PRAMANA
— journal of
physics

Vol. 59, No. 2
August 2002
pp. 221–228

Quantum cobwebs: Universal entangling of quantum states

ARUN KUMAR PATI
Institute of Physics, Bhubaneswar 751 005, India
Center for Philosophy and Foundation of Science, New Delhi, India
School of Informatics, University of Wales, Bangor LL57 1UT, UK
Email: akpati@iopb.res.in

Abstract. Entangling an unknown qubit with one type of reference state is generally impossible. However, entangling an unknown qubit with two types of reference states is possible. To achieve this, we introduce a new class of states called *zero sum amplitude* (ZSA) multipartite, pure entangled states for qubits and study their salient features. Using shared-ZSA states, local operations and classical communication, we give a protocol for creating multipartite entangled states of an unknown quantum state with two types of reference states at remote places. This provides a way of encoding an unknown pure qubit state into a multiqubit entangled state.

Keywords. Universal entangled states; zero sum amplitude states; multiparticle entangled states.

PACS Nos 03.67.-a; 03.65.Bz

1. Introduction

Quantum entanglement is generally regarded as a very useful resource for quantum information processing [1]. It can be used for teleportation [2], dense coding [3], entanglement swapping [4,5], remote state preparation (RSP) of special qubits [6] and general quantum states [7,8], quantum remote control [9], telecloning [10], remote information concentration [11] and many more important tasks.

Here, we find yet another startling application of quantum entanglement. Imagine that we need to distribute an unknown qubit to more than one party. If we could distribute to many parties without entangling (i.e., in product states) then that would violate no-cloning principle [12,13] (the other limitations are no-deleting against a copy [14], and no-complementing [15] of an unknown qubit). This means that the distributed state of a qubit with N parties must be in an entangled state. But creating an universal entangled state of an unknown state was shown to be impossible [16]. Similarly the reverse is also impossible [17]. However, the surprising fact is that if we drop the symmetric requirement then it is possible to create *two types of multiparticle entangled state of an unknown state*. This is important because our protocol can produce universal entangled states (which people have previously thought to be impossible) that works with unit probability of success. It is a hope that exact universal entangled states will play important roles for storage of quantum information against environmental decoherence [18].

To achieve this, we introduce a class of entangled states called *zero sum amplitude* (ZSA) entangled states which may have merit on their own. We present a protocol where upon using a special class of ZSA multipartite (with, say, N number of parties) shared entangled states, local operations and classical communication (LOCC), one can create two types of shared-entangled state of an unknown quantum state with $(N-1)$ qubits at remote places. The information about an unknown state is distributed with all the N parties concerned *in a non-local way*. Thus, remote shared-entangling of an unknown state with multiparties is a very secure way to preserve information about an unknown state (as long as the N parties can maintain their quantum correlation).

2. Zero sum amplitude entangled states for multiqubits

For the sake of generality, we introduce an arbitrary pure N-qubit ZSA entangled state $|\Phi\rangle_{12\ldots N} \in \mathscr{H}^2 \otimes \cdots \otimes \mathscr{H}^2$ (N times) given by $|\Phi\rangle_{12\ldots N} = \sum_{i=1}^{2^N} c_i |i\rangle_{12\ldots N}$, where $\{|i\rangle\}$ is an orthonormal basis for 2^N-dimensional Hilbert space, $\sum_{i=1}^{2^N} c_i = 0$ (i.e., all the complex amplitudes sum to zero) and $\sum_{i=1}^{2^N} |c_i|^2 = 1$ (i.e., the state is normalized to unity). The state space of a quantum system is the complex projective Hilbert space $\mathscr{P}$ with dimension $\dim \mathscr{P} = \dim \mathscr{H} - 1$. For a general ZSA state the dimension of the state space (viewed as a real manifold) is $D = (2^{N+1} - 3)$.

In the rest of the paper, we will consider a special class of ZSA states where the number of complex amplitudes is equal to the number of parties (and since each party posseses one qubit, it is also equal to the total number of qubits) involved, and N orthonormal states contain all zeros except at a single entry which contains one. For example, an N-partite ZSA state is given by

$$\begin{aligned} |\Psi\rangle_{123\ldots N} &= c_1|100\ldots0\rangle_{123\ldots N} + c_2|010\ldots0\rangle_{123\ldots N} + \cdots + c_N|00\ldots1\rangle_{123\ldots N} \\ &= \sum_{k=1}^{N} c_k |x_k\rangle_{123\ldots N}, \end{aligned} \tag{1}$$

where $|x_k\rangle$ $(k = 1, 2, \ldots, N)$ is an N-bit string containing all 0's except that the kth party contains 1 and the amplitudes obey the ZSA condition $\sum_k c_k = 0$ and the normalization $\sum_k |c_k|^2 = 1$. This class of states can be completely specified by $(2N-3)$ real parameters.

To appreciate the remarkable features of this class of states we first discuss the case of two parties. When the number of parties is two, the ZSA state is given by $|\Psi\rangle_{12} = c_1|10\rangle_{12} + c_2|01\rangle_{12}$. The ZSA and normalization conditions guarantee that the above state is nothing but an EPR singlet state $|\Psi^-\rangle_{12} = \frac{1}{\sqrt{2}}(|10\rangle_{12} - |01\rangle_{12})$, which is just one member of the Bell states. This state is known to be locally equivalent to other Bell states and can be used for successful quantum teleportation of an unknown qubit [19]. However, throughout the paper whenever we mention multiparticle state we will consider three or more qubits, i.e., $N \geq 3$.

Let us introduce a class of ZSA normalized entangled state of three qubits $|\Psi\rangle_{123} \in \mathscr{H}^2 \otimes \mathscr{H}^2 \otimes \mathscr{H}^2$ given by

$$|\Psi\rangle_{123} = c_1|100\rangle_{123} + c_2|010\rangle_{123} + c_3|001\rangle_{123}, \tag{2}$$

where c_i's $(i = 1,2,3)$ are non-zero amplitudes of the basis states where the *i*th qubit is in the state $|1\rangle$. These amplitudes obey $\sum_{i=1}^{3} c_i = 0$ and $\sum_{i=1}^{3} |c_i|^2 = 1$. Interestingly, states of the type (2), *but without ZSA condition*, have shown up in a variety of places in the literature [20,19,21,22]. We conjecture that the ZSA states are not locally equivalent to $|GHZ\rangle_{123\ldots N}$ and $|W\rangle_{123\ldots N}$ states, except for the trivial case of $N = 2$.

Note that this tripartite entangled state is not a maximally entangled state. The ZSA states are not maximally fragile [23], i.e., measurement of any one of the subsystems does not necessarily destroy the entanglement between the remaining qubits. For example, if we project the first qubit onto the computational basis $|0\rangle$, the state of the particles 2 and 3 is $c_2|10\rangle_{23} + c_3|01\rangle_{23}$, which is a non-maximally entangled state. But projection onto a basis $|1\rangle$ gives a disentangled state. This property holds with respect to all other qubits. The one-particle reduced density matrix $\rho_k \in \mathscr{H}^2$ for any one of the three particles is not completely random but a pseudo-pure state given by

$$\rho_k = |c_k|^2 I + (1 - 2|c_k|^2)|0\rangle\langle 0|, \;\; k = 1,2,3. \tag{3}$$

Further, if we trace out Alice's qubit, the two-qubit state at Bob and Charlie's place is a *mixed entangled* state given by

$$\begin{aligned}\rho_{23} = {} & |c_1|^2|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |c_2|^2|1\rangle\langle 1| \otimes |0\rangle\langle 0| + |c_3|^2|0\rangle\langle 0| \\ & \otimes |1\rangle\langle 1| + c_2c_3^*|1\rangle\langle 0| \otimes |0\rangle\langle 1| + c_2^*c_3|0\rangle\langle 1| \otimes |1\rangle\langle 0|.\end{aligned} \tag{4}$$

That (4) is inseparable can be seen by applying Peres–Horodecki criterion [24,25] which is a necessary and sufficient one in $\mathscr{H}^2 \otimes \mathscr{H}^2$. This says that if a density matrix ρ is separable then the partial transpose has only nonnegative eigenvalues. If T is a transposition on the space of bounded operators $\mathscr{B}(\mathscr{H})$, then the partial transpose PT (with respect to the second subsystem) on $\mathscr{B}(\mathscr{H}) \otimes \mathscr{B}(\mathscr{H})$ is defined as $\rho^{\mathrm{PT}}_{m\mu,n\nu} = \rho_{m\nu,n\mu}$. The eigenvalues of ρ^{PT}_{23} are $\lambda_1 = |c_2|^2$, $\lambda_2 = |c_3|^2$, $\lambda_3 = \frac{1}{2}(|c_1|^2 + \sqrt{|c_1|^4 + 4|c_2|^2|c_3|^2})$ and $\lambda_4 = \frac{1}{2}(|c_1|^2 - \sqrt{|c_1|^4 + 4|c_2|^2|c_3|^2})$. Though the first three eigenvalues are nonnegative, the last one is not (one can check that $\lambda_3\lambda_4$ is a negative number). Therefore, the two qubit density matrix ρ_{23} is inseparable. The same is true if we trace out any other qubit and look at the density matrix of the two-qubit system. A particular measure of entanglement for a mixed state is 'entanglement of formation' [26,27]. The entanglement of formation of ρ_{23} can be computed explicitly and it is given by

$$\begin{aligned}E_{23} = {} & -\frac{1}{2}\left(1 + \sqrt{1 - 4|c_2|^2|c_3|^2}\right) \log\left[\frac{1}{2}\left(1 + \sqrt{1 - 4|c_2|^2|c_3|^2}\right)\right] \\ & -\frac{1}{2}\left(1 - \sqrt{1 - 4|c_2|^2|c_3|^2}\right) \log\left[\frac{1}{2}\left(1 - \sqrt{1 - 4|c_2|^2|c_3|^2}\right)\right].\end{aligned} \tag{5}$$

We will see in §3 that the mixed state density matrix (4) is transformed to a pure state with LOCC. Next we come to the main result of our paper.

3. Universal entangling of unknown qubits with two parties

In what follows we give our protocol for creating two types of universal entangled states of an unknown state at two remote locations. Suppose Alice, Bob and Charlie at remote locations share an entangled state (2) and have access to particles $1,2$ and 3, respectively. An unknown qubit is given to Alice in the form $|\psi\rangle_a = \alpha|0\rangle_a + \beta|1\rangle_a$, where $\alpha = \cos\frac{\theta}{2}$ and $\beta = \sin\frac{\theta}{2}\exp(i\phi)$. We show that Alice can always *create an entangled state of any unknown state with a reference state* $|0\rangle$ or $|1\rangle$ shared between Bob and Charlie by sending two bits of information to both of them. The combined state of the input and the tripartite ZSA entangled state $|\psi\rangle_a \otimes |\Psi\rangle_{123}$ can be expressed in terms of Bell states of particle a and 1 as

$$\begin{aligned} |\psi\rangle_a \otimes |\Psi\rangle_{123} = \frac{1}{\sqrt{2}}[&|\Phi^+\rangle_{a1} \otimes (c_2\alpha|10\rangle_{23} + c_3\alpha|01\rangle_{23} \\ &+ c_1\beta|00\rangle_{23}) + |\Phi^-\rangle_{a1} \otimes (c_2\alpha|10\rangle_{23} + c_3\alpha|01\rangle_{23} - c_1\beta|00\rangle_{23}) \\ &+ |\Psi^+\rangle_{a1} \otimes (c_1\alpha|00\rangle_{23} + c_2\beta|10\rangle_{23} + c_3\beta|01\rangle_{23}) \\ &+ |\Psi^-\rangle_{a1} \otimes (c_1\alpha|00\rangle_{23} + c_2\beta|10\rangle_{23} - c_3\beta|01\rangle_{23})]. \end{aligned} \tag{6}$$

Using the ZSA property, i.e., $\sum_i c_i = 0$, we can rewrite the combined state as

$$\begin{aligned} |\psi\rangle_a \otimes |\Psi\rangle_{123} = \frac{1}{\sqrt{2}}[&|\Phi^+\rangle_{a1} \otimes (c_2 i\sigma_y|\psi\rangle_2|0\rangle_3 + c_3|0\rangle i\sigma_y|\psi\rangle_3) + |\Phi^-\rangle_{a1} \\ &\otimes (c_2\sigma_x|\psi\rangle_2|0\rangle_3 + c_3|0\rangle\sigma_x|\psi\rangle_3) - |\Psi^+\rangle_{a1} \otimes (c_2\sigma_z|\psi\rangle_2|0\rangle_3 \\ &+ c_3|0\rangle\sigma_z|\psi\rangle_3) + |\Psi^-\rangle_{a1} \otimes (c_2|\psi\rangle_2|0\rangle_3 + c_3|0\rangle|\psi\rangle_3)], \end{aligned} \tag{7}$$

where σ_x, σ_y and σ_z are Pauli matrices. Now Alice performs a joint measurement on particles a and 1. Since a Bell-basis measurement will give four possible outcomes $\{|\Phi^\pm\rangle, |\Psi^\pm\rangle\}$ she can get two classical bits of information. Then, Alice sends two classical bits to both Bob and Charlie, who in turn can apply certain local unitary operations to share an entangled state of an unknown state with reference states such as $|0\rangle$ or $|1\rangle$. For example, if the outcome is $|\Phi^+\rangle$ or $|\Phi^-\rangle$, then after receiving the classical information Bob and Charlie will apply $i\sigma_y \otimes i\sigma_y$ or $\sigma_x \otimes \sigma_x$, respectively. They will be sharing an entangled state given by

$$|\psi^{(1)}\rangle_{23} = c_2|\psi\rangle_2|1\rangle_3 + c_3|1\rangle|\psi\rangle_3, \tag{8}$$

where $|\psi^{(1)}\rangle_{23}$ is an *universal entangled state* of an unknown qubit with a reference state $|1\rangle$. This is universal because the protocol works perfectly for any input qubit $|\psi\rangle$. If the result is $|\Psi^+\rangle$ or $|\Psi^-\rangle$, then after receiving the classical communication Bob and Charlie will apply $\sigma_z \otimes \sigma_z$ or $I \otimes I$, respectively. In this case they will be sharing an entangled state given by

$$|\psi^{(0)}\rangle_{23} = c_2|\psi\rangle_2|0\rangle_3 + c_3|0\rangle|\psi\rangle_3, \tag{9}$$

where the states $|\psi^{(0)}\rangle_{23}$ is an *universal entangled state* of an unknown state with a reference state $|0\rangle$. For successful creation of universal entangled states $|\psi^{(0)}\rangle_{23}$ or $|\psi^{(1)}\rangle_{23}$ two classical bits are needed from Alice. Note that the states in eqs (8) and (9) are not normalized. The normalization constant for (8) is $N(\beta) = 1/\sqrt{|c_2|^2 + |c_3|^3 + 2|\beta|^2\mathrm{Re}(c_2^* c_3)}$ and for (9) is $N(\alpha)$, where $N(\alpha)$ can be obtained from $N(\beta)$ by replacing β with α.

An interesting observation is that if the state $|\psi\rangle$ is in a known state such as $|0\rangle$ or $|1\rangle$, then one may use this scheme for quantum cryptographic purposes. For example, if $|\psi\rangle = |0\rangle$, then $|\psi^{(0)}\rangle_{23}$ is not entangled but $|\psi^{(1)}\rangle_{23}$ is. Similarly, if $|\psi\rangle = |1\rangle$, then $|\psi^{(1)}\rangle_{23}$ is not entangled but $|\psi^{(0)}\rangle_{23}$ is. This may provide a way to generate a coded message (detailed discussions are beyond the scope of the present paper and the results will be reported elsewhere [28]).

Next, we quantify the amount of nonlocal quantum resources needed to create a remote shared-entangled state. For more than two parties there is no unique measure of quantum entanglement [29]. Note that the tripartite system can be partitioned in three different ways, viz., *A* vs. *BC*, *B* vs. *AC* and *C* vs. *AB*. Therefore, there are three different ways of calculating the bipartite entanglement. The amount of bipartite entanglement with respect to splitting of particles *A* vs. *BC* is given by the von Neumann entropy of the reduced density matrix ρ_1 [30] as $E(A \text{ vs. } BC) = -\mathrm{Tr}(\rho_1 \log \rho_1) = -(1-|c_1|^2)\log(1-|c_1|^2) - |c_1|^2 \log |c_1|^2$. Therefore, we can roughly say that using $E(\rho_1)$ amount of entanglement and communication of two classical bits to Bob and Charlie one can create two types of quantum universal entanglers for an unknown state. Thus a mixed entangled state (4) is converted to a pure universal entangled state after receiving classical communication from Alice. (Recall a similar situation in quantum teleportation, where a completely random mixture is converted to a pure unknown state).

We can also argue that no classical correlated state (CCS) can create a universal entangled state of an unknown state. If we could create an universal entangled state using CCS via local operations and classical communication then we could create some amount of entanglement between Bob and Charlie. But we know that via LOCC one cannot create any entanglement [29], hence CCS cannot create any universal entangled states.

We can actually quantify the amount of entanglement present in a bipartite universal entangled state (quantum cobweb). When the universal entangled state is of the type (9), then the reduced density matrix of qubit 2 at Bob's place is

$$\rho_2 = N(\alpha)^2[|c_2|^2|\psi\rangle\langle\psi| + |c_3|^2|0\rangle\langle 0| + c_2c_3^*\alpha|\psi\rangle\langle 0| + c_2^*c_3\alpha|0\rangle\langle\psi|]. \qquad (10)$$

In the bipartite case, the Schmidt decomposition theorem [31,32] guarantees that the eigenvalues of the reduced density matrices of *B* and *C* will be identical. They are given by $\eta_\pm = \frac{1}{2}(1 \pm \sqrt{1-4\varepsilon})$, where $\varepsilon = 4N(\alpha)^4|\beta|^4||c_2|^2||c_3|^2$. Therefore, the amount of entanglement will be $E(|\psi^{(0)}\rangle_{23}) = -\mathrm{Tr}(\rho_2 \log \rho_2) = -\mathrm{Tr}(\rho_3 \log \rho_3) = -\eta_+ \log \eta_+ - \eta_- \log \eta_-$.

As an example, if the amplitudes are cube roots of unity, then the ZSA entangled state is of the form $|\Psi\rangle_{123} = \frac{1}{\sqrt{3}}(|100\rangle_{123} + \mathrm{e}^{2\pi i/3}|010\rangle_{123} + \mathrm{e}^{-2\pi i/3}|001\rangle_{123})$. The reduced density matrix for each of the subsystem is the same and also has identical spectrum. It is given by $\rho_1 = \rho_2 = \rho_3 = \mathrm{diag}(2/3, 1/3)$. Therefore, the amount of bipartite entanglement between any partioning is $E(\rho_1) = E(\rho_2) = E(\rho_3) = 1 - (5 - 3\log 3)/3 = 0.9$ ebits. Thus, with the use of 0.9 ebits of entanglement and two cbits of communication one can create, for example, a universal entangled state of the form $|\Psi^{(0)}\rangle_{23} = \frac{1}{\sqrt{3}}(\mathrm{e}^{2\pi i/3}|\psi\rangle_2|0\rangle_3 + \mathrm{e}^{-2\pi i/3}|0\rangle_2|\psi\rangle_3)$, where $|\psi\rangle$ is entangled with $|0\rangle$.

4. Creating quantum cobweb

The states that we have created by this protocol are very special. One can check that there is no local unitary operation on $\mathscr{H}_2 \otimes \mathscr{H}_3$ that can disentangle the unknown state perfectly. Even if both parties come together and perform joint unitary and measurement operations they cannot disentangle the qubit perfectly. Since a general quantum operation is a positive, linear, trace-preserving map that has a unitary representation involving the ancilla, let us assume that there is a unitary operator that disentangles any arbitary qubit perfectly. The action of the unitary operator on a universal entangled state of $|\psi\rangle$ and $|\bar{\psi}\rangle = \alpha|1\rangle - \beta^*|0\rangle$ (with $\langle\psi|\bar{\psi}\rangle = 0$) will be given by

$$N(\alpha)(c_2|\psi\rangle_2|0\rangle_3 + c_3|0\rangle|\psi\rangle_3)|A\rangle \rightarrow |0\rangle|\psi\rangle|A'\rangle,$$
$$N(\beta)(c_2|\bar{\psi}\rangle_2|0\rangle_3 + c_3|0\rangle|\bar{\psi}\rangle_3)|A\rangle \rightarrow |0\rangle|\bar{\psi}\rangle|A''\rangle, \tag{11}$$

where $|A\rangle$ is the initial and $|A'\rangle, |A''\rangle$ are the final states of the ancilla $N(\alpha)$ and $N(\beta)$ are normalization constants for entangled states of $|\psi^{(0)}\rangle$ and $|\bar{\psi}^{(0)}\rangle$, respectively. Taking the inner product we have $2N(\alpha)N(\beta)\alpha\beta^*\mathrm{Re}(c_2c_3^*) = 0$ and this can never be satisfied for any non-zero values of c_2, c_3, α and β. Therefore, we cannot disentangle the state even by joint action and irreversible operations. Thus, the unknown state (containing some secret information) can remain simultaneously with two parties in a non-local manner. This class of states we call *quantum cobwebs* because once they are created the unknown state is trapped inside the multiparticle entangled state. However, here we would like to leave it as an open question whether these universal entangled states can be disentangled perfectly using entanglement assisted local operations and classical communication. We should mention that our scheme may not be the only way to generate universal entangled states (cobwebs). But the author has not yet found any other means of creating cobwebs.

It may be worth mentioning that if Bob and Charlie perform non-local unitary operations and measurement, then one of them can recover the state with unit fidelity in a *probabilistic manner*. For example, to disentangle $|\psi^{(0)}\rangle_{23}$ Bob and Charlie can come together and perform a CNOT operation followed by a measurement of particle 2 in the basis $\{|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\}$. When Bob gets $|+\rangle$, Charlie's qubit is in the state $|\psi\rangle$ and when Bob gets $|-\rangle$ Charlies qubit is not in the state $|\psi\rangle$, (i.e. there is an error in getting $|\psi\rangle$) so they can discard this. The probability of success is $P = [|c_2|^2 + |c_3|^3 + 2\mathrm{Re}(c_2^*c_3)]/2[|c_2|^2 + |c_3|^3 + 2\alpha^2\mathrm{Re}(c_2^*c_3)]$ which is greater than half (i.e., better than a random guess).

5. Universal entangled state for multiparties

We can generalize the universal quantum entangler for $(N-1)$ parties where an unknown qubit can be entangled with a reference state and shared with $(N-1)$ parties. Let there be N parties in a network of N nodes each having access to a single qubit. They share N-partite ZSA entangled state $|\Psi\rangle_{123\ldots N} \in \mathscr{H}^{2\otimes N}$ given by (1).

Now, we describe how Alice can create an $(N-1)$-partite entangled state of any unknown state with a reference state $|0\rangle$ or $|1\rangle$ shared between Bob, Charlie and Nancy by sending two bits of information to the concerned parties. The combined state of the unknown qubit and N-partite entangled state $|\psi\rangle_a \otimes |\Psi\rangle_{123\ldots N}$ can be expressed in terms of Bell states of particle a and 1 as (again using the ZSA property)

$$|\psi\rangle_a \otimes |\Psi\rangle_{123\ldots N} = \frac{1}{\sqrt{2}}\left[|\Phi^+\rangle_{a1} \otimes \sum_{k=2}^{N} c_k|(i\sigma_y)\psi_{(k)}^{(0)}\rangle_{23\ldots N} + |\Phi^-\rangle_{a1}\right.$$

$$\otimes \sum_{k=2}^{N} c_k |(\sigma_x)\psi_{(k)}^{(0)}\rangle_{23\ldots N} - |\Psi^+\rangle_{a1} \otimes \sum_{k=2}^{N} c_k |(\sigma_z)\psi_{(k)}^{(0)}\rangle_{23\ldots N} + |\Psi^-\rangle_{a1}$$
$$\otimes \sum_{k=2}^{N} c_k |\psi_{(k)}^{(0)}\rangle_{23\ldots N}\Bigg], \tag{12}$$

where $|\psi_{(k)}^{(0)}\rangle_{23\ldots N} = |0\rangle_2|0\rangle_3 \cdots |\psi\rangle_k \cdots |0\rangle_N$ is an $(N-1)$ qubit strings containing all qubits in the state $|0\rangle$ except that the kth party contains the unknown state $|\psi\rangle$. Alice performs a joint Bell-state measurement on particles a and 1. If the outcome is $|\Phi^+\rangle$ or $|\Phi^-\rangle$ then after sending classical communications to the concerned $(N-1)$ parties, they will apply $i\sigma_y \otimes \cdots \otimes i\sigma_y$ or $\sigma_x \otimes \cdots \otimes \sigma_x$, respectively. They will end up sharing an entangled state given by $|\psi^{(1)}\rangle_{23\ldots N} = \sum_{k=2}^{N} c_k |\psi_{(k)}^{(1)}\rangle_{23\ldots N}$, where $|\psi_{(k)}^{(1)}\rangle_{23\ldots N} = |1\rangle_2|1\rangle_3 \cdots |\psi\rangle_k \cdots |1\rangle_N$ is an $(N-1)$ qubit string that contains all qubits in the state $|1\rangle$ except that the kth party contains the unknown state $|\psi\rangle$. If the outcome is $|\Psi^+\rangle$ or $|\Psi^-\rangle$ then after receiving classical communications, $(N-1)$ parties will apply $\sigma_z \otimes \cdots \otimes \sigma_z$ or $I \otimes \cdots \otimes I$ (do nothing), respectively. Thus, they will end up sharing an entangled state given by $|\psi^{(0)}\rangle_{23\ldots N} = \sum_{k=2}^{N} c_k |\psi_{(k)}^{(0)}\rangle_{23\ldots N}$. Thus, with the use of ZSA entangled state and two classical bits one can create universal entangled states of an unknown state with two types of reference states that have been shared between $(N-1)$ parties at remote locations. Thus, $|\psi^{(0)}\rangle_{23\ldots N}$ and $|\psi^{(1)}\rangle_{23\ldots N}$ are $(N-1)$ node quantum cobwebs from which an unknown state cannot be disentangled perfectly by local or nonlocal unitary operations. The multiparty cobweb states are not normalized as expected.

As an example, if the amplitudes are Nth roots of unity, then the ZSA entangled state is given by $|\Psi\rangle_{123\ldots N} = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \mathrm{e}^{i2\pi k/N} |x_k\rangle_{123\ldots N}$. The reduced density matrix of each qubit is identical and is given by $\rho_k = \mathrm{diag}[(1-1/N), 1/N]$. Therefore, the amount of bipartite entanglement is independent of the choice of N possible bipartite partitioning. The bipartite entanglement E with respect to splitting between particle 1 and the rest $(N-1)$ qubits is $E = E(\rho_1) = -[(1-\frac{1}{N})\log(1-\frac{1}{N}) + \frac{1}{N}\log\frac{1}{N}]$. With the use of E bits of entanglement one can create a universal entangled state of $(N-1)$ qubits as $|\psi^{(0)}\rangle_{23\ldots N} = \frac{1}{\sqrt{N}} \sum_{k=2}^{N} \mathrm{e}^{i2\pi k/N} |\psi_{(k)}^{(0)}\rangle_{23\ldots N}$. If the number of parties N becomes very large $E \to 1/N$, i.e., it approaches zero. For large but finite N one can prepare a universal entangled state with the use of $O(1/N)$ ebits for $O(N)$ parties at remote locations.

6. Conclusions

To conclude, we have introduced a class of zero sum amplitude multipartite entangled states, studied their properties and presented a protocol to create two types of universal entangled states of an unknown state with reference states. This was thought to be an impossible task. This surprising feature exploits one property, that is the zero sum amplitude nature of the original shared entangled state between N parties. We hope that the nature of ZSA and universal entangled states will throw some new light on the nature of quantum information and role of entanglement. Some open questions include: Is our scheme the most simple scheme for creating cobwebs? Why are the ZSA states so special? In future, one can also explore if these multipartite ZSA entangled states can be employed for some other quantum information processing tasks.

Acknowledgements

The author thanks S Bose and A Chefles for useful remarks, S R Jain, Z Ahmed, H D Parab for discussions concerning the zero sum nature of complex numbers and T Brun for bringing some useful references to my notice.

References

[1] C H Bennett, H J Bernstein, S Popescu and B Schumacher, *Phys. Rev.* **A53**, 2046 (1996)
[2] C Bennett, G Brassard, C Crepeau, R Jozsa, A Peres, and W K Wooters, *Phys. Rev. Lett.* **70**, 1895 (1993)
[3] C H Bennett and S Wiesner, *Phys. Rev. Lett.* **69**, 2881 (1992)
[4] M Zukowski, A Zeilinger, M A Horne and E Ekert, *Phys. Rev. Lett.* **A71**, 4278 (1993)
[5] S Bose, V Vedral and P Knight, *Phys. Rev.* **A57**, 822 (1998)
[6] A K Pati, *Phys. Rev.* **A63**, 014320-1 (2001)
[7] H K Lo, *Phys. Rev.* **A62**, 012313 (2000)
[8] C H Bennett, D P DiVincenzo, J A Smolin, B M Terhal and W K Wootters, *Phys. Rev. Lett.* **87**, 077902 (2001)
[9] S F Huelga, J A Vaccaro, A Chefless and M B Plenio, *Phys. Rev.* **A63**, 042303 (2001)
[10] M Murao *et al*, *Phys. Rev.* **A59**, 156 (1999)
[11] M Murao and V Vedral, *Phys. Rev. Lett.* **86**, 352 (2001)
[12] W K Wootters and W H Zurek, *Nature* **299**, 802 (1982)
[13] D Dieks, *Phys. Lett.* **A92**, 271 (1982)
[14] A K Pati and S L Braunstein, *Nature* **404**, 164 (2000)
[15] V Buzek, M Hillery and R F Werner, *Phys. Rev.* **A60**, R2626 (1999)
[16] V Buzek and M Hillery, *Phys. Rev.* **A62**, 022303 (2000)
[17] V Buzek and M Hillery, *Phys. Rev.* **A62**, 052303 (2000)
[18] A Barenco, A Berthiaume, D Deutsch, A Ekert, R Jozsa and C Macchiavello, *SIAM J. Comput.* **26**, 1541 (1997)
[19] O Cohen and T Brun, *Phys. Rev. Lett.* **84**, 5908 (2000)
[20] V Coffman, J Kundu and W K Wootters, *Phys. Rev.* **A61**, 052306 (2000)
[21] T Brun and O Cohen, *Phys. Lett.* **A281**, 88 (2001)
[22] W Dür, G Vidal and J I Cirac, *Phys. Rev.* **A62**, 062314-1 (2000)
[23] N Gisin and H B Pasquinucci, *Phys. Lett.* **A246**, 1 (1998)
[24] A Peres, *Phys. Rev. Lett.* **77**, 1413 (1996)
[25] M Horodecki, P Horodecki and R Horodecki, *Phys. Lett.* **A223**, 1 (1996)
[26] C H Bennett, D P DiVincenzo, J Smolin and W K Wootters, *Phys. Rev.* **A54**, 3824 (1996)
[27] W K Wootters, quant-ph/9709029
[28] A K Pati, *Quantum cryptography using zero sum amplitude entangled states* (in preparation)
[29] G Vidal, W Dür and J I Cirac, *Phys. Rev. Lett.* **85**, 658 (2000)
[30] S Popescu and D Rohrlich, *Phys. Rev.* **A56**, R3319 (1997)
[31] A Ekert and P Knight, *Am. J. Phys.* **63**, 415 (1995)
[32] A K Pati, *Phys. Lett.* **A278**, 118 (2000)