(\vec{r}). \tag{10}$$

We now present an analysis of the equation in the absence of an external potential.

The time-independent equation also admits a variational interpretation, and can be regarded as an extremum of the functional

$$H[\phi(\vec{r})] = \frac{1}{2}\int \mathrm{d}^3r|\vec{\nabla}\phi(\vec{r})|^2 - \frac{C}{2}\int\int \frac{|\phi(\vec{r})|^2|\phi(\vec{r}_1)|^2}{|\vec{r}-\vec{r}_1|}\mathrm{d}^3r_1\mathrm{d}^3r, \tag{11}$$

with the normalization constraint $\int|\phi(\vec{r})|^2\mathrm{d}^3r = 1$.

It is useful to record that with this constraint the energy functional above, under the scaling of $r \to br$, has a scaling behaviour that yields a minimum under the variation of b, indicating that the solution is stable with respect to scaling.

We analyse one class of solutions of the free particle equation by doing an asymptotic analysis [19]. This procedure allows us to obtain eigenvalues E for these solutions, but the wave function is known only in the region of large r. The key observation is that if ϕ is taken to be an exponentially localized function about an arbitrary point, it generates an asymptotic potential V_{G} which is a monopole potential C/r with corrections that decay exponentially with r, where r measures the distance from the point around which ϕ is localized.

These remarks are best illustrated by taking $\phi(\vec{r})$ to be the ground-state hydrogen wave function $(1/\pi\sigma^3)^{1/2}\mathrm{e}^{-r/\sigma}$. The corresponding self-potential is

$$V_{\mathrm{G}}(r) = -C\left[\frac{1}{r} - \mathrm{e}^{-2r/\sigma}\left(\frac{1}{r}+\frac{1}{\sigma}\right)\right]. \tag{12}$$

Substituting the solution in eq. (10) and keeping terms to order $\mathrm{e}^{-r/\sigma}$, one finds,

$$E\mathrm{e}^{-r/\sigma} = \left(\frac{2}{\sigma r} - \frac{1}{\sigma^2}\right)\mathrm{e}^{-r/\sigma} - \frac{C}{r}\mathrm{e}^{-r/\sigma}, \tag{13}$$

which yields the value of E and σ to be

$$\sigma = \frac{2}{C}, \text{ and } E = -\frac{1}{\sigma^2} = -C^2/4. \tag{14}$$

The asymptotic wave function and the potential get exponential corrections that are exactly calculable while the eigenvalue is not changed (see [19] for details). Note that these results follow on writing the equation in a dimensionless form, i.e., by measuring 'r' or σ in a length unit a. In terms of this, the time is measured in units of $\tau = 2ma^2/\hbar$, and the energy is measured in units of $\varepsilon = \hbar/\tau = \hbar^2/2ma^2$.

4. Physical implications

In this section we consider some physical implications of the results obtained above. The above results show that the gravitational self-potential leads to a self-bound state of linear extent of order $1/C$. Again for a microscopic particle like the electron this means a localization over enormous distances of order 10^{43} A, with a binding energy of 10^{-87} Ryd. Thus on scales much smaller than σ, we do not expect the gravitational potential to affect things, and the usual quantum mechanics should apply.

4.1 *Superposition*

In the context of the stationary states of the NS equation we can examine the existence of spatial superpositions or cat states in somewhat greater detail. Note that the centre of the stationary localized states discussed above can be chosen arbitrarily as the free particle equation is translationally invariant. More specifically, we calculate the energy of a cat state consisting of two self-localized humps with separation much larger than σ. The wave function of such a state is

$$\phi_{\mathrm{c}}(r) = [\alpha\phi_{\mathrm{s}}(r) + \beta\phi_{\mathrm{s}}(|\vec{r}-\vec{b}|)]. \tag{15}$$

There is a subtle difference between the localized states of our NS equation and those of, for example, a hydrogen atom. In our case there is no separate centre of mass (CM) coordinate, whereas in the case of the hydrogen atom there is a localized stationary state for the relative coordinate and the CM wave function is free to be in a plane wave or localized state.

It would then be appropriate to identify our stationary wave function with a coordinate localized state of the CM of the hydrogen atom. Since the hydrogen atom has a very large mass compared to mass of the electron, localizing the atom costs far less energy than localizing the electron. If we then neglect the former, we can effectively (i.e., approximately) use the relative wave function by itself to describe the localized atom. This we do.

First consider the case of the usual quantum mechanics with linear operators which is based on the superposition principle (that is in the absence of the nonlinear gravitational interaction term). In this case if we identify $\phi_{\mathrm{s}}(r)$ as the usual lowest s-wave state of the hydrogen atom, that is a stationary state with energy eigenvalue E_0 centred at the origin, then $\phi_{\mathrm{s}}(|\vec{r}-\vec{b}|)$ is the same state centred at $\vec{b}$. If the separation $|b| \gg \sigma$, the scale of the wave function, the superposed state above has the same energy eigenvalue.

Now, consider the case in the presence of the nonlinear gravitational interaction term, and let ϕ_{s} denote the stationary state solution derived above. When $b \gg \sigma$ the function is normalized with $\alpha^2+\beta^2 = 1$. Under this condition the energy of the superposed state is not the same any more. This is because (i) the superposed state is no longer a solution of the nonlinear NS equation, and (ii) the gravitational potential energy between the lumps comes into play. In fact for asymptotic separation between the lumps it is always higher, given by

$$E_{\mathrm{c}} = T - (\alpha^4+\beta^4)K_{11} - 2\alpha^2\beta^2K_{12}, \tag{16}$$

where T and $K_{11} = V_{\mathrm{G}}$ are the kinetic and potential energies of the single stationary state, and K_{12} is the gravitational potential energy between the humps. Since K_{12} is of order

$1/b$, it is much smaller than K_{11} (for asymptotic separations) and can be neglected. It is then easily seen that energy of the superposed state is higher than the energy of the single stationary state and in this asymptotic case it is clear that this originates not from the gravitational potential energy between the humps, but from the normalization of the superposed state and the nonlinearity (in its dependence on the wave function) of the gravitational interaction.

We specifically choose $\alpha = \beta = 1/\sqrt{2}$ to conform to the example considered by Penrose [13]. In this case for asymptotic separation we get a factor of 2 in the numerator, corresponding to two lumps and a factor of $1/4$ from the normalization, giving a potential energy equal to $K_{11}/2$ (neglecting the K_{12} term as we consider asymptotic separations) for the superposed state, which is exactly half of that for the single lump stationary state.

It is clear from the solution that $|E| = |V_G/2| = |K_{11}/2|$; this is also a simple consequence of the virial theorem. The energy difference, ΔV, between the superposed state and the stationary state is then given by $\Delta V = |K_{11}/2| = |E|$. This is a measure of sorts of the breakdown of superposition in the NS equation, as it is zero for the case of linear quantum mechanics.

4.2 *The Penrose conjecture on the time of state reduction of superposed states*

Penrose [13] has argued that such superposed states are unstable due to energy uncertainty arising from the mismatch of the two space times due to the two lumps respectively that make up the superposition.

More specifically a typical energy scale is obtained by Penrose by integrating the modulus squared of the difference of the free-fall accelerations due to each of the two localized lumps that make the coordinate superposition of the stationary states. This energy, Δ, turns out to be the gravitational energy of the difference of the mass distributions due to the two lumps. Penrose chooses to term this the gravitational self-energy (GSE) of the difference. We will use the same acronym – GSE.

A typical time of superposed state reduction is then obtained by dividing $\hbar$ by this energy in analogy with the decay of an unstable particle. These observations are general, for any given mass distributions and not specific to those considered above for the NS particle.

It can be shown that there is a conditional proportionality between the GSE computed by Penrose [13] (and others) and the energy difference between the superposed state and the stationary state, ΔV, above. This follows after some simple algebra with the assumption (i) that the 'mass density' for a particle governed by the NS equation is given by $\rho_s(\vec{r}) = m|\phi_s(r)|^2$, and (ii) of the asymptotic limit, that is, $|b| \gg \sigma$.

There are problems with the above identification. We should note here that it is obvious that the mass density expression used in constructing the gravitational term for the NS equation does not in any sense mean that the electron mass is distributed according to its wave function – it can have only the usual quantum mechanical probabilistic interpretation. However, as the mass of the particle is increased, its behaviour becomes more and more classical and such an interpretation may become plausible. (For example, if the asymptotic separation parameter b is of the order of laboratory scales of centimeters and the stationary wave function spatially smaller, say 10^{-5} cm, this implies that we are dealing with particles of a mass in excess of 10^{12} Gev.)

For a particle of mass m, governed by the NS equation we importantly find that $\Delta V = |E|$ goes as m^5. This is indeed different from the classical expression for the GSE of a constant density mass distribution which goes as m^2/R, which in turn goes as $m^{5/3}$ for composite objects like droplets considered in [22]. (We observe that the expression for ΔV is proportional to the GSE for the single stationary state, because, for asymptotic separation, the gravitational potential energy between the lumps can be neglected.)

On assuming Penrose's conjecture that this is the energy uncertainty associated with an unstable particle, we can now explicitly find the time-of-state reduction for the superposed state of a NS particle as a function of m. This is $T_{\mathrm{SP}} = \hbar/\Delta V = \hbar/|E| = 2\hbar^3/G^2m^5$. For specificity this is (i) 10^{70} seconds for an electron, (ii) 10^{-40} seconds for a particle of a mass that is smaller than the limiting mass by a factor of 5 – which puts it safely in a nonrelativistic, stable and non-black hole regime (see the following section).

For comparison we can estimate the time-of-state reduction for a droplet of constant density, when the GSE given in [22] goes as $m^{5/3}$. $T_{\mathrm{SP}} = \hbar/\Delta = \hbar/Gm^2/R = \hbar/Gm^{5/3}$. This is 10^{-10} second for a particle of a mass that is smaller than the limiting mass by a factor of 5.

4.3 *Limiting mass of a particle*

It must be noted that the energy eigenvalue (which is negative for the bound state) goes as m^5. This results in an instability as the sum of the rest mass energy and the attractive binding energy becomes negative. At this point our Newton–Schrödinger description with only the gravitational interaction breaks down. The value of this limiting mass is obtained by putting the ratio of the modulus of the energy eigenvalue and the rest mass, $\Gamma = 1$. (Note that Γ must be small compared to unity for our analysis to work.)

$$m_{\mathrm{L}} = \left(\frac{2\hbar^2c^2}{G^2}\right)^{1/4}. \tag{17}$$

This is effectively the Planck mass (apart from a constant factor of order 1). Perhaps, this is not so surprising for a theory which has only gravitational interactions.

We would like to consider more of the physics as we approach this limiting mass. To this end we determine some relevant parameters for particles of mass that approaches the limiting mass and compare them to their counterparts for the electron. We note that our theory has only one dimensionless parameter, $\delta = (m^2G/\hbar c)$. All dimensionless quantities are then expressed in terms of δ.

(i) The parameter $\Gamma = |E|/mc^2$ above goes as $\frac{1}{2}\delta^2$. It is 1 for the limiting mass and 10^{-90} for the electron. Recall, that in our theory the breakdown of superposition is linked to the energy, ΔV, which is further equal to E, the energy eigenvalue of the bound state. Dividing by the rest mass energy yields, $\Gamma = |E|/mc^2$, which then is also a measure of sorts of the breakdown of superposition.

(ii) The typical size for a particle, R, may be evaluated by calculating the expectatation value of r, for the wave function $(1/\alpha^3\pi)^{1/2}\mathrm{e}^{-r/\alpha}$, where $\alpha = a\sigma = 2a/C = \hbar^2/Gm^3$. Thus, $R = 3\alpha/2$.

(iii) The horizon parameter is $2Gm/Rc^2 = 4/3\delta^2$, where we have substituted for R above. We should further notice that this parameter goes as m^4 or the square of δ. This parameter decides if a mass distribution is a black hole or not. Specifically, if it is greater than 1 we have a black hole. It is $8/3$ for the limiting mass and 10^{-90} for an electron. Clearly, at the limiting mass the particle is a black hole!

This analysis shows that in the presence of a gravitational modification of the Schrödinger equation as given above we find not only a stability problem with masses as we approach the limiting mass but that such a particle would be a black hole. Note however that a reduction of the mass by only a factor of 5 of the limiting mass gives a very sensible nonrelativistic description and brings down the horizon parameter by a factor of 125 taking it safely away from a black hole.

We have studied a certain conjectural nonlinear modification of the nonrelativistic Schrödinger equation due to gravity – the Newton–Schrödinger equation, to look at the quantum mechanics of such a particle. The implication is that in this description, as we approach the Planck mass, not only do we strongly violate the superposition principle but the particle becomes a black hole, thereby putting a limit on the mass of elementary particles.

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