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Information cloning of harmonic oscillator coherent states

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Abstract. We show that in the case of unknown *harmonic oscillator coherent states* it is possible to achieve what we call *perfect information cloning.* By this we mean that it is still possible to make arbitrary number of copies of a state which has *exactly* the same information content as the original unknown coherent state. By making use of this *perfect information cloning* it would be possible to estimate the original state through measurements and make arbitrary number of copies of the estimator. We define the notion of a *measurement fidelity* and calculate it for our case as well as for the Gaussian cloners.

Keywords. Cloning; coherent states.

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1. Introduction

Cerf and Iblisdir [1] have shown that there is an optimal fidelity to cloning coherent states. Fidelity of cloning was interpreted in [1] as $\langle\alpha|\rho_1|\alpha\rangle$, where $|\alpha\rangle$ is the unknown coherent that was cloned and ρ_1 is the one-particle reduced density matrix of the output. In the Gaussian-cloners of the type considered in [1] there are N-copies of ρ_1 which are all *mixed* states. We call the fidelity introduced by [1] the *overlap fidelity*. The formula for the optimal overlap fidelity of $N \to M$ cloning of coherent states [1] is

$$F_{N,M}^{\text{overlap}} = \frac{MN}{MN+N-M}. \tag{1}$$

We have presented an alternate route to the question of cloning coherent states [2]. We show that it is possible to make arbitrary number of copies of coherent states with exactly the same information content as the original unknown state. Complete information about a coherent state is contained in the complex coherency parameter α. Thus by information cloning what we mean is the ability to make arbitrary number of copies of coherent states whose coherency parameter is $c(N)\alpha$ where α is the coherency parameter of the *unknown* coherent state and $c(N)$ is a *known* constant depending on the number of copies made.

We consider $1+N$ systems of harmonic oscillators whose creation and annihilation operators are the set $(a,a^\dagger),(b_k,b_k^\dagger)$ (where the index k takes on values $1,\ldots,N$) satisfying the commutation relations

$$[a,a^\dagger]=1;\;\;[b_j,b_k^\dagger]=\delta_{jk};\;\;[a,b_k]=0;\;\;[a^\dagger,b_k]=0. \tag{2}$$

Coherent states parametrized by a complex number are given by

$$|\alpha\rangle = D(\alpha)\;|0\rangle, \quad D(\alpha)=\mathrm{e}^{\alpha a^\dagger-\alpha^* a}, \tag{3}$$

where $|0\rangle$ is the ground state. For the *cloning transformation* we consider the unitary transformation

$$U=\mathrm{e}^{t(a^\dagger\sum_j\kappa_j b_j - a\sum_j\kappa_j^* b_j^\dagger)}. \tag{4}$$

A crucial property of this transformation is that it evolves a *disentangled* set of coherent states into another *disentangled* set of coherent states. Let us consider a *disentangled* set of coherent states $|\alpha\rangle|\beta_1\rangle_1|\beta_2\rangle_2\ldots|\beta_N\rangle_N$ and the action of U on it.

The transformation U induces a transformation on the parameters (α,β_j) which can be represented by the matrix $\mathscr{U}$, i.e., $\alpha_a(t)=\mathscr{U}_{ab}\alpha_b$. We have introduced the notation α_a with $a=1,\ldots,N+1$ such that $\alpha_1=\alpha,\alpha_k=\beta_{k-1}$ $(k\geq 2)$. The explicit formula for $\mathscr{U}$ is given by

$$\mathscr{U}=\begin{pmatrix} \cos rt & \frac{r_1}{r}\,\mathrm{e}^{-i\delta_1}\sin rt & .. & .. & \frac{r_N}{r}\,\mathrm{e}^{-i\delta_N}\sin rt \\ -\frac{r_1}{r}\,\mathrm{e}^{i\delta_1}\sin rt & M_{11} & .. & .. & M_{1N} \\ .. & .. & .. & .. & \\ .. & .. & .. & .. & \\ -\frac{r_N}{r}\,\mathrm{e}^{i\delta_N}\sin rt & M_{N1} & .. & .. & M_{NN} \end{pmatrix}, \tag{5}$$

where

$$M_{jk}\;=\;\delta_{jk}-\mathrm{e}^{i\delta_j-i\delta_k}\frac{r_jr_k}{r^2}(1-\cos rt). \tag{6}$$

In eq. (5) we have used $\kappa_j=r_j\mathrm{e}^{i\delta_j}$.

Now the coefficients of α in all $\beta_k(t)$ must be made to have the same magnitude implying $r_1=r_2=\cdots=r_n$. With the choice $\beta_1=\beta_2=\cdots=\beta_N$, one gets

$$\beta_k(t)=-\mathrm{e}^{i\delta_k}\frac{\sin rt}{\sqrt{N}}\alpha. \tag{7}$$

With the optimal choice of $\sin rt=-1$ and using appropriate unitary transformations to remove known phases, one gets N copies of the state $|\alpha/\sqrt{N}\rangle$. In the general formalism of Grosshans and Grangier [3], this corresponds to the case of *quantum cloning with gain* or *duplicator* with the gain $g=1/\sqrt{N}$.

2. Information cloning

Thus we are able to produce N-copies *not* of the original state $|\alpha\rangle$ but of a state of the form $|\alpha/\sqrt{N}\rangle$ which has the *same information content* as $|\alpha\rangle$ in the sense that a complete determination of the latter is equivalent to a complete determination of the former. This is what we would like to call *cloning of information* in contrast to *cloning of the quantum* *state itself*. It is quite plausible that in many circumstances of interest cloning in this more restricted sense may suffice.

Superficially this may appear to be a triviality in the sense that one can always apply known unitary transformations on unknown quantum states to produce states with the same information content in the sense used above. But what is *nontrivial* in our construction is that *arbitrary* number of copies of such information-equivalent states can be produced.

In contrast to the Gaussian cloners of [1] our *information cloning* produces N-copies which are *pure* states. The overlap fidelity for our information cloning is

$$F_{\text{info}}^{\text{overlap}} = \mathrm{e}^{-|\alpha|^2(1-\frac{1}{\sqrt{N}})^2}. \tag{8}$$

Not only can this be very small, it is also not *universal*.

We introduce another notion of fidelity which we call *measurement fidelity* by which we mean the best reconstruction of the original unknown state that can be achieved through actual measurements performed in some optimal way. We now propose using the copies of the information-equivalent states to estimate the parameter α. Normally when the available number of copies of a state is very large, one can estimate the state quite accurately and use that to create arbitrary number of clones of the original coherent state. However, in our proposal for information cloning even though the number of copies N can be arbitrarily *large*, the coherency parameter given by $\alpha/\sqrt{N}$ becomes *arbitrarily small* while the *variances* in α remain the same as in the original state. This raises the question as to how best the original state can be reconstructed and about the statistical significance of our information cloning procedure.

On introducing momentum and position operators $\hat{p}, \hat{x}$ through $\hat{x} = (a + a^{\dagger})/\sqrt{2}$, $\hat{p} = (a - a^{\dagger})/\sqrt{2}i$, the *probability distributions* in position and momentum representations are given by

$$|\psi_{\text{clone}}(x)|^2 = \frac{1}{\sqrt{\pi}}\mathrm{e}^{-(x-\sqrt{\frac{2}{N}}\alpha_R)^2}, \qquad |\psi_{\text{clone}}(p)|^2 = \frac{1}{\sqrt{\pi}}\mathrm{e}^{-(p-\sqrt{\frac{2}{N}}\alpha_I)^2}. \tag{9}$$

Let us distribute our N-copies into two groups of $N/2$ each and use one to estimate α_R through position measurements and the other to estimate α_I through momentum measurements. Let y_N denote the average value of the position obtained in $N/2$ measurements and let z_N denote the average value of momentum also obtained in $N/2$ measurements. The *central limit theorem* states that the probability distributions for y_N, z_N are given by

$$f_x(y_N) = \sqrt{\frac{N}{2\pi}}\mathrm{e}^{-\frac{N}{2}(y_N-\sqrt{\frac{2}{N}}\alpha_R)^2}, \quad f_p(z_N) = \sqrt{\frac{N}{2\pi}}\mathrm{e}^{-\frac{N}{2}(z_N-\sqrt{\frac{2}{N}}\alpha_I)^2}. \tag{10}$$

The *estimated* value of α is

$$\alpha_{\text{est}} = \frac{y_N + iz_N}{\sqrt{2}}\sqrt{N}. \tag{11}$$

The *measurement fidelity* F^{meas} can be understood as the quantity $|\langle\alpha|\alpha_{\text{est}}\rangle|^2$: The probability distribution for F is given by

$$p(F)\mathrm{d}F = \int \mathrm{d}z_N\,\mathrm{d}y_N\;\delta\left(z_N^2 + y_N^2 - \frac{2}{N}|\alpha_{\text{est}}|^2\right) f_x(y_N)\,f_p(z_N). \tag{12}$$

It is straightforward to show that

$$p(F)\mathrm{d}F = \mathrm{d}F. \tag{13}$$

Consequently the average value of F^{meas} is

$$\bar{F}^{\text{meas}}_{1,N} = 1/2.$$

Now we generalize our results to the $M' \to N'$ case. We start with M copies and let each copy be information cloned to N copies, so we have MN copies finally.

The position and momentum distributions are still given by eq. (9) but now $NM/2$ measurements are carried out for position and momentum. Consequently,

$$f_x(y_{MN}) = \sqrt{\frac{MN}{2\pi}}\,\mathrm{e}^{-\frac{MN}{2}(y_{MN}-\sqrt{\frac{2}{N}}\alpha_R)^2},$$
$$f_p(z_{MN}) = \sqrt{\frac{MN}{2\pi}}\,\mathrm{e}^{-\frac{MN}{2}(z_{MN}-\sqrt{\frac{2}{N}}\alpha_I)^2}. \tag{14}$$

The *estimated* value of α is still given by eq. (11). One finally obtains

$$p_{M,MN}(F)\mathrm{d}F = MF^{M-1}\mathrm{d}F. \tag{15}$$

The average measurement fidelity in this case is given by

$$\bar{F}^{\text{meas}}_{M,MN} = \frac{M}{M+1}. \tag{16}$$

This approaches 1 as $M \to \infty$.

These fidelities should not be directly compared with eq. (1). As emphasized by Massar and Popescu [4] there can be many notions of fidelities and two schemes should be compared only with the *same* criterion for fidelity. So we compute the measurement fidelity for Gaussian cloners. Each copy is the Gaussian mixture

$$\rho = \int \mathrm{d}2\alpha \frac{A_{M,MN}}{\pi}\mathrm{e}^{-A_{M,MN}|\alpha|^2}|\alpha_0+\alpha\rangle\langle\alpha_0+\alpha|, \tag{17}$$

where $A_{M,MN} = MN/(N-1)$ is such that it reproduces eq. (1). The resulting measurement fidelity distribution is

$$p^{\text{Gauss}}_{M,MN}(F)\mathrm{d}F = F^{\frac{MNA_{M,MN}}{2(A_{M,MN}+2)}-1}\mathrm{d}F, \tag{18}$$

while the average measurement fidelity is

$$\bar{F}^{\text{Gauss}}_{M,MN} = \frac{M^2N^2}{M^2N^2+2MN+4N-4}. \tag{19}$$

For $M=1, N=2$ the measurement fidelities for Gaussian and information cloning are $1/3$ and $1/2$, respectively. For $M=1, N=4$ these become $4/9$ and $1/2$. For $M=2, N=2$ these are $4/7$ and $2/3$, while for $M=2, N=4$ they become $16/23$ and $2/3$, respectively.

3. Conclusion

In this paper we have demonstrated the concept of *information cloning* for harmonic oscillator coherent states. The principal difference with the *Gaussian cloning* of [1,5] is that in our case the outputs are *pure* and *disentangled* states. The coherency parameter for the output states is reduced by the factor $1/\sqrt{N}$ where N is the number of copies. The variances are unchanged. We have also introduced the notion of *measurement fidelity* which is different from the notion of fidelity introduced in [1,5]. For purposes of comparison we have calculated the measurement fidelities for Gaussian cloners also. In the case of d-level quantum states a formula is available giving the fidelity that can be achieved given N copies [6]. Our formula (16) is such a relation for coherent states.

Quantum theory denies any statistical significance to single quantum states. This is true for generic quantum states and is due to the fact that in the course of any measurement the outcomes are always *random* and the state itself changes irrepairably. The no-cloning theorem gives a very subtle and remarkable consistency to this view. While this is so for generic states, in the case of coherent states the fact that both Gaussian cloning [1] which results in copies of mixtures with the same average coherency parameter as the original unknown coherent state, as well as our information cloning [2] where the copies are pure states are possible means there is a meaning to the statistical significance of these states. This issue will be elaborated elsewhere.

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