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Bianchi type I inflationary universe in general relativity

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Abstract. In this paper, we have investigated Bianchi type I inflationary universe in the presence of massless scalar field with a flat potential. To get an inflationary solution, we have considered a flat region in which potential V is constant. The inflationary scenario of the model is discussed in detail.

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1. Introduction

Inflationary universes play an important role in solving a number of outstanding problems in cosmology like the homogeneity, the isotropy and the flatness of the observed universe. The idea of an early inflationary phase was introduced by Guth [1] in the context of grand unification theories. Several versions of inflationary scenario exist as investigated by Linde [2], Abrecht and Steinhardt [3], Linde [4], Abbott and Wise [5], Mataresse and Luechin [6], Mijic *et al* [7], and La and Steinhardt [8], but in most of the cases, the implementation of the model is made in the framework provided by FLRW universes, i.e., within a model which is already homogeneous and isotropic. While in these models, the flatness problem is well understood and solved, the situation is not so clear about isotropy and homogeneity. Indeed what the FRW models can do, is to solve the horizon problem, a necessary, but not sufficient condition for the solution of the isotropy and homogeneity problems. Rothman and Ellis [9] have investigated that we can have a solution of the isotropy problem if we work with anisotropic metrics and show that they can be isotropized and inflated under very general circumstances. We assume a homogeneous space due to the obvious difficulties in dealing with non-homogeneous metrics.

Using the concept of Higgs field ϕ with potential $V(\phi)$, inflation will take place if $V(\phi)$ has a flat region and the ϕ field evolves slowly but the universe expands in an exponential way due to vacuum field energy [10]. It is assumed that the scalar field will take sufficient time to cross the flat region so that the universe expands sufficiently to become homogeneous and isotropic on the scale of the order of the horizon size.

In this paper, we have investigated Bianchi type I inflationary cosmological model in the presence of massless scalar field with a flat potential. To get a determinate solution, we have considered a flat region in which potential is constant. The inflationary scenario of the model is discussed in detail. We consider the Bianchi type I metric in the form

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + A^2\,\mathrm{d}x^2 + B^2(\mathrm{d}y^2 + \mathrm{d}z^2), \tag{1}$$

where A and B are functions of t alone. The Lagrangian will be that of gravity minimally coupled to a scalar field $V(\phi)$ [10],

$$S = \int \sqrt{-g}\left[R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right]\mathrm{d}^4x. \tag{2}$$

[Notations have their usual meanings and units are taken so that $8\pi G = C = 1$.] Now from the variation of S with respect to the dynamical fields, we obtain the Einstein field equation

$$R_{ij} - \frac{1}{2}Rg_{ij} = -T_{ij} \tag{3}$$

with

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \left[\frac{1}{2}\partial_\rho\phi\partial^\rho\phi + V(\phi)\right]g_{\mu\nu}, \tag{4}$$

$$\frac{1}{\sqrt{-g}}\partial_\mu[\sqrt{-g}\,\partial^\mu\phi] = -\frac{\mathrm{d}V(\phi)}{\mathrm{d}\phi}. \tag{5}$$

The Einstein field equation (3) for the metric (1) are given by

$$2\frac{B_{44}}{B} + \frac{B_4^2}{B^2} = -\frac{1}{2}\dot{\phi}^2 + V(\phi), \tag{6}$$

$$\frac{A_{44}}{A} + \frac{B_{44}}{B} + \frac{A_4B_4}{AB} = -\frac{1}{2}\dot{\phi}^2 + V(\phi), \tag{7}$$

$$2\frac{A_4B_4}{AB} + \frac{B_4^2}{B^2} = \frac{1}{2}\dot{\phi}^2 + V(\phi), \tag{8}$$

and the equation for the scalar field is

$$\ddot{\phi} + \left(\frac{A_4}{A} + 2\frac{B_4}{B}\right)\dot{\phi} + \frac{\mathrm{d}V}{\mathrm{d}\phi} = 0. \tag{9}$$

2. Solution of the field equations

Since we are interested in inflationary solutions, the flat region is considered where potential is constant [10], i.e.

$$V(\phi) = \text{constant } = V_0 \text{ (say)}. \tag{10}$$

Equation (9) on integration leads to

$$\phi = \frac{\lambda}{AB^2}, \tag{11}$$

where λ is the constant of integration.

From eqs (6) and (7), we have

$$\frac{A_{44}}{A} - \frac{B_{44}}{B} + \frac{A_4 B_4}{AB} - \frac{B_4^2}{B^2} = 0 \tag{12}$$

which leads to

$$\left(\frac{A_4}{A} - \frac{B_4}{B}\right)_4 + \left(\frac{A_4}{A} - \frac{B_4}{B}\right)\left(\frac{A_4}{A} + \frac{2B_4}{B}\right) = 0, \tag{13}$$

which on integration gives

$$\frac{A_4}{A} - \frac{B_4}{B} = \frac{K}{AB^2}, \tag{14}$$

where k is a constant.

Now eqs (7) $+\frac{1}{2}$ (6) $-\frac{1}{2}$ (8) together with eq. (11) leads to

$$\frac{A_{44}}{A} + \frac{2B_{44}}{B} = V_0 - \frac{\lambda^2}{A^2 B^4}, \tag{15}$$

$$\left(\frac{A_4}{A} + \frac{2B_4}{B}\right)_4 + \frac{A_4^2}{A^2} + 2\frac{B_4^2}{B^2} = V_0 - \frac{\lambda^2}{A^2 B^4}. \tag{16}$$

Using $AB^2 = \xi$ in eq. (16), we get

$$\left(\frac{\xi_4}{\xi}\right)_4 + \frac{A_4^2}{A^2} + 2\frac{B_4^2}{B^2} = V_0 - \frac{\lambda^2}{\xi^2}. \tag{17}$$

Now $AB^2 = \xi$ leads to

$$\frac{\xi_4}{\xi} = \frac{A_4}{A} + 2\frac{B_4}{B}. \tag{18}$$

From eq. (14), we have

$$\frac{A_4}{A} - \frac{B_4}{B} = \frac{K}{\xi}. \tag{19}$$

Equations (18) and (19) lead to

$$\frac{3A_4}{A} = \frac{\xi_4}{\xi} + \frac{2K}{\xi} \tag{20}$$

and

$$3\frac{B_4}{B} = \frac{\xi_4}{\xi} - \frac{K}{\xi}. \tag{21}$$

Using eqs (20) and (21) in eq. (17), we have

$$\left(\frac{\xi_4}{\xi}\right)_4 + \frac{1}{9}\left(\frac{\xi_4}{\xi} + \frac{2K}{\xi}\right)^2 + \frac{2}{9}\left(\frac{\xi_4}{\xi} - \frac{K}{\xi}\right)^2 = V_0 - \frac{\lambda^2}{\xi^2} \tag{22}$$

which leads to

$$\xi\xi_{44} - \frac{2}{3}\xi_4^2 - V_0\xi^2 + \frac{2}{3}K^2 + \lambda^2 = 0. \tag{23}$$

Let us use the transformation

$$\xi_4 = F(\xi) \tag{24}$$

so that

$$\xi_{44} = F\frac{\mathrm{d}F}{\mathrm{d}\xi}. \tag{25}$$

Equation (23) now becomes

$$\frac{\mathrm{d}F^2}{\mathrm{d}\xi} - \frac{4}{3\xi}F^2 = 2V_0\xi - \frac{\frac{4}{3}K^2 + 2\lambda^2}{\xi} \tag{26}$$

which leads to

$$F^2\xi^{-4/3} = 3V_0\xi^{2/3} + \left(K^2 + \frac{3}{2}\lambda^2\right)\xi^{-4/3} + L,$$

where L is a constant of integration.

Thus

$$\xi_4^2 = F^2 = 3V_0\xi^2 + \left(K^2 + \frac{3}{2}\lambda^2\right) + L\xi^{4/3}. \tag{27}$$

We had four equations (6)–(9) in four unknowns A, B, ϕ and V. The condition $V =$ constant $= V_0$, gives us four equations in three unknowns which is over determined set. It is easy to verify that these equations are consistent and have a solution when $L = 0$. Thus eq. (27) leads to

$$\xi_4 = \sqrt{3V_0\xi^2 + \left(K^2 + \frac{3}{2}\lambda^2\right)} \tag{28}$$

which leads to

$$\frac{\mathrm{d}\xi}{\sqrt{3V_0\xi^2 + \left(K^2 + \frac{3}{2}\lambda^2\right)}} = \mathrm{d}t \tag{29}$$

and this leads to

$$\xi = n \sinh \sqrt{3V_0}(t+\alpha), \tag{30}$$

where

$$\frac{K^2 + \frac{3}{2}\lambda^2}{3V_0} = n^2 \tag{31}$$

and α is a constant.

Thus from (20) and (21), we have

$$\frac{3A_4}{A} = \frac{\xi_4 + 2K}{\xi} = \frac{n\sqrt{3V_0}\cosh\sqrt{3V_0}(t+\alpha) + 2K}{n\sinh\sqrt{3V_0}(t+\alpha)} \tag{32}$$

and

$$\frac{3B_4}{B} = \frac{\xi_4 - K}{\xi} = \frac{n\sqrt{3V_0}\cosh\sqrt{3V_0}(t+\alpha) - 2K}{n\sinh\sqrt{3V_0}(t+\alpha)}. \tag{33}$$

Equations (32) and (33), on integration lead to

$$A = \beta^{1/3}\sinh^{1/3}\sqrt{3V_0}(t+\alpha)\left[\tanh\left\{\frac{1}{2}\sqrt{3V_0}(t+\alpha)\right\}\right]^{2K/3n\sqrt{3V_0}} \tag{34}$$

and

$$B = \gamma^{1/3}\sinh^{1/3}\sqrt{3V_0}(t+\alpha)\left[\tanh\left\{\frac{1}{2}\sqrt{3V_0}(t+\alpha)\right\}\right]^{-K/3n\sqrt{3V_0}}. \tag{35}$$

Thus

$$\phi = \frac{\lambda}{AB^2} = \frac{\lambda}{\xi} = \frac{\alpha}{n\sinh\sqrt{3V_0}(t+\alpha)}. \tag{36}$$

Using the transformations,

$$t+\alpha = T, \quad \beta^{1/3}x = X, \quad \gamma^{1/3}y = Y, \quad \gamma^{1/3}z = Z,$$

the metric (1) leads to

$$\begin{aligned} \mathrm{d}s^2 = -\mathrm{d}T^2 &+ \sinh^{2/3}\left(\sqrt{3V_0}T\right)\tanh^{2k/3n\sqrt{3V_0}}\left(\frac{1}{2}\sqrt{3V_0}T\right)\mathrm{d}X^2 \\ &+ \sinh^{2/3}\left(\sqrt{3V_0}T\right)\tanh^{-K/3n\sqrt{3V_0}}\left(\frac{1}{2}\sqrt{3V_0}T\right)(\mathrm{d}Y^2 + \mathrm{d}Z^2). \end{aligned} \tag{37}$$

3. Some physical and geometrical features

The scalar of expansion (θ) and the shear (σ) for the model (37) are given by

$$\theta = \frac{A_4}{A} + \frac{2B_4}{B} = \frac{\xi_4}{\xi} = \coth\left(\sqrt{3V_0}T\right) \tag{38}$$

$$\sigma = \frac{1}{\sqrt{3}}\left(\frac{A_4}{A} - \frac{B_4}{B}\right) = \frac{K}{\sqrt{3}n\sinh\left(\sqrt{3V_0}T\right)}. \tag{39}$$

The rate of expansion H_i (Hubble parameters) in the direction of X, Y, Z are given by

$$H_1 = \frac{n\sqrt{3V_0}\cosh\left(\sqrt{3V_0}T\right) + 2K}{3n\sinh\left(\sqrt{3V_0}T\right)} \tag{40}$$

and

$$H_2 = H_3 = \frac{n\sqrt{3V_0}\cosh\left(\sqrt{3V_0}T\right) - K}{3n\sinh\left(\sqrt{3V_0}T\right)}. \tag{41}$$

The Higg's field (ϕ) is given by eq. (36),

$$\phi = \frac{\lambda}{n\sinh\left(\sqrt{3V_0}R\right)}. \tag{42}$$

The spatial volume V is given by

$$V = \sinh\left(\sqrt{3V_0}T\right). \tag{43}$$

The spatial volume increases with time. When $T \to \infty$ then spatial volume $V \to \infty$. Thus inflation is possible in Bianchi type I model with a massless scalar field in the potential which has flat space region. The expansion $(\theta) \to \infty$, when $T \to 0$ and $\theta \to$ finite quantity when $T \to \infty$. The quantity K/n measures anisotropy in the model. When $T \to \infty$, then $\sigma \to 0$. From (31), we find that $\lambda/n = \sqrt{2V_0 - (2K^2/3n^2)}$, where λ is a constant of integration. Clearly $V_0 > K^2/3n^2$. Thus the quantity K/n has physical relevance. The quantity λ/n appears in Higg's field ϕ. The model (37) starts with a big-bang at $T = 0$ and tends to 1 when $T \to \infty$.

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