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One-soliton solutions from Laplace's seed

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Abstract. One-soliton solutions of axially symmetric vacuum Einstein field equations are presented in this paper. Two sets of Laplace's solutions are used as seed and it is shown that the derived solutions reduce to some already known solutions when the constants are properly adjusted. An analysis of the solutions in terms of the Ernst potential is also presented. It is found that the solutions do not reduce to the Euclidean form at spatial infinity. However, in the static limit, Weyl solutions are obtained for half integral ∂-values.

Keywords. General relativity; one-soliton solutions; diagonal seed.

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1. Introduction

The field equations in general relativity are highly nonlinear in nature and only simple solutions are obtained by direct method. Different transformation techniques are thus developed by assuming certain symmetry properties of space-time, which give new solutions from the old ones. The inverse scattering method (ISM) developed by Belinskii and Zakharov [1–3] is a useful technique of obtaining the solutions of nonlinear field equations in general relativity. For some nonlinear differential equations it is possible to construct a linear eigenvalue problem; the unknown functions of the original nonlinear equations are included as the potential terms in the linear equations and integrability conditions of the linear equations lead to the nonlinear equations [4].

Two important characteristics of the soliton technique are the 'pole trajectories' and the background metric, called the 'seed' [5]. The solutions are identified by the number of poles they contain. For example, $N = 1$ is a simple pole (i.e. one-soliton solutions), $N = 2$ is a double pole (i.e. two-soliton solutions) and so on. The background metric is chosen suitably and one can get the N-soliton solutions from the given seed.

In §2, the inverse scattering method of Belinskii and Zakharov [1–3] is described in brief. In §3, two sets of one-soliton solutions are presented which are obtained from two different Laplace's seed. It is known that in ISM, the determination of Ω (the eigenfunction) is not easy for a general seed metric. Verdaguer [6] obtained one-soliton solutions from a

Euclidean seed and solved for eigenfunction Ω. In this paper, we have taken two different Laplace's solutions as seed in the general static axially symmetric metric and obtained two different sets of one-soliton solutions in the Verdaguer form. It is noted that the inclusion of Laplace's solution ψ in the general seed metric is reflected just as an additive term in the Verdaguer result, obtained from Euclidean seed. Verdaguer's result is thus generalized to include a general static metric as seed. When certain restrictions are imposed on the constants, the derived solutions reduce to the one-soliton solutions of Letelier [7] and under a complex coordinate transformations these solutions correspond to the Van Stockum metric [8]. An analysis of the solutions in terms of the Ernst potential is given in §4. Our conclusion follows in §5.

2. Inverse scattering method

The general stationary axially symmetric metric can be represented by

$$\mathrm{d}s^2 = g_{ab}\mathrm{d}x^a\mathrm{d}x^b + f(\mathrm{d}r^2 + \mathrm{d}z^2), \tag{1}$$

where the indices a and b take the values 1, 2 and $t, \phi = x^1, x^2$. The g_{ab} and f are functions of r and z only. The Einstein equations for the metric (1) is obtained, in matrix form, as [2]

$$U_r + V_z = 0, \tag{2}$$

$$[\ln(rf)]_r = (4r)^{-1}\mathrm{Tr}(U^2 - V^2), \tag{3}$$

$$[\ln(rf)]_z = (2r)^{-1}\mathrm{Tr}(UV), \tag{4}$$

where

$$U = rg_r g^{-1}, \quad V = rg_z g^{-1}. \tag{5}$$

The subscripts r and z denote partial differentiations.

The idea behind ISM is that if we give the data of the analytic structure we can determine the functional form of the potential which in turn gives rise to nonlinear terms of the field equations. The steps followed in the method of solutions are:

(i) construct a linear eigenvalue equation where nonlinear terms are included in the potential function,
(ii) solve the direct scattering problem with initial conditions,
(iii) from a known solution of the eigenvalue, reconstruct the potential function.

The integrability condition of eqs (3), (4) is eq. (2). Therefore, once g is known, the other metric coefficient f can be calculated from (3), (4). So the main problem is to find a solution of (2) related to an eigenvalue – eigenfunction problem for some linear differential operators. Such a system will depend on a complex parameter, say λ. The solutions of the matrices g, U and V will then be determined by the possible types of analytic structure of the eigenvalues in the λ-plane. However, there does not exist any general algorithm for obtaining solutions of such a system. For stationary axially symmetric problems, Belinskii and Zakharov introduced two differential operators D_1 and D_2 in the form [1,2]:

$$D_1 = \partial_r + \frac{2\lambda r}{\lambda^2 + r^2}\partial_\lambda, \tag{6a}$$

$$D_2 = \partial_z - \frac{2\lambda^2}{\lambda^2 + r^2}\partial_\lambda, \tag{6b}$$

where ∂ with a subscript denotes partial differentiation. The operators D_1 and D_2 satisfy the commutation relation

$$[D_1, D_2] = 0.$$

The next step of ISM is to look for a potential function Ω, also called eigenfunction, which depends on r,z and the complex parameter λ. BZ showed that the eigenfunction Ω satisfy the following eigenvalue equations [1,2]:

$$D_1\Omega = \frac{rU + \lambda V}{\lambda^2 + r^2}\Omega, \tag{7a}$$

$$D_2\Omega = \frac{rV - \lambda U}{\lambda^2 + r^2}\Omega. \tag{7b}$$

The eigenfunction Ω is a two-dimensional matrix which reduces to $g(r,z)$ when the spectral parameter λ is set equal to zero i.e.,

$$\Omega(r,z,\lambda)|_{\lambda=0} = g(r,z). \tag{8}$$

For a known seed g_0, U and V can be calculated from eq. (5) and the eigenfunction Ω is evaluated from eqs (7), (8). Once Ω is known, physically realistic g_{ab}^{ph} can be calculated and thus new metric coefficients are determined. For odd N-soliton solutions, in order to preserve the physically viable signature of space-time, one has to choose a non-physical seed, because the odd soliton solutions produce a change of signature of space-time.

The results leading to the new solutions are summarized as follows:

For physically acceptable solutions we have the supplementary condition

$$\det g_0 = -r^2, \tag{9}$$

where g_0 is a 2×2 matrix associated to g_{ab}.

The constructed stationary metric coefficients g'_{ab} are obtained from the following relations [2,6]:

$$g'_{ab} = \left(I - \sum_{k=1}^{N} \frac{R_k}{\mu_k}\right) g_0, \tag{10}$$

where I is a unit matrix and

$$(R_k)_{ab} = n_a^{(k)} m_b^{(k)}, \tag{11}$$

$$m_a^{(k)} = m_c^{0(k)} \lfloor \Omega_0^{-1}(\mu_k, r, z) \rfloor_{ca}, \tag{12}$$

$m_c^{0(k)} =$ arbitrary constants,

$$\Gamma_{kl} \cdot n_a^{(l)} = \frac{1}{\mu_k} m_c^{(k)} (g_0)_{ca}, \tag{13}$$

$$\Gamma_{kl} = \frac{m_c^{(k)} (g_0)_{ca} m_a^{(l)}}{\mu_k \mu_l + r^2}. \tag{14}$$

The poles μ_k are defined by

$$\mu_k = \omega_k - z \pm [(\omega_k - z)^2 + r^2]^{1/2}, \tag{15}$$

where ω_k are arbitrary constants, may be called the origin of solitons and either of the plus or minus sign before the square bracket in eq. (15) is allowed. Here, N is the number of solitons i.e. the number of poles that appears in the scattering matrix.

The derived metric g'_{ab} given in (10) is a solution of eq. (12) but in general, it does not satisfy eq. (9). Physically acceptable g_{ab}^{ph} are defined by [2]

$$g_{ab}^{\mathrm{ph}} = -r(-\det g'_{ab})^{-1/2} g'_{ab}, \tag{16}$$

where

$$\det g'_{ab} = (-1)^N r^{2N} \left(\prod_{k=1}^{N} \frac{1}{\mu_k^2} \right) \det g_0. \tag{17}$$

The other metric coefficient f is calculated from the relation [6]:

$$f = C f_0 r^{-N^2/2} \left(\prod_{k=1}^{N} \mu_k \right)^{N+1} \left(\prod_{\substack{k,l=1 \\ k>l}}^{N} (\mu_k - \mu_l)^2 \right)^{-1} \det \Gamma_{kl}, \tag{18}$$

where C is an arbitrary constant and f_0 is the f-function of the static metric. For one-soliton solutions (i.e. $N = 1$) the term $\left(\Pi_{\substack{k,l=1 \\ k>l}}^{N} (\mu_k - \mu_l)^2 \right)^{-1}$ in eq. (18) becomes unity.

The problem of using inverse scattering method concentrates on the determination of Ω, the eigenfunction, which is not easy for a general static axially symmetric metric used as seed. However, as one requires the value of Ω along the pole trajectories $\lambda = \mu_k$, the problem of finding Ω is somewhat simplified [7]. We have taken advantage of that.

3. One-soliton solutions (diagonal seed)

Verdaguer [6] obtained one-soliton solutions starting from Euclidean seed metric. It is then natural to examine the case when one chooses a static axisymmetric metric instead of Euclidean metric as seed. In this section, we have taken two different static axisymmetric metrics as seed and shown that it just introduces an extra additive term in the final expression for the derived metrics as given by Verdaguer. It is also shown that our solutions correspond to some already known solutions. These will be mentioned in proper places.

We choose the diagonal seed solution of the metric (1) as

$$g_0 = \mathrm{diag}\ \left(r^{1-b}\mathrm{e}^{\psi},\ r^{1+b}\mathrm{e}^{-\psi}\right), \tag{19}$$

where b is a constant and $\psi = \psi(r,z)$ is a solution of Laplace's equation

$$\psi_{rr} + \psi_{zz} + \frac{\psi_r}{r} = 0. \tag{20}$$

Subscripts r and z denote partial differentiations. With $\psi = 0$ and $b = 1$, the seed solution becomes Euclidean.

Since the seed solution is diagonal, one might expect the corresponding eigenfunction to be diagonal as well. The eigenfunction is then given in the following form

$$\Omega = \mathrm{diag}\left[(r^2 - 2\lambda z - \lambda^2)^{(1-b)/2}\mathrm{e}^{F}, \quad (r^2 - 2\lambda z - \lambda^2)^{(1+b)/2}\mathrm{e}^{-F}\right], \tag{21}$$

where F is a function of r, z, λ and

$$F(r,z,\lambda)|_{\lambda=0} = \psi(r,z). \tag{22}$$

Equation (22) together with (21) satisfies (8).

With the help of eqs (10)–(21), the one-soliton solution of metric (1) is obtained as

$$\begin{aligned}
\mathrm{d}s^2 &= (-2)^{b-1}C_0\frac{r^{b^2/2}}{(r^2+z^2)^{1/2}}\cosh\left[\chi - \frac{1}{2}b\eta + D\right](\mathrm{d}r^2 + \mathrm{d}z^2) \\
&\quad + \frac{1}{\cosh\left[\chi - \frac{1}{2}b\eta + D\right]}\left[-r^{1-b}\mathrm{e}^{\psi}\sinh\left\{\chi - \frac{1}{2}(b-1)\eta + D\right\}\mathrm{d}t^2\right. \\
&\quad \left. + r^{1+b}\mathrm{e}^{-\psi}\sinh\left\{\chi - \frac{1}{2}(b+1)\eta + D\right\}\mathrm{d}\phi^2 + (-1)^b 2r\cosh\left(\frac{1}{2}\eta\right)\mathrm{d}t\,\mathrm{d}\phi\right],
\end{aligned} \tag{23}$$

where

$$\chi = 2F - \psi, \tag{24}$$

is a function of (r, z, λ); C_0 and D are new arbitrary constants, while the function η is defined by

$$\mathrm{e}^{\eta} = (\mu/r)^2. \tag{25}$$

It is evident that the solution presented in (23) corresponds to the one-soliton solution of Letelier [7] for $b=1$. On substituting $\psi=0$ and introducing two parameters q_1 and q_2 in the manner

$$q_1 = \frac{1}{2}(1-b), \quad q_2 = \frac{1}{2}(1+b), \tag{26}$$

our solution (23) reduces to that of Verdaguer [6]. It is interesting to note that the inclusion of the term ψ, a solution of Laplace's equation, in the seed metric g_0, has been reflected in the Verdaguer's result as an additive term in the hyperbolic function.

The static limit of solution (23) corresponds to $D\to\infty$. On substituting $\psi=0$ and $b=1$, our solution in the static limit, reduces to the Weyl solution [9] with $\partial = 1/2$.

With $\psi=0$ and redefining the parameter b according to (26), the solution (23) corresponds to the one-soliton cosmological type solution obtained from the Kasner seed. However, for $\psi=0$ and $b=1$, one obtains the solutions for an Euclidean seed metric.

It is observed that $D=0, \psi=0$ and $b=1$, the derived metric (23), under the complex coordinate transformations $t\to i\phi$ and $\phi\to it$, corresponds to the Van Stockum metric [8]

$$\mathrm{d}s^2 = C_0 r^{-1/2}(\mathrm{d}r^2+\mathrm{d}z^2) + 2r\,\mathrm{d}t\,\mathrm{d}\phi - 2(z-\omega)\mathrm{d}t^2. \tag{27}$$

It appears from (23) and (24) that if the Laplace's solution ψ and the function F are known, the stationary metric (23) is then completely given for a static metric as seed. F satisfies the relations [7]

$$(r\partial_r - \lambda\partial_z + 2\lambda\partial_\lambda)F = r\psi_r, \tag{28}$$

$$(r\partial_z + \lambda\partial_r)F = r\psi_z. \tag{29}$$

Since the solutions are required at the pole $\lambda=\mu$; F depends on the value of μ. We take the plus sign before the square bracket in eq. (15) and consider real pole trajectories.

Prolate spheroidal coordinates (x,y) are used in the following analysis as we have taken the Laplace's solution ψ in that coordinate system. Prolate spheroidal coordinates are defined by

$$\begin{aligned} r^2 &= K^2(x^2-1)(1-y^2), \\ z &= z_1 + Kxy, \end{aligned} \tag{30}$$

where K and z_1 are constants. The arbitrary constant ω_k in eq. (15) can be replaced by shifting the origin along the z-axis by means of

$$\omega_k = z_1 + K. \tag{31}$$

For one-soliton solutions, it is found that

$$\mu = K(x+1)(1-y). \tag{32}$$

The relations satisfied by F are

$$F_x = \frac{(1+y)}{2(x-y)}[(x-1)\psi_x + (1-y)\psi_y], \tag{33}$$

$$F_y = \frac{(x-1)}{2(x-y)}[(1+y)\psi_y - (x+1)\psi_x]. \tag{34}$$

The subscripts x and y denote partial differentiations. Here we have taken the values of the eigenfunction Ω and hence F on the pole trajectories only.

In the following, an application of the above simplified but general technique is given and two sets of one-soliton solutions are constructed.

Set 1: We take the Laplace's solution in prolate spheroidal coordinates as

$$\psi = \alpha_0(x+y)^{-1}, \tag{35}$$

where α_0 is a constant.

From eqs (33), (34) one gets

$$F = \frac{\alpha_0}{2}(1+y)(x+y)^{-1}, \tag{36}$$

and from (24),

$$\chi = \alpha_0 y(x+y)^{-1}. \tag{37}$$

The metric coefficient g_{11}^{ph} is thus found to be

$$g_{11}^{\text{ph}} = -r^{1-b}\exp\left[\frac{\alpha_0}{(x+y)}\right] \frac{\sinh\left[\alpha_0 y(x+y)^{-1} - \frac{1}{2}(b-1)\eta + D\right]}{\cosh\left[\alpha_0 y(x+y)^{-1} - \frac{1}{2}b\eta + D\right]}, \tag{38}$$

where

$$\eta = \ln\left[\frac{(x+1)(1-y)}{(x-1)(1+y)}\right]. \tag{39}$$

It is observed that with $b=1$, the solution (38) is not well behaved at spatial infinity, i.e, it does not reduce to the Euclidean form. With $b=1$, when $x\to\infty$ and $D\to\infty$, eq. (38) takes the form

$$g_{11}^{\text{ph}} = -\left(\frac{1-y}{1+y}\right)^{1/2}. \tag{40}$$

With the symmetry of Ernst equation [10], one may interpret (40) as Weyl solution [8] for half integer value of ∂.

Set 2: In this case let the Laplace's solution be

$$\psi = \alpha_0 xy. \tag{41}$$

It is found that

$$F = \frac{\alpha_0}{2}(x+xy-y), \tag{42}$$

$$\chi = \alpha_0(x-y), \tag{43}$$

and

$$g_{11}^{\rm ph} = -r^{1-b}{\rm e}^{\alpha_0 xy}\frac{\sinh\left[\alpha_0(x-y)-\frac{1}{2}(b-1)\eta+D\right]}{\cosh\left[\alpha_0(x-y)-\frac{1}{2}b\eta+D\right]}, \tag{44}$$

where η is defined by (39). Now for $b=1$ this solution also does not reduce to the Euclidean form at spatial infinity.

4. Ernst potential of the solutions

In this section an analysis of the solutions presented in eq. (23) is given in terms of the Ernst potential by proper adjustment of the constants.

For an axially symmetric stationary metric

$$ds^2 = A^{-1}[{\rm e}^{2\gamma}({\rm d}r^2+{\rm d}z^2)+r^2{\rm d}\phi^2] - A({\rm d}t - B{\rm d}\phi)^2, \tag{45}$$

the Ernst potential is given by [10,11]

$$E = A + i\Phi, \tag{46}$$

where Φ is known as the twist potential and it can be derived from the relations [12]:

$$\Phi_r = r^{-1}A^2B_z, \tag{47}$$

$$\Phi_z = -r^{-1}A^2B_r, \tag{48}$$

where the subscripts r and z denote partial differentiations.

From eqs (1), (45) and (23) one obtains

$$A = r^{1-b}{\rm e}^{\psi}\frac{\sinh\left[\chi-\frac{1}{2}(b-1)\eta+D\right]}{\cosh\left[\chi-\frac{1}{2}b\eta+D\right]}, \tag{49}$$

$$B = (-1)^{1+b}r^b{\rm e}^{-\psi}\frac{\cosh\left(\frac{1}{2}\eta\right)}{\sinh\left[\chi-\frac{1}{2}(b-1)\eta+D\right]}. \tag{50}$$

Due to the presence of the superposing field ψ in eqs (49) and (50), the evaluation of Φ becomes a complicated task. The real part of Ernst potential is obtained as

$${\rm Re}\,E = \mu r^{-b}{\rm e}^{\psi}(1-\mu^2r^{-2}p)(1+p)^{-1}, \tag{51}$$

where

$$P = \left(\frac{\mu}{r}\right)^{2b}{\rm e}^{-2(\chi+D)}. \tag{52}$$

In the static limit $D\to\infty$, one obtains the Ernst potential of the metric (23) as

$$E = \mu r^{-b} e^{\Psi}. \tag{53}$$

With vanishing superposing field i.e. with $\alpha_0 = 0$ and substituting $b = 1 - 2q_1$, the solution (53) reduces to that of Verdaguer equation (2.19) of ref. [6]. Further, when $D \to \infty$ and $\alpha_0 = 0$, if one substitutes $b = -d$, the solution presented in (51) corresponds to the one-soliton solution $(s = 1, h_1 = 1)$ of Carot and Verdaguer for real pole trajectories (eq. (9) of ref. [13]). The parameters d, s, and h_1 are defined in ref. [13].

For Euclidean seed i.e., for $b = 1$ and $\alpha_0 = 0$, the Ernst potential of the solution (23) is obtained as

$$E = [\mu r^{-1}(1 - \mu^2 r^{-2} q) + in(1 - \mu^2 r^{-2})](1 + q)^{-1}, \tag{54}$$

where

$$q = \left(\frac{\mu}{r}\right) e^{-2D} \tag{55}$$

and n is a constant given by

$$n^{-1} = e^{D} \coth D.$$

In the static limit i.e. at $D \to \infty$, the solution presented in (54) takes the form

$$E = \mu r^{-1}. \tag{56}$$

Now changing the coordinates (r, z) to (ρ, θ) in the following manner:

$$r = \rho \sin\theta, \quad z - \omega = \rho \cos\theta, \tag{57}$$

one obtains

$$E = \left(\frac{1 - \cos\theta}{1 + \cos\theta}\right)^{1/2}. \tag{58}$$

Substituting $y = \cos\theta$.

$$E = \left(\frac{1 - y}{1 + y}\right)^{1/2}. \tag{59}$$

Since the Ernst potential is symmetric in x and y, one can construct a new solution

$$E = i\left(\frac{x - 1}{x + 1}\right)^{1/2},$$

and it is only a trivial task to verify the static solution

$$E = \left(\frac{x - 1}{x + 1}\right)^{1/2}. \tag{60}$$

The solution (60) is not the Schwarzschild solution but it represents an asymptotically flat Weyl solution with deformation parameter $\partial = \frac{1}{2}$. If the above solution is generalized for N-odd soliton solutions, Das [14] has shown that E takes the form

$$E=\left(\frac{x-1}{x+1}\right)^{N/2}. \tag{61}$$

This is the Weyl solution with $\partial = N/2$. The deformation parameter ∂ is a measure of the deviation from Schwarzschild space-times. A series of Weyl solutions can be obtained for different values of ∂. The deformation parameter ∂ can take any positive value in the Weyl metric but it is restricted only to the integer values for Tomimatsu–Sato metrics [15].

Verdaguer [6] obtained a solution for $\partial = 1/2$. Das extended the solution for $\partial = N/2$, where N is an odd integer. Both of them used flat metrics as seed. We, in this paper, have used a general static axisymmetric metric as seed and constructed a stationary metric which can be reduced to their metrics when proper adjustment is made among the constants appearing in the solutions.

The asymptotic expansion of E in eq. (60) is obtained as

$$E=1-\frac{1}{x}+\frac{1}{2x^2}+\cdots. \tag{62}$$

In Boyer–Lindquist coordinates (R,Θ), which coincide with the spherical coordinates (ρ,θ) for large R,

$$Kx=R-m,\quad y=\cos\theta,\quad m=\text{ constant}, \tag{63}$$

eq. (62) assumes the form

$$E=1-\frac{K}{R}+\frac{K(K-2m)}{2R^2}+\cdots. \tag{64}$$

The solution is thus asymptotically flat.

The real part of Ernst potential for the solution presented in Set 1, takes the asymptotic form, with $b=1$ as

$$\operatorname{Re}E=\left(\frac{1-y}{1+y}\right)^{1/2}\left[1+\frac{(1+\alpha_0)}{x}+\frac{1}{2}\left\{\alpha_0(\alpha_0-2y+1)+1\right\}\frac{1}{x^2}+\cdots\right]. \tag{65}$$

The solution is not spatially well-behaved at infinity. The derived solution (65) is singular on the symmetry axis $(y=\pm 1)$ which represents a line source along the axis.

However, for $y=0$, one obtains from (65),

$$\operatorname{Re}E=1+\frac{(1+\alpha_0)}{x}+\frac{\{\alpha_0(\alpha_0+1)+1\}}{2x^2}+\cdots. \tag{66}$$

The solution (66) is asymptotically flat and contains monopole, dipole and other higher mass multipole terms with monopole mass term $(1+\alpha_0)/2$. The constant α_0 may be interpreted as a measure of the strength of the superposing field.

5. Conclusions

We have presented in this paper the one-soliton solutions for two different Laplace seed in the general axisymmetric metric. It is found that the derived metrics (eqs (38) and

(44)) do not, in general, reduce to Euclidean form at spatial infinity. However, when one truncates the solutions for static limit, Weyl solutions are obtained for half integral ∂-values. Further, on imposing some restrictions on the constants appearing in the solutions, our stationary solution (23) corresponds to the solutions of Verdaguer [6] and to the one-soliton solutions of Letelier [7]. The inclusion of Laplace's solution ψ in the general seed metric is reflected just as an additive term in the Verdaguer's result. The generated solution after some readjustment of the parameters reduces to the one-soliton solution of Carot and Verdaguer [13] for real pole trajectories.

The solution presented in Set 1, is singular on the symmetry axis $(y=\pm 1)$, thereafter representing a line source along the axis. This solution becomes asymptotically flat for $b=1$ and $y=0$ and contains the mass multipole terms.

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References

[1] V A Belinskii and V E Zakharov, *Sov. Phys. JETP*, **48**, 985 (1978)
[2] V A Belinskii and V E Zakharov, *Sov. Phys. JETP* **50**, 1 (1979)
[3] G A Alekseev and V A Belinskii, *Sov. Phys. JETP* **51**, 655 (1980)
[4] A Tomimatsu and H Sato, *Suppl. Prog. Theor. Phys.* **70**, 215 (1981)
[5] B J Carr and E Verdaguer, *Phys. Rev.* **D28**, 2995 (1983)
[6] E Verdaguer, *J. Phys.* **A15**, 1261 (1982)
[7] P S Letelier, *J. Math. Phys.* **26**, 467 (1985)
[8] W J Van Stockum, *Proc. R. Soc. Edinburg* **A57**, 135 (1973)
[9] H Weyl, *Ann. Phys. (Leipz)* **54**, 117 (1917)
[10] D Kramer, H Stephani, E Herlt, M MacCallum and E Schumtzer, *Exact solutions of Einstein's field equations* (Cambridge University Press, Cambridge, 1980)
[11] F J Ernst, *Phys. Rev.* **167**, 1175 (1968)
[12] Ts I Gutsunaev and V S Manko, *Class. Quantum Gravit.* **6**, L137 (1989)
[13] J Carot and E Verdaguer, *Class. Quantum Gravit.* **6**, 845 (1989)
[14] K C Das, unpublished
[15] A Tomomatsu and H Sato, *Prog. Theor. Phys. Jpn.* **50**, 95 (1973)