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A solution of the DGLAP equation for gluon at low *x*

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Abstract. We obtain a solution of the DGLAP equation for the gluon at low x first by expanding the gluon in a Taylor series and then using the method of characteristics. We test its validity by comparing it with that of Glück, Reya and Vogt. The convergence criteria of the approximation used are also discussed. We also calculate $\partial F_2(x,Q^2)/\partial \ln Q^2$ using its approximate relations with the gluon distribution at low x. The predictions are then compared with the HERA data.

Keywords. DGLAP equation for gluon; method of characteristics; scaling violation of F_2.

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1. Introduction

It is known for a long time [1] that with reasonable boundary conditions, perturbative QCD predicts a universal growth of the gluon structure function at large t $(t=\ln(Q^2/\Lambda^2)$ and small x $(x=Q^2/2p\cdot q)$ faster than any power of $\ln(1/x)$ but slower than any inverse power of x. More recently this perturbative prediction was brought to the phenomenological front [2] through double asymptotic scaling. One of the recent important observations [3] from this prediction is the rise of the exponent of the structure function with increasing t $(t=\ln(Q^2/\Lambda^2)$, consistent with HERA data at low x. In recent years, an approximate method of solving DGLAP equations [4] at low x has been pursued [5–7] with considerable phenomenological success. In that approach the DGLAP equation is expressed as a partial differential equation in x and t using the Taylor series expansion valid to be at low x. One of the limitations of that approach is that the solutions reported are not unique [8]. They are selected as the simplest ones [6,7] with a single boundary condition on the nonperturbative x-distribution of the structure function at some $Q^2=Q_0^2$. However, complete solution of DGLAP equation with two differential variables generally needs two boundary conditions [9], one at $x\to 0, t\longrightarrow\infty$ limit of double asymptotic scaling [2] and the other at any fixed $t=t_0$. An alternative way of obtaining a unique solution with one single initial condition is through the application of the method of characteristics [10]. In this method the coordinates (x,t) are transformed to an appropriate new set of coordinates (s,τ). As a result the partial differential equation of the gluon in (x,t) becomes an ordinary differential equation in a single variable s along the characteristic curves, which can be solved by ordinary method with a single initial condition. The aim of the present paper is precisely this: we will obtain a solution of the DGLAP equation for gluon distribution at low x using the method of characteristics [10]. The result is different from DLLA one [1] which predicts a definite t dependence of the exponent of x in the structure function. In §2, we summarize the theory while in §3, we discuss our results.

2. Formalism

2.1 *Gluon momentum density*

The DGLAP equation for gluon structure function $G(x,Q^2)(= xg(x,Q^2))$ taking only the leading term of the gluonic kernel [4,11] and neglecting the singlet structure function $F_2(x,Q^2)$ is

$$\frac{\partial G(x,t)}{\partial t} = 3\frac{\alpha_s(t)}{\pi}\left[\begin{array}{c}\left\{\left(\frac{11}{12}-\frac{n_f}{18}\right)+\ln(1-x)\right\}G(x,t)\\ +\int_x^1 \mathrm{d}z\left\{\left(\frac{zG(x/z,t)-G(x,t)}{(1-z)}\right)+\left(z(1-z)+\frac{1-z}{z}\right)G(x/z,t)\right\}\end{array}\right]. \tag{1}$$

Here

$$\alpha_s(t) = \frac{4\pi}{\beta_0 t} = \frac{\gamma^2\pi}{3t}, \tag{2}$$

where $t = \ln(Q^2/\Lambda^2)$, $\gamma = \sqrt{12/\beta_0}$ and $\beta_0 = 11 - (\frac{2}{3})n_f, n_f$ being the number of flavours. Equation (1) can be written as

$$\frac{\partial G(x,t)}{\partial t} - \left(\frac{\gamma^2}{t}\right)\left[\left\{\left(\frac{11}{12}-\frac{n_f}{18}\right)+\ln(1-x)\right\}G(x,t)+I_g(x,t)\right] = 0, \tag{3}$$

where

$$I_g(x,t) = \int_x^1 \mathrm{d}z\left\{\left(\frac{zG(x/z,t)-G(x,t)}{(1-z)}\right)+\left\{z(1-z)+\frac{1-z}{z}\right\}G(x/z,t)\right\}. \tag{4}$$

Let us introduce the variable

$$u = 1-z. \tag{5}$$

Since $x < z < 1$, so $0 < u < 1-x$ and hence we can approximate x/z as

$$\frac{x}{z} = \frac{x}{1-u} \approx x(1+u). \tag{6}$$

Using eq. (6) we expand $G(x/z,t)$ in a Taylor series as

$$G\left(\frac{x}{z},t\right) = G(x,t) + xu\frac{\partial G(x,t)}{\partial x} + \frac{1}{2}x^2u^2\frac{\partial^2 G(x,t)}{\partial x^2} + \cdots \tag{7}$$

which covers the whole range of u, i.e., $0 < u < 1-x$. In order to check the convergence of the series (7) let us use a general form of the gluon density

$$G(x,t) \approx x^{\alpha(x,t)}, \tag{8}$$

where the slope function $\alpha(x,t)$ is in general x and t dependent. Convergence of the series (7) requires that

$$\left| xu\frac{1}{2}\frac{\partial^2 G/\partial x^2}{\partial G/\partial x}\right| < 1. \tag{9}$$

This leads to

$$u < u_0, \tag{10}$$

where

$$u_0 = \frac{2\left|\ln x\frac{\partial\alpha(x,t)}{\partial x} + \frac{\alpha(x,t)}{x}\right|}{x\left|\left(\ln x\frac{\partial\alpha(x,t)}{\partial x} + \frac{\alpha(x,t)}{x}\right)^2 + \left(\ln x\frac{\partial^2\alpha(x,t)}{\partial x^2} + \frac{2}{x}\frac{\partial\alpha(x,t)}{\partial x} - \frac{\alpha(x,t)}{x^2}\right)\right|} \tag{11}$$

which gives the upper limit of u up to which the series (7) is convergent. However, as $0 < u < 1-x$, eq. (9) yields an upper limit of x as well

$$x_{\max} < 1 - u_0. \tag{12}$$

For x independent α, eq. (11) simplifies to

$$u_0 = \frac{2}{|\alpha - 1|}. \tag{13}$$

The above analysis demonstrates that series (7) is convergent so long x does not exceed the limit (12).

In case α is x dependent, (11) indicates that both u_0 and $x_{\max}$ also develop x dependence. In that case only the region of x satisfying

$$x \leq x_{\max} \tag{14}$$

for the particular value of Q^2 will have validity of the present formalism. On the other hand for simpler case when α is x independent then (13) implies that convergence is assumed only for $3 < \alpha < -1$, suggesting its limited utility.

Assuming the validity of convergence and neglecting the higher order terms $O(u^2)$ eq. (7) has the form [7]

$$G\left(\frac{x}{z},t\right) \approx G(x,t) + xu\frac{\partial G(x,t)}{\partial x}. \tag{15}$$

Using (6) and (15) in (4) and performing the u integration we obtain

$$I_g(x,t) = R_g(x)G(x,t) + P_g(x)\frac{\partial G(x,t)}{\partial x}, \tag{16}$$

where we have used the identity

$$\sum_{k=1}^{\infty}\frac{u^k}{k} = \ln\frac{1}{1-u}. \tag{17}$$

$R_g(x)$ and $P_g(x)$ have the explicit forms

$$R_g(x) = -2(1-x) + \frac{1}{2}(1-x^2) - \frac{1}{3}(1-x^3) + \ln\left(\frac{1}{x}\right) \tag{18}$$

and

$$P_g(x) = \left(-\frac{11}{12} + 2x - \frac{3}{2}x^2 - \frac{2}{3}x^3 - \frac{x^4}{4} + \ln\frac{1}{x}\right)x. \tag{19}$$

Using (16) in (3) we will have the DGLAP equation for the gluon at low x to be

$$t\frac{\partial G(x,t)}{\partial t} - \gamma^2 P_g(x)\frac{\partial G(x,t)}{\partial x} - \gamma^2\left\{\left(\frac{11}{12} - \frac{n_\mathrm{f}}{18}\right) + \ln(1-x) + R_g(x)\right\}G(x,t) = 0. \tag{20}$$

We note that eq. (20) has the derivative $\partial G(x,t)/\partial x$ which enables one to calculate the x evolution at low x [7] beyond its traditional use in t evolution only. In order to solve (20) by the method of characteristics [10], we recast it in the form

$$a(x,t)\frac{\partial G(x,t)}{\partial x} + b(x,t)\frac{\partial G(x,t)}{\partial t} + c(x,t)G(x,t) = 0, \tag{21}$$

where

$$a(x,t) = \gamma^2 P_g(x), \tag{22}$$

$$b(x,t) = -t \tag{23}$$

and

$$c(x,t) = \gamma^2\left\{\left(\frac{11}{12} - \frac{n_\mathrm{f}}{18}\right) + \ln(1-x) + R_g(x)\right\}. \tag{24}$$

As an initial value to the problem, we set

$$G(x,t)\,|_{t=t_0} = G(x) \tag{25}$$

which would correspond to some specific nonperturbative inputs [12–14] available in the current literature. Let us now introduce two new variables s and τ such that

$$\frac{\mathrm{d}x}{\mathrm{d}s} = a(x,t) \tag{26}$$

and

$$\frac{\mathrm{d}t}{\mathrm{d}s} = b(x,t). \tag{27}$$

On using (26) and (27), eq. (21) becomes

$$\frac{\mathrm{d}G(s,\tau)}{\mathrm{d}s} + c(\tau,s)G(s,\tau) = 0. \tag{28}$$

Solving the parametric equations (26) and (27) and setting along the characteristics at $s = 0$,

$$x(s=0) = \tau \quad \text{and} \quad t(s=0) = t_0, \tag{29}$$

we get the transformation equations as

$$s = -\ln\left(\frac{t}{t_0}\right) \tag{30}$$

and

$$\ln\tau + \frac{11}{12} = \left(\ln\frac{1}{x} + \frac{11}{12}\right)\left(\frac{t_0}{t}\right)^{\gamma^2}, \tag{31}$$

where we have approximated $P_g(x)$ and $R_g(x)$ in the limit $x \to 0$ as

$$P_g(x) \approx x\left(\ln\frac{1}{x} - \frac{11}{12}\right) \tag{32}$$

and

$$R_g(x) \approx \left(\ln\frac{1}{x} - \frac{11}{6}\right). \tag{33}$$

Using (30) and (31) we can express $c(x,t)$ defined in eq. (24) in terms of τ and s as

$$c(\tau,s) = -\gamma^2\left\{\left(\ln\tau + \frac{11}{12}\right)\exp(-\gamma^2 s) + \frac{n_\mathrm{f}}{18}\right\}. \tag{34}$$

Puting (34) in (28) and solving the ordinary differential equation yields

$$G(\tau,s) = G(\tau)\exp\left[-\left(\ln\tau + \frac{11}{12}\right)\exp(-\gamma^2 s) + \left(\frac{\gamma^2 n_\mathrm{f}}{18}\right)s + \left(\ln\tau + \frac{11}{12}\right)\right]. \tag{35}$$

Now transforming back to the original variables (x,t) with the help of the transformation equations (30) and (31) we get

$$G(x,t) = G(\tau) x^{-\{1-(t_0/t)^{\gamma^2}\}} \left(\frac{t_0}{t}\right)^{\gamma^2 n_f/18} \exp\left[-\frac{11}{12}\left\{1-\left(\frac{t_0}{t}\right)^{\gamma^2}\right\}\right], \qquad (36)$$

where τ appearing in $G(\tau)$ is to be expressed in terms of x and t using eq. (31) as

$$\tau = \exp\left[\left(-\ln\frac{1}{x} + \frac{11}{12}\right)\left(\frac{t_0}{t}\right)^{\gamma^2} - \frac{11}{12}\right]. \qquad (37)$$

Here it is to be noted that $G(\tau)$ is the initial condition of the problem that modifies due to the method used and so can be obtained as mentioned after eq. (25). Equation (36) is our main result.

This is to be compared with the known DLLA form

$$G(x,t) = G(x,t_0) \exp\left[\sqrt{4\gamma^2 \ln\left(\frac{t}{t_0}\right) \ln\left(\frac{1}{x}\right)}\right] \qquad (38)$$

provided the gluon is not singular at $t = t_0$.

Our present result is different from the solution of (20) using the method reported earlier [6,7] which is

$$G(x,t) = G(x,t_0)\left(\frac{t}{t_0}\right) \qquad (39)$$

or that of ref. [15] with the factorization ansatz, which is

$$G(x,t) = G(x,t_0)\left(\frac{t}{t_0}\right)^{12/\beta_0 \ln(1/x)}. \qquad (40)$$

In our analysis, we study the quantitative differences of these alternative forms of gluons.

2.2 *Slope of the structure function at low x from gluon density*

There are different formulae [16–18] that relate the gluon density to the scaling violation of $F_2(x,Q^2)$ at low x.

Prytz [16] formula reads

$$\frac{\partial F_2(x,Q^2)}{\partial \ln Q^2} \approx \frac{10\alpha_s}{27\pi} G(2x), \qquad (41)$$

while that of Bora and Choudhury [17] suggests

$$\frac{\partial F_2(x,Q^2)}{\partial \ln Q^2} \approx \frac{5\alpha_s}{3\pi} G\left(\frac{4}{3}x\right). \qquad (42)$$

Recently Gay Ducati and Goncalves [18] have obtained a general relation which incorporates both the above by expanding the gluon $G(x/1-z)$ at an arbitrary point of expansion $z = \alpha$. In the small x limit their formula reads

$$\frac{\partial F_2(x,Q^2)}{\partial \ln Q^2} \approx \frac{10\alpha_s}{27\pi} G\left[\frac{x}{1-\alpha}\left(\frac{3}{2}-\alpha\right)\right], \tag{43}$$

where $\alpha < 1$.

Using (43) we will estimate the logarithmic slope of the structure function from the proposed gluon distribution at several points of expansion and compare with data [19] at $Q^2 = 20$ GeV^2 where the data on the slope are available.

3. Results and discussion

In the present paper, we have obtained a new description of gluon distribution given by eq. (36). Already existing forms referred in the literature are eqs (38), (39) and (40). While eq. (38) is the standard form in QCD [1], eq. (39) is based on the solution of Lagrange's auxillary system of equations [8] in x and t. Similarly (40) is derived on the *ad hoc* assumption of factorizability of gluon distribution in x and t. In the new approximation (36), the inherent difficulty of nonunique nature of solutions of Lagrange's auxillary system of equations [6,7] while deriving (39) as well as the *ad hoc* assumption of factorizability in (40) are avoided. In this way, the new approximation is a definite improvement over the two forms (39) and (40).

In figure 1 we compare (36) at $Q^2 = 20$ GeV^2 with the already existing descriptions, eqs (38), (39) and (40) taking the MRSA [14] input gluon at $Q_0^2 = 4$ GeV^2. While comparing (36) with other forms, we need $G(\tau)$ as input rather than $G(x,t_0)$ as occurred in eqs (38), (39) or (40). We note that $G(\tau)$ contains both x and t variables through eq. (37). At $t = t_0$,

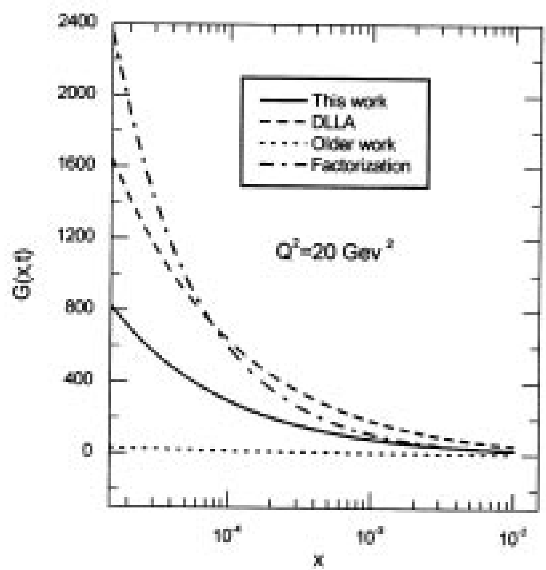

Figure 1. Gluon distributions eqs (36), (38), (39) and (40) at $Q^2 = 20$ GeV^2 taking MRSA input [14] at $Q_0^2 = 4.0$ GeV^2. In the labels of the figures, eqs (36), (38), (39) and (40) are respectively referred as this work, DLLA, older work and factorization.

τ reduces to x and $G(\tau)$ reduces to $G(x)$. However there is no additional arbitrariness in the choice of $G(\tau)$ as compared with $G(x,t_0) = G(x)$ of eqs (38), (39) and (40) since it is obtained by a formal replacement $x \to \tau$ in the same MRSA input [14]. We also note that taking some explicit parametrization for $G(\tau)$ from some other sources e.g. GRV(LO) [12] is undoubtedly unjustified. In such a case there would have been no predictive value at all for this approximation.

To test the predicted gluon density given by eq. (36) further, we use the method of ref. [18] and calculate the slopes $\mathrm{d}F_2(x,Q^2)/\mathrm{d}\ln Q^2$ using (43) for several points of expansion $\alpha = 0.0, 0.5, 0.7$ and 0.8. To that end, we take $n_\mathrm{f} = 4$ and start with MRSA [14] input with $Q_0^2 = 4$ GeV^2 and compare our results (figure 2) with the data from H1 and ZEUS [19] at $Q^2 = 20$ GeV^2 where the results of the slopes are available. We note that the data favor eq. (36) when the point of expansion of the gluon density is chosen around $\alpha \approx 0.7$.

In figure 3, we have plotted the slopes using different gluon densities given by eqs (36), (38), (39) and (40) with the expansion point at $\alpha = 0.7$ and compared with the same set of data at $Q^2 = 20$ GeV^2.

The question naturally arises whether other values of α yield better agreement for eqs (38), (39) and (40) compared with eq. (36). To explore this, we vary the point of expansion α from very low value of 0.2 to 0.99 and calculate the slopes of the structure function using all the eqs (36), (38), (39) and (40). We observe that the x dependence of the slopes flatten out with increasing values of α. We display in figures 4a–d four such graphs for values of $\alpha = 0.5, 0.7, 0.9$, and 0.98 respectively along with the H1 and ZEUS [19] data for the slopes at $Q^2 = 20$ GeV^2. We note that the value of α for which the calculated slopes agree with the experimental data is different for different equations. Equation (38), i.e., the DLLA result gives better agreement for $\alpha = 0.98$, whereas eq. (40), i.e., the factorization

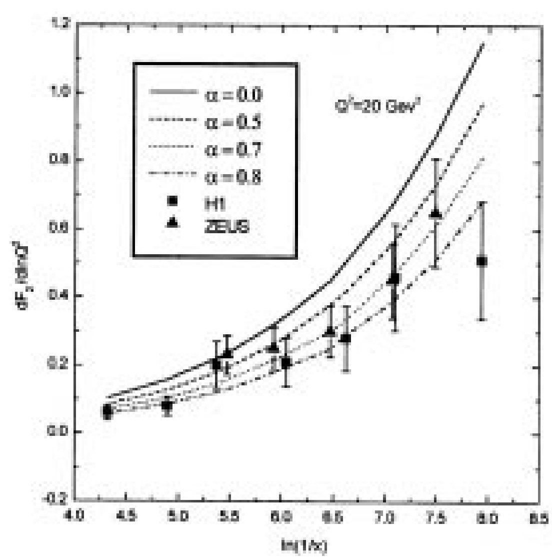

Figure 2. Slope of the structure function $\mathrm{d}F_2(x,Q^2)/\mathrm{d}\ln Q^2$ using eq. (36) and the relation eq. (43) obtained at several points of expansion α. Data from H1 and ZEUS [19].

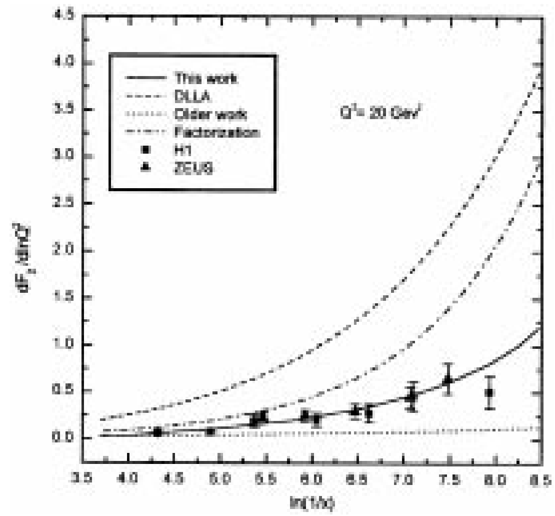

Figure 3. Slope of the structure function $dF_2(x,Q^2)/d\ln Q^2$ using eqs (36), (38), (39) and (40) at point of expansion $\alpha = 0.7$. Data from ref. [19].

result is good when the value of α is chosen around 0.9, while the older result (eq. (39)) fails. This work, i.e., eq. (36) is good when the value of α is around 0.7.

Let us therefore consider the relative merits of eq. (36) over the others, in the light of analysis of Gay Ducati and Goncalves [18], where points $0.5 \leq \alpha < 0.8$ are favored. Since the longitudinal momentum of the gluon x_g is given by the quantity within the parenthesis of G occurred in eq. (43) i.e. $x_g = \frac{x}{1-\alpha}\left(\frac{3}{2} - \alpha\right)$, it implies that x_g is more than twice the value of the longitudinal momentum of the probed quark (or antiquark) in DIS. In conformity with the results obtained from Glauber-Müeller approach [20] that includes shadowing corrections, it is concluded in ref. [18] that the more suitable points of expansion are in the range $0.5 \leq \alpha < 0.8$ (i.e. $2x \leq x_g < 3.5x$). This conclusion also agrees with that of Ryskin *et al* [21] where the estimated value of the longitudinal gluon momentum x_g is shown to be approximately three times larger than the Bjorken x_B, that corresponds to the expansion at $\alpha = 0.75$.

In the light of these analyses of refs [18,20,21], eq. (36) is definitely preferred over eqs (39) and (40). The approximate form (36) is admissible only if it helps in explaining general trend of data or else explains the data precisely in a limited kinematical range. One also needs to know how the approximate formula (36) compares with the exact ones. In figure 5 we present the gluon density obtained from eq. (36) with MRSA input at $Q_0^2 = 4$ GeV^2 and compare with the exact results whose analytic forms are given in [12]. We compare the two at representative values of $Q^2 = 3.5, 5.0, 10, 20, 40$, and 80 GeV^2 and for $10^{-5} < x < 10^{-1}$. At low Q^2, the difference between the two is much prominent specially for $x < 10^{-4}$. For $Q^2 < 20$ GeV^2, our predicted gluon density (36) lies below GRVLO. As Q^2 increases the difference between the two gradually shrinks and for $Q^2 \geqslant 20$ GeV^2, our predicted gluon rises faster than the GRV gluon. Above $Q^2 = 20$ GeV^2, the deviation from the exact result is much prominent in the small x regime $x < 10^{-4}$ and the difference

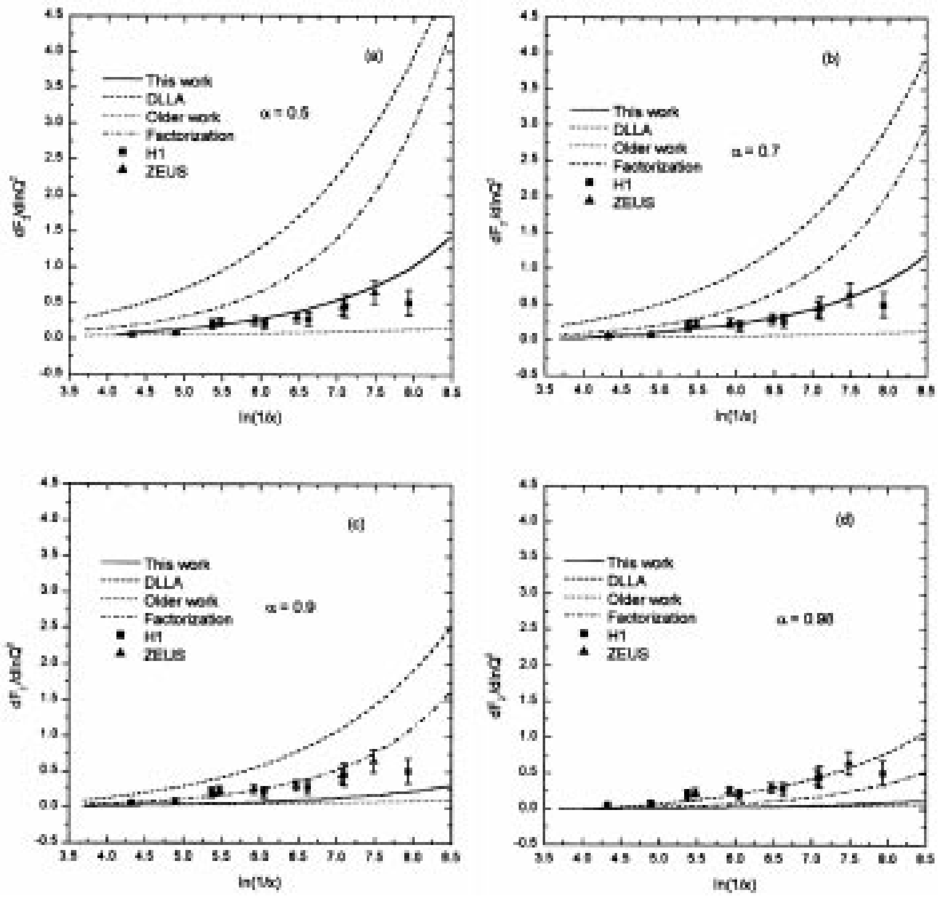

Figure 4. (**a–d**) Slope of the structure function using eqs (36), (38), (39) and (40) for value of $\alpha = 0.5, 0.7, 0.9$, and 0.98 respectively and comparison with data from H1 and ZEUS [19].

increases with Q^2. From the study of the cross over points we infer that for $3.5 < Q^2 < 80$ GeV^2 and $x > 10^{-3}$ our prediction lies within about 10 per cent of the exact result.

To conclude, we have proposed a form of the gluon distribution (36) at low and moderate x which is theoretically and phenomenologically favored over the earlier approximate forms eqs (39) and (40). We have also seen that the scaling violation relationship eq. (43) which was earlier used to extract the gluon momentum density can also be used to describe the logarithmic slope of the structure function F_2 data using our proposed gluon momentum density if we properly choose the point of expansion of $G(x/1-z)$ in conformity with the approach of refs [20,21]. The approximate formula has a limited range of validity and breaks down particularly at small x which is to be expected in the DGLAP framework. However, it will be interesting to study the distribution beyond the leading order using the method used in this paper.

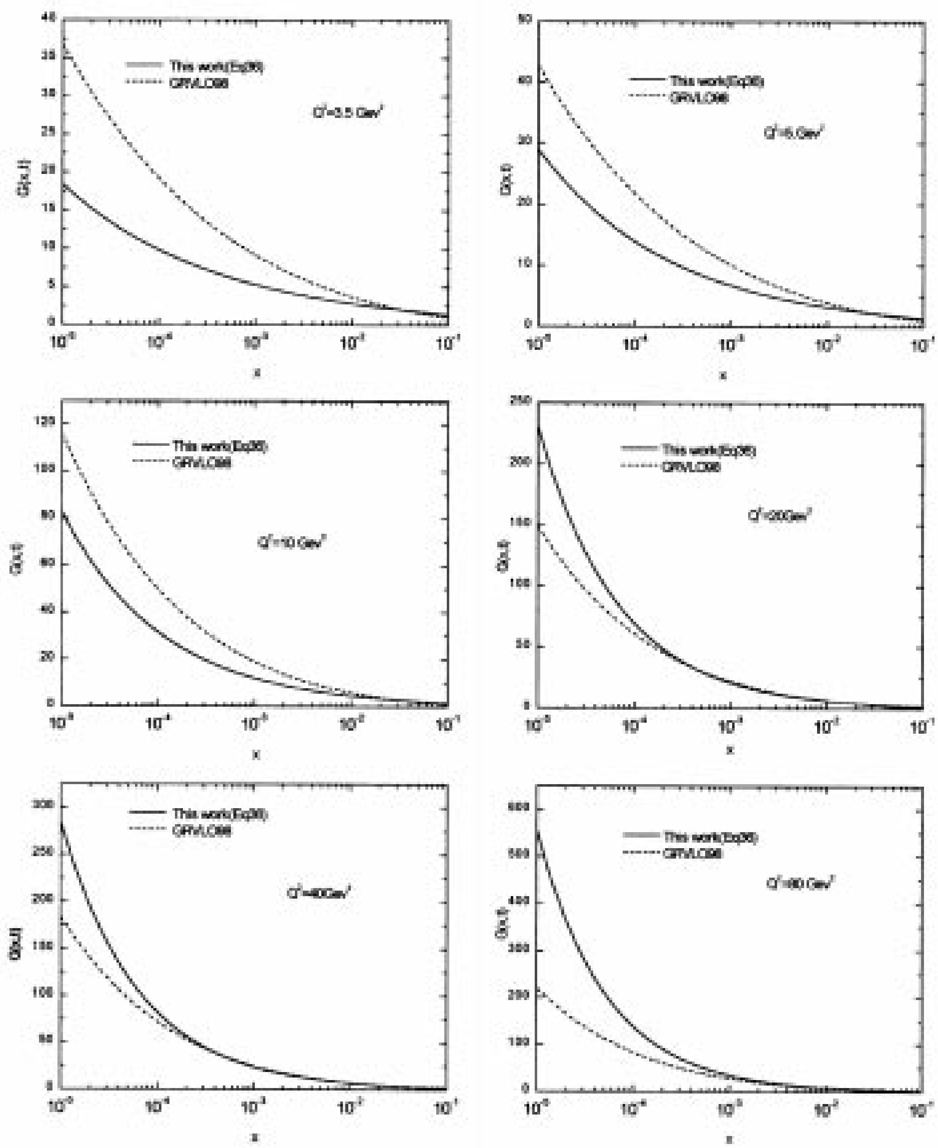

Figure 5. Gluon distribution obtained from eq. (36) and comparison with GRVLO (98) at $Q^2 = 3.5, 5.0, 10, 20, 40$, and 80 GeV^2. Input for eq. (36) is the MRSA [14] gluon.

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