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A direct link between the Lie group SU(3) and the singular hypersurface $X^3 + \cdots = 0$ via quantum mechanics

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Abstract. A classical phase space with a suitable symplectic structure is constructed together with functions which have Poisson brackets algebraically identical to the Lie algebra structure of the Lie group SU(3). It is shown that in this phase space there are two spheres which intersect at one point. Such a system has a representation as an algebraic curve of the form $X^3 + \cdots = 0$ in $\mathscr{C}^3$. The curve introduced is singular at the origin in the limit when the radii of the spheres go to zero. A direct connection between the Lie groups SU(3) and a singular curve in $\mathscr{C}^3$ is thus established. The key step needed to do this was to treat the Lie group as a quantum system and determine its phase space.

Keywords. Lie groups; singularities; classical phase space.

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1. Introduction

It has long been known that there is an intriguing algebraic correspondence between the Cartan matrix of simply laced Lie groups and the intersection matrix of spheres that appear when certain simple singularities are resolved [1]. That this correspondence might be more than a mathematical curiosity was established when it was shown that duality in string theory made effective use of such a link [2]. A type 2A string compactified on a K3 surface (a four dimensional surface) was conjectured to be dual to a heterotic string compactified on T^4 (the four torus). A test of this conjecture required the zero mass excitations of the two theories to match. The zero mass excitations at the type 2A end came from certain singular points that appear on the K3 surface in a certain limit while those at the heterotic string end came from gauge excitations associated with a SU(3) Lie group. The excitations at the type 2A end were 'classical' solitonic type excitations while those at the heterotic end were 'quantum' gauge excitations [3]. This result suggests that a link between a singular hypersurface and the 'classical limit' of simply laced Lie groups might exist.

In this talk I will establish such a link. I will show how starting from the Lie groups SU(2) and SU(3), a direct path to the intersection of spheres present in an algebraic surface can be established. Intersecting spheres also appear when certain singular points are resolved on the hypersurface $X_0X_1 - X_2^n = 0$ where $(X_0, X_1, X_2) \in \mathscr{C}^3$, the space of three complex variables. Our approach is constructive and shows in a precise sense how resolved singularites are classically related to Lie groups. The extension of our approach to the general case of SU(n) is straightforward. An understanding of why these two very different mathematical systems have such a correspondence has been found within the framework of algebraic geometry and algebraic groups [4]. The novelty of our treatment lies in thinking of the Lie group as a quantum system, constructing the classical phase space for the system where the Lie structure gets represented as Poisson brackets for suitable functions in this classical phase space. It is in this space that a link with singularity theory is established. The link is thus between quantum objects and features present in the corresponding classical phase space.

2. Resolving singularities

We start with a mathematical problem. Consider the hypersurface V_n in $\mathscr{C}^3$ defined by the algebraic equation:

$$V_n(X_0, X_1, X_2) = X_0 X_1 - X_2^n = 0, \tag{1}$$

where n is an integer ≥ 2 and $(X_0, X_1, X_2) \in \mathscr{C}^3$. We have the following definition [3]:

Definition. A point $(X_0, X_1, X_2) \in V_n = 0$ is a singular point of the hypersurface if $\partial_{X_i} V_n = 0$ at that point for all X_i.

It follows from the definition that the point $(0,0,0)$, i.e., the origin is a singular point of the hypersurface $V_n \equiv X_0 X_1 - X_2^n = 0$, for $n \geq 2$. Indeed a simple way of seeing that this hypersurface is an orbifold is possible. To see this set $X_0 = \xi^n$, $X_1 = \eta^n$, $X_2 = \xi\eta$, where $(\xi, \eta) \in \mathscr{C}^2$. We note that in terms of these variables ξ, η, the equation $X_0 X_1 - X_2^n = 0$ is identically satisfied, i.e., ξ, η parametrize the hypersurface $V_n = 0$. There is, however, one restriction on the variables ξ, η when they are on the hypersurface $V_n = 0$, namely, the point (ξ, η) must be identified with $(\omega\xi, \omega\eta)$, where $\omega^n = 1$. Thus the hypersurface $V_n = 0$ can be identified with the orbifold $\mathscr{C}^2/\mathscr{Z}_n$ with the $\mathscr{Z}_n$ action defined by $(\xi, \eta) \longrightarrow (\omega\xi, \omega\eta)$, $\omega^n = 1$.

There is a standard method of minimally resolving this singularity, i.e., constructing a globally well-defined hypersurface which is in 1–1 correspondence with the original hypersurface $V_n = 0$ except at the point $(0,0,0)$. The singular point is 'blown up'. We describe this procedure first for the case $n = 2$ and $n = 3$, and then we state the result for the general case when $n > 3$.

Let us introduce the space $\mathscr{C}^3 \times \mathscr{P}^2$, where $\mathscr{P}^2$ is the complex projective two space. Points in $\mathscr{C}^3 \times \mathscr{P}^2$ can be written as the pair $((X_0, X_1, X_2), [s_0, s_1, s_2])$, where $(X_0, X_1, X_2) \in \mathscr{C}^3$ and $[s_0, s_1, s_2]$ is an element of $\mathscr{P}^2$, i.e., it represents the equivalence class of points (s_0, s_1, s_2) under the equivalence relation $(s_0, s_1, s_2) \sim \lambda(s_0, s_1, s_2)$, where λ is a complex number $\neq 0$. Next we introduce the space $\mathscr{C}^3(\mathscr{P}^2, \mathscr{R})$. This is defined as the set:

$$\mathscr{C}^3(\mathscr{P}^2, \mathscr{R}) = \{(X_0, X_1, X_2), [s_0, s_1, s_2] | X_i s_j = X_j s_i, \forall i, j\}.$$

Geometrically the restriction $X_i s_j = X_j s_i$ gives a space consisting of points (X_0, X_1, X_2) in $\mathscr{C}^3$ and lines through the origin and these points. These lines are elements of $\mathscr{P}^2$. Thus for all points in $\mathscr{C}^3$, other than the origin, the element of $\mathscr{P}^2$ is uniquely fixed by

(X_0, X_1, X_2). There is thus a 1–1 correspondence between points in $\mathscr{C}^3$ and the pair of points in $\mathscr{C}^3(\mathscr{P}^2, \mathscr{R})$. For the origin, however, the situation is different. When $X_0 = X_1 = X_2 = 0$, there is no restriction on $[s_0, s_1, s_2]$. Thus the origin of $\mathscr{C}^3$ is replaced by the entire $\mathscr{P}^2$ in $\mathscr{C}^3(\mathscr{P}^2, \mathscr{R})$: it is 'blown up'. Let us now study the way the hypersurface $X_0 X_1 - X_2^2 = 0$ behaves in $\mathscr{C}^3(\mathscr{P}^2, \mathscr{R})$. To study the way the singular point in $V_n = 0$ in $\mathscr{C}^3$ gets mapped in $\mathscr{C}^3(\mathscr{P}^2, \mathscr{R})$ we approach the origin in $\mathscr{C}^3$. This is done by scaling the points (X_0, X_1, X_2) in $\mathscr{C}^3$ by t and letting $t \longrightarrow 0$. Note that the constraints $X_i s_j = X_j s_i \; \forall i, j$ imply that $X_i = k s_i$ (where k =constant). Thus $(tX_0, tX_1, tX_2) = tk(s_0, s_1, s_2)$, i.e., in the $t \longrightarrow 0$ limit, $[s_0, s_1, s_2] \in \mathscr{P}^2$ points on $V_n = 0$ satisfy $s_0 s_1 - s_2^2 = 0$ in $\mathscr{P}^2$. We now have the following theorem.

Theorem [4]. *A polynomial equation of degree n in* $\mathscr{P}^2$ *describes a compact Riemann surface of genus g with* $g = \frac{1}{2}(n-1)(n-2)$.

In our case the polynomial equation $s_0 s_1 - s_2^2 = 0$ in $\mathscr{P}^2$ is of degree 2. Hence the surface in $\mathscr{P}^2$ is a genus zero surface, i.e., topologically it is a sphere. Thus the singular point of the hypersurface $X_0 X_1 - X_2^2 = 0$ in $\mathscr{C}^3$ is replaced by a sphere in $\mathscr{C}^3(\mathscr{P}^2, \mathscr{R})$. The singularity has been resolved by a process of 'blowing up' turning the singular point into a sphere.

We next consider the case $n = 3$. Repeating the procedure for the $n = 2$ case we find the points $[s_0, s_1, s_2]$ in $V_3 = 0$ now have to satisfy the polynomial equation $t^2(s_0 s_1 - t s_2^3) = 0$, i.e., the equation $s_0 s_1 = 0$ ($s_2 \neq 1$) in the $t \longrightarrow 0$ limit. This gives a pair of spheres corresponding to setting $s_0 = 0$, $s_1 = 0$. These two spheres intersect at the point $(0, 0, s_2)$. The self intersection of these spheres can be shown to be given by -2. The intersection matrix for these two spheres is thus

$$\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}.$$

This is exactly equal to the negative of the Cartan matrix for SU(3). The pattern found for $n = 3$ persists for $n > 3$ (for details, see [1]).

3. The classical phase space for the Lie groups SU(2), SU(3)

We now propose to look at the classical origins of the Lie groups SU(2), SU(3) in the following sense. The groups have a local structure encoded by their Lie algebras. We will call the associated phase space, defined with a suitable symplectic structure, the classical counterpart of the Lie group if functions on the phase space can be constructed which have Poisson brackets algebraically identical to the Lie algebra structure of the Lie group. The construction we will describe is standard [6]. It involves coherent states associated with the Lie group of interest. We start by quickly summarizing the results for SU(2). This simple example contains a crucial ingredient needed for our subsequent analysis. Let us introduce the lowest weight representation for SU(2), which we write as the vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The coherent state $|\lambda\rangle$ is then defined as

$$|\lambda\rangle = \mathrm{e}^{\lambda J_+}\begin{pmatrix}0\\1\end{pmatrix},\ \text{where } J_+ = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} \equiv e_{12},$$

and λ is a complex variable. Then $\langle\lambda|\lambda\rangle = (1+\lambda\bar{\lambda})$.

We then construct the Kähler potential $V(\lambda,\bar{\lambda}) = k\ \log(1+\lambda\bar{\lambda})$. This is then used to define a symplectic form $\omega_{\lambda\bar{\lambda}} = \partial_\lambda \partial_{\bar{\lambda}} V$, that is

$$\omega = k\begin{pmatrix} 0 & \frac{1}{(1+\lambda\bar{\lambda})^2} \\ -\frac{1}{(1+\lambda\bar{\lambda})^2} & 0 \end{pmatrix},$$

where k is a constant. We next note that

$$X_+ = \frac{\langle\lambda|J_+|\lambda\rangle}{\langle\lambda|\lambda\rangle} = \frac{2\lambda}{1+\lambda\bar{\lambda}},$$

$$X_- = \frac{\langle\lambda|J_-|\lambda\rangle}{\langle\lambda|\lambda\rangle} = \frac{2\bar{\lambda}}{1+\lambda\bar{\lambda}},$$

$$X_0 = \frac{\langle\lambda|J_0|\lambda\rangle}{\langle\lambda|\lambda\rangle} = \frac{1-\lambda\bar{\lambda}}{1+\lambda\bar{\lambda}},$$

are functions on the phase space described by the complex variable λ and the symplectic form ω. Furthermore, $X_+X_- + {X_0}^2 = 1$, i.e., these functions represent points on S^2. Also,

$$\{X_+,X_-\} = (\omega^{-1})_{\lambda\bar{\lambda}}\partial_\lambda X_+ \partial_{\bar{\lambda}} X_- + (\omega^{-1})_{\bar{\lambda}\lambda}\partial_{\bar{\lambda}} X_+ \partial_\lambda X_- = 2iX_0,$$

$$\{X_0,X_\pm\} = \pm X_\pm,$$

for suitable choice of k. Thus the expectation values of the operators $J_\pm$, J_0 in the normalized state vector $|\lambda\rangle$ represent the classical functions whose quantization, achieved by replacing Poisson brackets by commutators, as postulated by quantum mechanics, leads to the Lie algebra structure. The classical phase space of SU(2) is thus S^2. Note that the presence of S^2 could be spotted simply by evaluating $\int \omega \mathrm{d}\lambda \wedge \mathrm{d}\bar{\lambda} = 4\pi$ and noting that the curvature of the phase space manifold is constant and positive. Also the metric on the phase space derived from the Kähler potential can be seen to be precisely the metric of S^2. The emergence of S^2 for the Lie group SU(2) is the key observation we want to record. For the group SU(3) we will construct appropriate Kähler form which describes the classical phase space associated with this group. It will then be demonstrated that this phase space contains two spheres. The spheres can easily be identified by the presence in this phase space of nontrivial cycles with $\int\omega = 4\pi$ and a sphere metric in an appropriate subspace. The intersection properties of these spheres can then be determined by using the methods of differential topology [7]. We demonstrate by these means that corresponding to the Lie groups SU(3) there is a classical phase space that mirror its Lie algebra structure. The spheres in such a phase space intersect each other. In the case of SU(n) they form a complex curve in $\mathscr{C}^{n-1}$ since the phase space is $\mathscr{P}^{n-1}$ for SU(n). We now note the following theorem.

Theorem [7]. *A complex curve in $\mathscr{C}^n$ can always be embedded in $\mathscr{C}^3$.*

Thus the intersecting spheres, present in $\mathscr{C}^{n-1}$, can be embedded in $\mathscr{C}^3$. On the other hand we saw that the resolution of a singular curve in $\mathscr{C}^3$ leads to intersecting spheres. Thus the two different mathematical objects: the singular curve and the algebraic curve obtained from the phase space of a compact Lie group, both live in $\mathscr{C}^3$ and both contain spheres.

Finally we note that a sphere is $\mathscr{P}^1$ and intersections of spheres correspond to intersections of $\mathscr{P}^1$'s. Each $\mathscr{P}^1$ has an algebraic description as a polynomial equation of order 1 or 2 in $\mathscr{P}^2$. Then n intersecting $\mathscr{P}^1$'s in $\mathscr{P}^2$ can correspond to a polynomial equation of order $n+1$ in $\mathscr{P}^2$. Such a polynomial equation provides an equivalent description of the system and gives an algebraic curve. Thus we have sketched how starting from the Lie group SU(3) and considering its classical representation in terms of Poisson brackets in phase space lead to an algebraic curve. The fact that the phase space associated with the general Lie group SU(n) is compact (it is P^{n-1}) implies that the system has an algebraic geometry description. This follows from the theorem:

Chow [7]. *A compact hypersurface can always be represented by an algebraic variety in a higher dimensional projective space.*

Now some details. For SU(3) we again work with the fundamental representation and introduce the coherent state

$$|v_1,v_2,v_3\rangle = \mathrm{e}^{v_1 e_{13}}\mathrm{e}^{v_2 e_{23}}\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}v_1\\v_2\\1\end{pmatrix}.$$

Note that we have taken only two generators out of three for SU(3), which do not annihilate the lowest weight state. Since SU(3) has rank 2, only two generators, namely e_{12} and e_{23} are Lie algebraically independent, the other one e_{13} follows from the Lie bracket of the above two. Had we taken e_{12} and e_{23} in the above state we would have obtained a state of the same form. The construction described here corresponds to the mathematical result that G/T is an even-dimensional manifold, where G is a Lie group and T its maximal torus. Furthermore $G/T = G_{\mathrm{c}}/B$, where G_{c} is the complexification of G and B the Borel subgroup. Here G_{c}/B corresponds to working with the lowest weight vector with B the subgroup of the lower triangular lowering operators [8].

The Kähler potential is given by

$$V(v_1,v_2,\bar{v}_1,\bar{v}_2) = k\,\log\langle v_1,v_2|\bar{v}_1,\bar{v}_2\rangle = k\,\log(1+v_1\bar{v}_1+v_2\bar{v}_2) = k\,\log\langle v|v\rangle,$$

and the symplectic structure determined by $\omega_{i\bar{j}} = \partial_{v_i}\partial_{v_j}V$ is precisely that of $\mathscr{P}^2$. It is then easy to verify that the commutation relations of SU(3) are reflected in the Poisson bracket between the functions $\langle e_{ij}\rangle$ and $\langle e_{kl}\rangle$, where

$$\langle e_{ij}\rangle = \frac{\langle v_1 v_2|e_{ij}|v_1 v_2\rangle}{\langle v_1 v_2|v_1 v_2\rangle}.$$

Note

$$[\omega] = \begin{pmatrix} 0 & a & 0 & b\\ -a & 0 & -b' & 0\\ 0 & b' & 0 & c\\ -b & 0 & -c & 0 \end{pmatrix},$$

where

$$a = \partial_{v_1}\partial_{\bar{v_1}} \log\langle v|v\rangle,$$

$$b = \partial_{v_1}\partial_{\bar{v_2}} \log\langle v|v\rangle,$$

$$c = \partial_{v_2}\partial_{\bar{v_2}} \log\langle v|v\rangle,$$

and prime stands for complex conjugate. It is also straightforward to check that

$$\int_{v_2=0} \omega \mathrm{d}v_1 \wedge \mathrm{d}\bar{v_1} = 4\pi,$$

$$\int_{v_1=0} \omega \mathrm{d}v_2 \wedge \mathrm{d}\bar{v_2} = 4\pi.$$

There are thus two spheres in the phase space. The form of the metric with $v_1 = 0$ or $v_2 = 0$ also clearly establishes that this is the case. The intersection of the spheres can be studied using the following theorems.

Theorem [9]. *For SU(n) the phase space constructed is* $\mathscr{P}^{n-1}$.

Theorem [10]. *The de Rham cohomology and Dolbeault cohomology groups for* $\mathscr{P}^n$ *are the same.*

Theorem [10]. *The intersection of the two surfaces* $\mathscr{C}_i$ *and* $\mathscr{C}_j$ *are given by* $\mathscr{C}_i \cdot \mathscr{C}_j = \frac{1}{(4\pi)^2}\int \omega_i \wedge \omega_j$, *where the integration is in the space containing the surface* $\mathscr{C}_i$ *and* $\mathscr{C}_j$ *and is a space of dimension four and* ω_i, ω_j *are (de Rham cohomology) elements of* $H^2_{dR}(M,R)$.

The cohomology groups associated with the symplectic form constructed are Dolbeault cohomology groups while the intersection is valid for de Rham cohomology groups. However, for $\mathscr{P}^n$ they are the same. We can thus determine intersection of spheres by simply evaluating $\int \omega \wedge \omega$. We find using the ω determined from the Kähler potential,

$$\int \omega \wedge \omega = (4\pi)^2.$$

Thus the two spheres intersect once. The self-intersection of spheres can again be shown to be 2. Thus for SU(3) the intersection matrix for the spheres present in the phase space is

$$\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix},$$

justifying its identification with the singular cubic curve. The procedure outlined can be repeated for SU(N).

4. Conclusions

We have shown in an example, viz., for SU(3) how a classical phase space can be associated with the group. In this phase space there are spheres present which intersect at one point.

Such a system has an algebraic description as a cubic polynomial in $\mathscr{C}^3$. An explicit correspondence between Lie groups and algebraic surfaces in $\mathscr{C}^3$ has thus been established. The result explains the 'classical' nature of the algebraic surface in this classical-quantum correspondence. Details of the constructions sketched are contained in [5].

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