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Decoherence and infrared divergence

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Abstract. The dynamics of a particle which is linearly coupled to a boson field is investigated. The boson field induces superselection rules for the momentum of the particle, if the field is infrared divergent. Thereby the Hamiltonian of the total system remains bounded from below.

Keywords. Quantum mechanics; decoherence; infrared divergence.

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1. Introduction

Decoherence and superselection rules are the basis for understanding how classical physics can emerge within the quantum theory. The mathematical structure of quantum mechanics and of quantum field theory provides only a few superselection rules, the most important being the charge superselection rule related to the gauge invariance of the electromagnetic field, see, e.g., [1] and references therein. But there are definitively not enough of these superselection rules to understand the classical appearance of the world within quantum theory. A possible solution of this problem is the emergence of superselection rules due to decoherence caused by the interaction with the environment [2]. We investigate the dynamics of a particle which is linearly coupled to a boson field. The Hamiltonian of the total system is bounded from below. This talk concentrates on a specific aspect of decoherence: decoherence due to infrared divergence of the boson field. In the case of velocity coupling the infrared divergence induces a superselection rule for the momentum with uniform estimates for the emergence of the superselection sectors [3]. For the model with local coupling the infrared divergence also causes a strong decoherence between states of different momenta, but the superselection sectors are not uniformly induced. In this talk only some indications are given about how these results can be derived. The detailed calculations will be published in [4]. The models considered here have quadratic Hamiltonians, and they can be solved in an explicit way. Despite their unrealistic simplicity, such Hamiltonians have been successfully used to construct heat bath models [5], and for understanding the approach to thermal equilibrium [6,7], or for calculating decoherence effects due to radiation [8]. The structure of quadratic Hamiltonians is therefore rich enough for obtaining interesting results.

2. The models

We consider a massive spinless particle coupled to a boson field of mass zero. The method presented here can be applied to a particle in $\mathbf{R}^d$ with $d \geq 1$. But to simplify the notations we restrict ourselves to one dimension. The Hilbert space of the theory is the tensor space $\mathscr{H}_{\mathrm{S}} \otimes \mathscr{H}_{\mathrm{F}} = \mathscr{L}^2(\mathbf{R}) \otimes \mathscr{F}(\mathscr{H}_0)$ of the Hilbert space for the particle $\mathscr{H}_{\mathrm{S}} = \mathscr{L}^2(\mathbf{R})$ and of the Fock space $\mathscr{H}_{\mathrm{F}} = \mathscr{F}(\mathscr{H}_0)$ for the boson field, where $\mathscr{H}_0$ is the one-particle Hilbert space of the bosons. The Hamiltonian without interaction between the particle and the field,

$$H_{(0)} = \frac{1}{2}\left(P^2 + \omega^2 Q^2\right) \otimes I_{\mathrm{F}} + I_{\mathrm{S}} \otimes H_{\mathrm{F}} \tag{1}$$

is the sum of the Hamiltonian of a harmonic oscillator or a free particle ($\omega = 0$) on the Hilbert space $\mathscr{L}^2(\mathbf{R})$ and the Hamiltonian H_{F} of a free field on the Fock space $\mathscr{F}(\mathscr{H}_0)$. As interaction, we take

– a velocity coupling $H_P = P \otimes \Phi(h)$ without harmonic force ($\omega = 0$),
– a local interaction $H_Q = Q \otimes \Phi(h)$ together with a harmonic force ($\omega > 0$),

where $\Phi(h)$ is the boson field operator smeared with a suitably chosen test function h. For these quadratic Hamiltonians the field equations are linear and can be solved explicitly. To investigate the reduced dynamics we use the Heisenberg picture and take the algebra generated by the Weyl operators as the algebra of observables.

3. Weyl operators and classical phase space

3.1 *Quantum mechanics of the particle*

As Hilbert space of the particle we take $\mathscr{H}_{\mathrm{S}} = \mathscr{L}^2(\mathbf{R})$. The Weyl operators are defined as exponentials of the canonical variables, the position operator Q and the momentum operator P, as $W_{\mathrm{S}}(a,b) = \exp i\,(aP + bQ)$ with the parameters $(a,b) \in \mathbf{R}^2$. In the momentum representation on the Hilbert space $\mathscr{L}^2(\mathbf{R})$ the Weyl operators are given by

$$\left(W_{\mathrm{S}}(a,b)f\right)(k) = \exp\left(\frac{i}{2}ab + iak\right) f(k-b). \tag{2}$$

From this identity follows that the arguments $(a,b) \in \mathbf{R}^2$ can be interpreted as the position and the momentum variables of the classical phase space of the particle. The closure of the linear span of Weyl operators with the operator norm is the Weyl algebra. This algebra is smaller than the algebra of all bounded operators on $\mathscr{L}^2(\mathbf{R})$. But the closure of the Weyl algebra with respect to the weak or the strong operator topology includes all bounded operators. For a one-dimensional harmonic oscillator with Hamiltonian $H = \frac{1}{2}\left(P^2 + \omega^2 Q^2\right), \omega > 0$, and unitary group $U(t) = \exp(-iHt)$, the time dependence of the canonical variables is $U^+(t)PU(t) = P\cos\omega t - Q\omega\sin\omega t$ and $U^+(t)QU(t) = P\omega^{-1}\sin\omega t + Q\cos\omega t$. The dynamics of the Weyl operators then follows as $U^+(t)W_{\mathrm{S}}(a,b)U(t) = W_{\mathrm{S}}(a(t),b(t))$ with

$$\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = R(t)\begin{pmatrix} a \\ b \end{pmatrix},\ R(t) = \begin{pmatrix} \cos\omega t & \omega^{-1}\sin\omega t \\ -\omega\sin\omega t & \cos\omega t \end{pmatrix}, \tag{3}$$

where $R(t)$ is a group of rotations on $\mathbf{R}^2$. For $\omega \to +0$, the transformations $R(t)$ have a well-defined limit, which corresponds to the dynamics $P(t) = P(0) = P$ and $Q(t) = Q(0) + Pt$ or $a(t) = a + bt$ and $b(t) = b$ of a free particle.

3.2 *Field operators*

The basic Hilbert space for the boson field is $\mathscr{H}_0 = \mathscr{L}^2(\mathbf{R}^n)$ with inner product $\langle f \mid g \rangle = \int \overline{f(k)} g(k) d^n k$ and norm $\|f\| = \sqrt{\langle f \mid f \rangle}$. The operator M is the one-particle Hamilton operator $(Mf)(k) = \omega(k)\ f(k)$ with $\omega(k) = \sqrt{k^2}$, $k^2 = \sum_{\mu=1}^{n} k_\mu^2$, $k = (k_1, ..., k_n) \in \mathbf{R}^n$. The operator M and its inverse M^{-1} are unbounded operators. To define real vectors we introduce the involution $f^*(k) := \overline{f(-k)}$. The operator M is a real operator $(Mf)^* = Mf^*$ with respect to this involution. The one-particle space $\mathscr{H}_0$ generates the Fock space $\mathscr{F}(\mathscr{H}_0)$. Let us denote with $f \circ g$ the symmetric tensor product. Then the normalizations of the Fock space $\mathscr{F}(\mathscr{H}_0)$ are chosen such that the inner product of exponential vectors $\exp f = 1_{\text{vac}} + f + \frac{1}{2} f \circ f + \cdots \in \mathscr{F}(\mathscr{H}_0)$, $f \in \mathscr{H}_0$, is $\langle \exp f \mid \exp g \rangle = \exp \langle f \mid g \rangle$. The normalized exponential vector $\exp\left(f - \frac{1}{2}\|f\|^2\right)$, $f \in \mathscr{H}_0$, is usually called a coherent state. We use the notation $\sigma(f)$ for the projection operator onto that state.

The creation operators: $a^+(h)$, $h \in \mathscr{H}_0$, are uniquely defined by $a^+(h) \exp f := h \circ \exp f$, and the annihilation operators $a(h)$, $h \in \mathscr{H}_0$ are the adjoint operators $a(h) = (a^+(h))^+$. These operators are normalized to $[a(f),\ a^+(g)] = \langle f \mid g \rangle$.

The Hamilton operator H_F of the free field is uniquely determined by $H_\text{F} \exp f = Mf \circ \exp f$. The Hilbert space of real vectors $f = f^* \in \mathscr{H}_0$, is denoted by $\mathscr{H}_\mathbf{R}$. It is convenient to introduce a test function space $\mathscr{E}_\mathbf{R} \subset \mathscr{H}_\mathbf{R}$ on which arbitrary powers of M are defined, $\mathscr{E}_\mathbf{R} = \left\{ f \in \mathscr{H}_\mathbf{R} \mid M^n f \in \mathscr{H}_\mathbf{R},\ n = -1, 0, 1, ... \right\}$. For real test functions $f \in \mathscr{E}_\mathbf{R}$, we define the *field operator* $\Phi(f)$ and the *conjugate momentum field* $\Pi(f)$,

$$\begin{aligned} \Phi(f) &:= \frac{1}{\sqrt{2}} \left(a^+(M^{-1/2} f) + a(M^{-1/2} f) \right), \\ \Pi(f) &:= i[H, \Phi(f)] = \frac{i}{\sqrt{2}} \left(a^+(M^{1/2} f) - a(M^{1/2} f) \right) \end{aligned} \tag{4}$$

see, e.g., [9]. These fields are self-adjoint and they satisfy the canonical commutation relations $[\Phi(f), \Phi(g)] = [\Pi(f), \Pi(g)] = 0$ and $[\Phi(f), \Pi(g)] = i \langle f \mid g \rangle$. From these definitions, it follows that the arguments of the canonical fields $\Pi(u)$ and $\Phi(v)$ can be extended to vectors $u \in \mathscr{H}_{-1\mathbf{R}}$ and $v \in \mathscr{H}_{1\mathbf{R}}$, where $\mathscr{H}_{\pm 1\mathbf{R}}$ are the real Hilbert spaces with the inner products $\langle f \mid g \rangle_{\pm 1} = \left\langle f \mid M^{\mp 1} g \right\rangle$. The spaces $\mathscr{H}_{1\mathbf{R}}$ and $\mathscr{H}_{-1\mathbf{R}}$ are dual spaces with respect to the inner product $\langle f \mid g \rangle$ of $\mathscr{H}_\mathbf{R}$. The Hamiltonians M and H_F generate the unitary groups $U_0(t) = \exp(-iMt)$ on $\mathscr{H}_0$ and $U_\text{F}(t) = \exp(-iH_\text{F} t)$ on $\mathscr{F}(\mathscr{H}_0)$, respectively. The time evolution of the field $\Phi(v)$ and of the canonical momentum field $\Pi(u)$ is

$$\begin{aligned} \Phi(v,t) &= U_\text{F}^+(t) \Phi(v) U_\text{F}(t) = \Phi(\cos(Mt) v) + \Pi\left(M^{-1} \sin(Mt) v\right), \\ \Pi(u,t) &= U_\text{F}^+(t) \Pi(u) U_\text{F}(t) = -\Phi(M \sin(Mt) u) + \Pi(\cos(Mt) u). \end{aligned} \tag{5}$$

These identities are defined for test functions $(u, v) \in \mathscr{H}_{-1\mathbf{R}} \times \mathscr{H}_{1\mathbf{R}}$.

3.3 *Weyl operators in QFT*

We define the unitary Weyl operators $W_{\mathrm{F}}(u,v),\ (u,v) \in \mathscr{H}_{-1\mathbf{R}} \times \mathscr{H}_{1\mathbf{R}}$, by

$$W_{\mathrm{F}}(u,v) := \exp\left(i\Pi(u) + i\Phi(v)\right). \tag{6}$$

The $\mathbf{R}$-linear space of the arguments $(u,v) \in \mathscr{H}_{-1\mathbf{R}} \times \mathscr{H}_{1\mathbf{R}}$ can be interpreted as the classical phase space of the theory. The expectation of the Weyl operators between exponential vectors is

$$\begin{aligned}&\left\langle \exp f \mid W_{\mathrm{F}}(u,v) \exp f \right\rangle \\ &\quad = \exp\left(\|f\|^2 - \frac{1}{4}\|u\|_{-1}^2 - \frac{1}{4}\|v\|_1^2 + i\mathrm{Im}\left\langle M^{1/2}u + iM^{-1/2}v \mid f \right\rangle \right)\end{aligned} \tag{7}$$

which includes the vacuum expectation. The expectation of $W_{\mathrm{F}}(u,v)$ in a K(ubo)–M(artin)–S(chwinger) state of inverse temperature $\beta > 0$ is [10a]

$$\left\langle W_{\mathrm{F}}(u,v)\right\rangle_\beta = \exp\left(-\frac{1}{4}\left\langle u \mid \coth\frac{\beta M}{2} u \right\rangle_{-1} - \frac{1}{4}\left\langle v \mid \coth\frac{\beta M}{2} v\right\rangle_1 \right). \tag{8}$$

The dynamics of the Weyl operators follows immediately from (5) as $U_{\mathrm{F}}^{+}(t)W_{\mathrm{F}}(u,v)$ $U_{\mathrm{F}}(t) = W_{\mathrm{F}}\left(u(t),v(t)\right)$ with

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = R(t)\begin{pmatrix} u \\ v \end{pmatrix},\ R(t) = \begin{pmatrix} \cos(Mt) & M^{-1}\sin(Mt) \\ -M\sin(Mt) & \cos(Mt) \end{pmatrix}. \tag{9}$$

The mapping $R(t)$ is a rotation of the phase space $\mathscr{H}_{-1\mathbf{R}} \times \mathscr{H}_{1\mathbf{R}}$ with respect to the quadratic form $\left\langle u \mid M^{1/2}u\right\rangle + \left\langle v \mid M^{-1/2}v\right\rangle = \|u\|_{-1}^2 + \|v\|_1^2$ of the vacuum functional.

4. Dynamics of the system of particle and field

4.1 *The flow on the classical phase space*

The Hilbert space of the total system of particle and field is $\mathscr{H}_{\mathrm{S}} \otimes \mathscr{H}_{\mathrm{F}} = \mathscr{L}^2(\mathbf{R}) \otimes \mathscr{F}(\mathscr{H}_0)$. The classical phase space can be represented by $(a,u,b,v) \in \mathbf{R} \times \mathscr{H}_{-1\mathbf{R}} \times \mathbf{R} \times \mathscr{H}_{1\mathbf{R}}$ and the Weyl operators are

$$\begin{aligned}W(a,u,b,v) &= \exp i\left(aP + bQ\right) \otimes I_{\mathrm{F}} + I_{\mathrm{S}} \otimes \left(\Pi(u) + \Phi(v)\right) \\ &= W_{\mathrm{S}}(a,b) \otimes W_{\mathrm{F}}(u,v) \text{ with } (a,u,b,v) \in \mathbf{R}\times\mathscr{H}_{-1\mathbf{R}} \times \mathbf{R}\times\mathscr{H}_{1\mathbf{R}}.\end{aligned} \tag{10}$$

The dynamics of the fields is uniquely determined by the dynamics of the Weyl operators

$$U^{+}(t)W(a,u,b,v)U(t) = W(a(t),u(t),b(t),v(t)), \tag{11}$$

where the flow $(a(t),u(t),b(t),v(t))$ on the classical phase space $\mathbf{R} \times \mathscr{H}_{-1\mathbf{R}} \times \mathbf{R} \times \mathscr{H}_{1\mathbf{R}}$ is given by a one-parameter group $R(t) = \exp Lt$ of symplectic transformations. The generator L of the group $R(t)$ is defined on the restricted phase space $\mathbf{R} \times \mathscr{E}_{\mathbf{R}} \times \mathbf{R} \times \mathscr{E}_{\mathbf{R}}$ and can be read off from the linear field equations of the theory.

4.2 *Reduced dynamics*

We assume that the initial state of the total system is given by

$$\rho = \rho_{\mathrm{S}} \otimes \rho_{\mathrm{F}} \in \mathscr{D}(\mathscr{H}_{\mathrm{S}} \otimes \mathscr{H}_{\mathrm{F}}), \tag{12}$$

where $\rho_{\mathrm{S}} \in \mathscr{D}(\mathscr{H}_{\mathrm{S}})$ is the initial state of the particle and $\rho_{\mathrm{F}} \in \mathscr{D}(\mathscr{H}_{\mathrm{F}})$ is the reference state of the field. The dynamics of any observable $A \in \mathscr{B}(\mathscr{H}_{\mathrm{S}})$ of the particle is then

$$A \to A(t) = \Phi_t[A] := \mathrm{tr}_{\mathrm{F}} U^{+}(t) \left(A \otimes I_{\mathrm{F}}\right) U(t) \rho_{\mathrm{F}}. \tag{13}$$

If A is the Weyl operator $W_{\mathrm{S}}(a,b)$ of the particle, then $W_{\mathrm{S}}(a,b) \otimes I_{\mathrm{F}} = W(a,0,b,0)$ is a Weyl operator of the total system as defined in §4.1 with $(a, u=0, b, v=0) \in \mathbf{R} \times \mathscr{H}_{-1\mathbf{R}} \times \mathbf{R} \times \mathscr{H}_{1\mathbf{R}}$. For quadratic Hamiltonians the time evolution $U^{+}(t) \left(W_{\mathrm{S}}(a,b) \otimes I_{\mathrm{F}}\right) U(t)$ yields the Weyl operator (eq. (11)) $W(a(t),u(t),b(t),v(t))$. Thereby the trajectories $(a(t),u(t),b(t),v(t))$ start from the initial conditions $(a(0),u(0),b(0),v(0)) = (a,0,b,0)$. Since $W(a,u,b,v) = W_{\mathrm{S}}(a,b) \otimes W_{\mathrm{F}}(u,v)$ the reduced dynamics (13) is simply calculated as

$$\Phi_t\left[W_{\mathrm{S}}(a,b)\right] = W_{\mathrm{S}}(a(t),b(t))\, \chi(a,b;t), \tag{14}$$

with the function $\chi(a,b;t) = \mathrm{tr}_{\mathrm{F}} W_{\mathrm{F}}(u(t),v(t)) \rho_{\mathrm{F}}$. A decrease of the trace $\chi(a,b;t)$ for large t indicates decoherence effects due to the environment of the bosons. If ρ_{F} is the projection operator $\sigma(g)$ onto the coherent state $\exp\left(g - (1/2)\|g\|^2\right), g \in \mathscr{H}_0$, the trace $\chi(a,b;t) = \mathrm{tr}_{\mathrm{F}} W_{\mathrm{F}}(u(t),v(t))\sigma(g)$ is bounded by, see (7),

$$|\chi(a,b;t)| = \exp\left(-\frac{1}{4}\|u(t)\|_{-1}^2 - \frac{1}{4}\|v(t)\|_1^2\right). \tag{15}$$

The choice of the coherent states simplifies the calculations considerably. But the statements about decoherence are qualitatively not affected if one takes another statistical operator as state of the environment [11]. If the boson field is in a state of inverse temperature $\beta > 0$, the function $\chi(a,b;t)$ is given by (8)

$$\begin{aligned} \chi_\beta(t) &= \exp\left(-\frac{1}{4}\left\langle u(t) \mid \coth\frac{\beta M}{2} u(t)\right\rangle_{-1} - \frac{1}{4}\left\langle v(t) \mid \coth\frac{\beta M}{2} v(t)\right\rangle_{1}\right) \\ &< \exp\left(-\frac{1}{4}\|u(t)\|_{-1}^2 - \frac{1}{4}\|v(t)\|_1^2\right). \end{aligned} \tag{16}$$

Remark. The technique to calculate the dynamics of a subsystem with the help of Weyl operators has been used by Davies to derive the approach of thermal equilibrium in a heat bath [6].

4.3 *Evaluation of the dynamics and induced superselection rules*

4.3.1 *Independent systems*: We first consider a free or harmonically bound particle on the real line and a free field with Hamiltonian (1). The operators

$$R_0(t) = \begin{pmatrix} \cos(\widehat{M}_0 t) & \widehat{M}_0^{-1}\sin(\widehat{M}_0 t) \\ -\widehat{M}_0 \sin(\widehat{M}_0 t) & \cos(\widehat{M}_0 t) \end{pmatrix} \text{ with } \widehat{M}_0 = \begin{pmatrix} \omega & 0 \\ 0 & M \end{pmatrix} \tag{17}$$

are rotations on the phase space $\mathbf{R} \times \mathscr{H}_{-1\mathbf{R}} \times \mathbf{R} \times \mathscr{H}_{1\mathbf{R}}$. This rotation is the tensor product of (3) and (9).

4.3.2 *Interacting systems*: (a) *Velocity coupling*. The Hamiltonian with the velocity coupling,

$$\begin{aligned} H &= \frac{1}{2}P^2 \otimes I_{\mathrm{F}} + P \otimes \Phi(h) + I_{\mathrm{S}} \otimes H_{\mathrm{F}} \\ &= \frac{1}{2}\left(P \otimes I_{\mathrm{F}} + I_{\mathrm{S}} \otimes \Phi(h)\right)^2 + I_{\mathrm{S}} \otimes \left(H_{\mathrm{F}} - \frac{1}{2}\Phi^2(h)\right)^2 \end{aligned} \tag{18}$$

is bounded from below if $\left\|M^{-1}h\right\| \leq 1$. The momentum P is a conserved quantity and it is possible to calculate the reduced dynamics of this model [3]. The bosonic part of (18) is the van Hove model [12]. If h has low energy contributions such that h is no longer in the domain of the operator $M^{-3/2}$, $h \notin \mathscr{D}(M^{-3/2})$, the ground state of the van Hove model disappears in the continuum, see [13], and §6.1 of [14]. Moreover, in the van Hove model and in our model the mean boson number diverges due to the soft bosons. As a consequence of this infrared catastrophe the selection rule of the momentum of the particle becomes a superselection rule [3]. Here we want to understand this effect with the help of the flow on the classical phase space. Assuming the stronger constraints

$$\left\|M^{-1}h\right\| < 1, \quad \text{and} \quad h \in \mathscr{D}(M^{-\frac{3}{2}}) \subset \mathscr{H}_{\mathbf{R}}, \tag{19}$$

the dynamics is unitarily equivalent to the free dynamics, and the flow can be easily calculated, see [4]. Starting from the initial values $(a(0), u(0), b(0), v(0)) = (a, 0, b, 0)$ we obtain

$$\begin{pmatrix} a(t) \\ u(t) \\ b(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} a + b\left\langle M^{-1}h \mid (\sin Mt)M^{-2}h\right\rangle + \alpha^2 bt \\ b(I - \cos Mt)M^{-2}h \\ b \\ b(\sin Mt)M^{-1}h \end{pmatrix}. \tag{20}$$

Since M has an absolutely continuous spectrum, the vector functions $u(t) \in \mathscr{H}_{-1\mathbf{R}}$ and $v(t) \in \mathscr{H}_{1\mathbf{R}}$ remain bounded for $t \to \infty$. These expressions have been derived under the conditions (19). But the identities (20) remain meaningful under the weaker condition $\left\|M^{-1}h\right\| \leq 1$. If the low energy contributions in h are strong enough such that $h \notin \mathscr{D}(M^{-3/2})$, there is a qualitative change in the time evolution: the Hamiltonian (18) is no longer unitarily equivalent to the free Hamiltonian, and the norms

$$\begin{aligned} \|u(t)\|_{-1} &= b^2 \left\|(I - \cos Mt)M^{-2}h\right\|_{-1} = b^2 \left\|(I - \cos Mt)M^{-\frac{3}{2}}h\right\| \\ \|v(t)\|_1 &= b^2 \left\|(\sin Mt)M^{-1}h\right\|_1 = b^2 \left\|(\sin Mt)M^{-\frac{3}{2}}h\right\| \end{aligned} \tag{21}$$

diverge for $t \to \infty$ if $b \neq 0$. As a consequence the decoherence function decreases to zero in both the cases of an environment given by coherent states (15) and an environment given by a thermal state (16). The formulae (20) then imply the uniform bound

$$\left\|\Phi_t\left[W_{\mathrm{S}}(a,b)\right]\right\| \leq \exp\left(-b^2\varphi(t)\right) \tag{22}$$

with a positive non-decreasing function $\varphi(t)$ which diverges for $t\to\infty$. If $b=0$, relations (14) and (20) imply $\Phi_t\left[W_{\mathrm{S}}(a,b=0)\right] = W_{\mathrm{S}}(a,b=0)$. Let $\mathscr{L}^2(\mathbf{R})$ be the Hilbert space of the particle in the momentum representation. For any interval $\mathbf{I}\subset\mathbf{R}$ of the real line we denote with $P(\mathbf{I})$ the projection operator, which maps $\mathscr{L}^2(\mathbf{R})$ into $\mathscr{L}^2(\mathbf{I})$, i.e., $f(x)\in\mathscr{L}^2(\mathbf{R})$ is mapped onto $(P(\mathbf{I})f)(x) = f(x)$ if $x\in\mathbf{I}$, and $(P(\mathbf{I})f)(x)=0$ if $x\notin\mathbf{I}$. From (2) it follows that

$$P(\mathbf{I}_2)W_{\mathrm{S}}(a,b)P(\mathbf{I}_1) = 0 \quad \text{if } b\notin \mathbf{I}_2-\mathbf{I}_1, \tag{23}$$

for any two non-overlapping intervals $\mathbf{I}_1$ and $\mathbf{I}_2$ of the real line. The relations (14), (22) and (23) yield $\left\|P(\mathbf{I}_2)\Phi_t\left[W_{\mathrm{S}}(a,b)\right]P(\mathbf{I}_1)\right\| \leq \exp\left(-\delta^2\varphi(t)\right)$, where $\delta = \mathrm{dist}(\mathbf{I}_1,\mathbf{I}_2)>0$ is the distance between the intervals. Moreover, any linear combination $A=\sum c_j W_{\mathrm{S}}(a_j,b_j)$ of Weyl operators satisfies an estimate $\left\|P(\mathbf{I}_2)\Phi_t[A]P(\mathbf{I}_1)\right\| \leq C_A\exp\left(-\delta^2\varphi(t)\right)$ with some constant C_A. These estimates imply: for all operators A of the Weyl algebra we have

$$\lim_{t\to\infty}\left\|P(\mathbf{I}_2)\Phi_t[A]P(\mathbf{I}_1)\right\| = 0, \tag{24}$$

and for all bounded observables A the strong convergence

$$\lim_{t\to\infty}P(\mathbf{I}_2)\Phi_t[A]P(\mathbf{I}_1)f = 0 \text{ for all } f\in\mathscr{L}^2(\mathbf{R}) \tag{25}$$

follows. Hence the reduced dynamics leads to a superselection rule for the momentum.

Remark. A uniform estimate of the type (24) has been derived for all bounded observables A using other methods in [3].

(b) *Position coupling*. For the Hamiltonian with the position coupling

$$H = \frac{1}{2}\left(P^2+\omega^2Q^2\right)\otimes I_{\mathrm{F}} + Q\otimes\Phi(h) + I_{\mathrm{S}}\otimes H_{\mathrm{F}}, \tag{26}$$

the flow on the phase space is – as in the free case – a rotation $R(t)$ given by (17). But the square of the energy operator $\widehat{M}$ is now given by

$$\widehat{M}^2 = \begin{pmatrix} \omega^2 & \langle h\mid\cdot\rangle \\ h & M^2 \end{pmatrix}. \tag{27}$$

The Hamiltonian (26) has a lower bound if and only if the operator (27) is positive. This condition is fulfilled if $h\in\mathscr{H}_{\mathbf{R}}$ satisfies the norm bound $\left\|M^{-1}h\right\|\leq\omega$. The operator (27) corresponds to the Hamiltonian of the Friedrichs model, see [15,16], and § 6.2 in [14]. If the function h satisfies some smoothness and support restrictions, the operator $\widehat{M}$ has an absolutely continuous spectrum. Then one expects that either the harmonic oscillator decays into the ground state (if the boson field is in a normal state), or the harmonic oscillator thermalizes into the canonical ensemble (if the boson field is in a KMS state with positive temperature). But in the case $\left\|M^{-1}h\right\|^2 = \left\langle h\mid M^{-2}h\right\rangle = \omega^2$, the spectrum of the operator $\widehat{M}$ includes zero. If, moreover, the low energy contributions dominate, the mean boson number diverges and strong decoherence effects emerge, which cause a non-uniform superselection rule for the momentum of the particle. The details of the calculations will be given in [4].

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