PRAMANA — journal of physics | Vol. 58, No. 3, March 2002, pp. 449–456

On the generation techniques of axially symmetric stationary metrics

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MS received 18 October 2000; revised 1 June 2001

Abstract. In the present paper, a relationship between the method of Gutsunaev–Manko and the soliton technique (for two-soliton solutions) of Belinskii–Zakharov, for generating solutions of axially symmetric stationary space-times in general relativity is discussed.

Keywords. General relativity; Einstein's equations; relation between solution generating techniques; soliton technique; method of Gutsunaev–Manko.

PACS Nos 04.20.Jb; 04.20.-q; 04.20.Cv

1. Introduction

Einstein's general theory of relativity is the most successful theory in describing the behaviour of astrophysical objects and the exact solutions of Einstein and Einstein–Maxwell field equations with or without external gravitational or electromagnetic fields are of much interest. Since the theory is highly nonlinear, only a few physically acceptable solutions were obtained by the conventional method of solving the tensor equations directly. Different transformation techniques were then developed one after another by utilizing the hidden symmetry of the field equations [1]. These were useful for generating new solutions as well as rediscovering some already known ones. The relations between some of these techniques were given by Cosgrove [2,3].

Belinskii and Zakharov [4,5], developed a new technique, known as 'soliton technique' of generating stationary axially symmetric solutions of Einstein's field equations in general relativity. Their method is based on the 'inverse scattering transformation'. The solutions are characterized by the number of poles they contain. For example, if the solution contains a simple pole, it is called a one-soliton solution, for double poles it is called a two-soliton solution and so on. The background metric is chosen suitably and general N-soliton solutions were constructed. The seed metric may be a diagonal or a non-diagonal one. The problem is easier to solve when the seed metric is diagonal because an explicit integration of the linear partial differential equations is easy in this case.

Gutsunaev–Manko and their coworkers [6–8] also developed a method of generating solutions for axially symmetric stationary space-times with Laplace's solution as seed. Their method is based on the nonlinear superposition of Kerr metric with an arbitrary external gravitational field.

In the present paper, we derive a formal relation between the method of Gutsunaev–Manko (GM) and the soliton technique of Belinskii–Zakharov (BZ) for diagonal two-soliton solutions. Using the same seed in both the methods, we analyse the similarities between the solutions obtained. Moreover, the generated solutions become identical when the constants are properly adjusted [9]. In §2, the method of GM and the soliton technique of BZ are briefly described. The relationship between the methods is discussed in §3.

2. Brief description of the methods

A. *Method of Gutsunaev–Manko*

Consider an axially symmetric stationary line element in prolate spheroidal coordinates (x,y):

$$\begin{aligned} \mathrm{d}s^2 = K^2 f^{-1} \left[\mathrm{e}^{2\gamma}(x^2-y^2)\left(\frac{\mathrm{d}x^2}{x^2-1}+\frac{\mathrm{d}y^2}{1-y^2}\right)+(x^2-1)(1-y^2)\mathrm{d}\phi^2\right] \\ -f(\mathrm{d}t-\omega\mathrm{d}\phi)^2, \end{aligned} \tag{1}$$

where K is a constant and the metric coefficients, f,ω and γ are functions of x,y only.

Ernst reformulated Einstein's field equations and showed that the metric coefficients of the line element (1) can be derived from a complex function ε, known as Ernst potential, which satisfy the following relations [10]

$$\varepsilon = f + i\Phi, \tag{2}$$

$$(\varepsilon+\varepsilon^*)\nabla^2\varepsilon = 2(\nabla\varepsilon\cdot\nabla\varepsilon), \tag{3}$$

Φ is known as the twist potential and ε^* is the complex conjugate of ε.

Here we use prolate spheroidal coordinates (x,y), since in this coordinate system the Ernst potential is symmetric in x and y. Prolate spheroidal coordinates (x,y) are related to Papapetrou coordinates (r,z) by

$$\begin{aligned} r^2 &= K^2(x^2-1)(1-y^2), \\ z &= Kxy. \end{aligned} \tag{4}$$

Gutsunaev and Manko showed that a solution to eq. (1) can be obtained from a Laplace's solution 2ψ and they derived ε in the form [6,7]

$$\varepsilon = \mathrm{e}^{2\psi}\frac{x(1+ab)+iy(b-a)-(1-ia)(1-ib)}{x(1+ab)+iy(b-a)+(1-ia)(1-ib)}, \tag{5}$$

where ψ satisfies the equation

$$(x^2-1)\psi_{xx}+(1-y^2)\psi_{yy}+2x\psi_x-2y\psi_y=0, \tag{6}$$

and a and b are two real functions which satisfy the following first order linear partial differential equations:

$$\begin{aligned}
\frac{a_x}{a} &= \frac{2}{(x-y)}[(xy-1)\psi_x+(1-y^2)\psi_y],\\
\frac{a_y}{a} &= \frac{2}{(x-y)}[-(x^2-1)\psi_x+(xy-1)\psi_y],\\
\frac{b_x}{b} &= -\frac{2}{(x+y)}[(xy+1)\psi_x+(1-y^2)\psi_y],\\
\frac{b_y}{b} &= -\frac{2}{(x-y)}[-(x^2-1)\psi_x+(xy+1)\psi_y],
\end{aligned} \tag{7}$$

where the subscripts x and y denote partial differentiations.

Amenedo and Manko [7] derived the metric coefficients f,ω and γ as

$$f=\mathrm{e}^{2\psi}AB^{-1}, \tag{8}$$

$$\omega=2K\mathrm{e}^{-2\psi}CA^{-1}+k_1, \tag{9}$$

$$\mathrm{e}^{2\gamma}=k_2(x^2-1)A\mathrm{e}^{2\gamma'}, \tag{10}$$

where k_1,k_2 are arbitrary constants, γ' is the γ-functin of the static metric with

$$2\psi'=\ln[(x-1)/(x+1)]+2\psi, \tag{11}$$

and A,B,C are given by

$$A=(x^2-1)(1+ab)^2-(1-y^2)(b-a)^2, \tag{12}$$

$$B=[(x+1)+(x-1)ab]^2+[a(1+y)+b(1-y)]^2, \tag{13}$$

$$\begin{aligned}C=&(x^2-1)(1+ab)[(b-a)-y(a+b)]\\&+(1-y^2)(b-a)[1+ab+x(1-ab)].\end{aligned} \tag{14}$$

It is evident from the above discussions that for a known Laplace's solution 2ψ, the metric given by eq. (1) can be solved completely.

B. *The soliton technique of Belinskii–Zakharov*

Consider an axially symmetric stationary metric

$$\mathrm{d}s^2=g_{AB}\mathrm{d}x^A\mathrm{d}x^B+\mathrm{e}^{\nu}(\mathrm{d}r^2+dz^2), \tag{15}$$

where g_{AB} is a 2×2 matrix, A,B take values 1, 2; $t,\ \phi=x^1,\ x^2$ respectively and ν is a function of r,z only. According to BZ, the Einstein's field equations, for the metric (15), to be solved are [5]

$$v_r = -r^{-1} + (4r)^{-1}\mathrm{Tr}\,(U^2 - V^2), \tag{16}$$

$$v_z = (2r)^{-1}\mathrm{Tr}\,(UV), \tag{17}$$

$$U_r + V_z = 0, \tag{18}$$

where

$$U = rg_r g^{-1}, \quad V = rg_z g^{-1}. \tag{19}$$

Soliton solution to the metric (15) is obtained by solving the following eigenvalue equations:

$$\begin{aligned}\left(\partial_r + \frac{2\lambda r}{\lambda^2 + r^2}\partial_\lambda\right)\Omega &= \frac{rU + \lambda V}{\lambda^2 + r^2}\Omega,\\ \left(\partial_z - \frac{2\lambda^2}{\lambda^2 + r^2}\partial_\lambda\right)\Omega &= \frac{rV - \lambda U}{\lambda^2 + r^2}\Omega,\end{aligned} \tag{20}$$

where λ is a complex spectral parameter and the eigenfunction $\Omega(r, z, \lambda)$ is a two-dimensional complex matrix which satisfy the relation

$$\Omega(r, z, \lambda)|_{\lambda=0} = g(r, z). \tag{21}$$

The seed metric $(g_0)_{AB}$ may be chosen with the supplementary condition

$$\det g_0 = -r^2. \tag{22}$$

The new metric coefficients are obtained from the following relations [5]:

$$g'_{AB} = (g_0)_{AB} - \sum_{k,l=1}^{N} \Gamma^{-1}_{kl}\mu_k^{-1}\mu_l^{-1}N_A^k N_B^l, \tag{23}$$

$$N_A^k = m_C^k (g_0)_{CA}, \tag{24}$$

$$\Gamma_{kl} = \frac{m_C^k (g_0)_{CA} m_A^l}{\mu_k \mu_l + r^2}, \tag{25}$$

$$m_A^k = m_{0C}^k \left[\Omega^{-1}(r, z, , \lambda = \mu_k)\right]_{CA}, \tag{26}$$

$$\mu_k = q_k - z + [(q_k - z)^2 + r^2]^{1/2}, \tag{27}$$

$$v_N = v_0 + \ln\left[r^{-N^2/2}\left(\prod_{k=1}^{N}\mu_k\right)^{N+1}\left\{\prod_{\substack{k,l=1\\k>l}}^{N}(\mu_k - \mu_l)^{-2}\right\}\det\Gamma_{kl}\right] + \ln C_N, \tag{28}$$

where m_{0C}^k, q_k and C_N are arbitrary constants, v_0 is the v-function of the static metric.

The new metric g'_{AB} given by eq. (23), is a solution to eq. (18) but, in general, it does not satisfy eq. (22). Physically acceptable $g = g_{AB}^{\mathrm{ph}}$ is defined by the relation

$$g_{AB}^{\mathrm{ph}} = -r(-\det g')^{-1/2} g'_{AB}, \tag{29}$$

where

$$\det g' = (-1)^N r^{2N} \left(\prod_{k=1}^{N} \frac{1}{\mu_k^2} \right) \det g_0. \tag{30}$$

Here N is the number of solitons i.e. the number of poles that appears in the scattering matrix. For $N = 1$, it is called a simple pole i.e. one-soliton solutions; $N = 2$, a doube pole i.e. two-soliton solutions and so on. Depending on the choice of the seed metric g_0, the eigenfunction Ω may be either diagonal or non-diagonal one.

Letelier showed [11] that in the case of a diagonal static metric of the form

$$\mathrm{d}s^2 = \mathrm{e}^{v_0}(\mathrm{d}r^2 + \mathrm{d}z^2) + r^2\mathrm{e}^{-\psi}\mathrm{d}\phi^2 - \mathrm{e}^{\psi}\mathrm{d}t^2, \tag{31}$$

as seed, where ψ is a Laplace's solution, the function v_0 is obtained from the relation

$$v_0(\psi) = -\psi + \frac{1}{2}\int r[(\psi_r^2 - \psi_z^2)\mathrm{d}r + 2\psi_r\psi_z\mathrm{d}z]. \tag{32}$$

The seed metric g_0 is given by (from eq. (31))

$$g_0 = \mathrm{diag}(-\mathrm{e}^{\psi}, r^2\mathrm{e}^{-\psi}). \tag{33}$$

Since the seed metric is diagonal, one may expect that the corresponding eigenfunction would be diagonal as well. We thus find an additional condition

$$\det\Omega_0|_{\lambda=0} = -r^2. \tag{34}$$

Without loss of generality, the eigenfunction may be taken as

$$\Omega_0 = \mathrm{diag}(-\mathrm{e}^{F_k}, (r^2 - 2\lambda z - z^2)\mathrm{e}^{-F_k}), \tag{35}$$

where

$$F_k = F_k(r, z, \lambda) \quad \text{and} \quad F(r, z, \lambda)|_{\lambda=0} = \psi(r, z). \tag{36}$$

The function F_k satisfies the relations [11]

$$\begin{aligned} &(r\partial_r - \lambda\partial_z + 2\lambda\partial_\lambda)F_k = r\psi_r, \\ &(r\partial_\lambda + \lambda\partial_r)F_k = r\psi_z, \end{aligned} \tag{37}$$

where the subscripts r and z denote partial differentiations.

It, therefore, appears that if the Laplace's solution ψ is known then the stationary metric (15) can be completely solved.

3. Relation between the methods

Manko and his coworkers obtained a solution for the Ernst potential in terms of Laplace's solution. In §2B, we find that soliton solutions to Einstein's equations may be constructed using Laplace's solution too. In fact it is easy to see that eqs (6) and (18) are equivalent, when expressed in the same coordinate system. Moreover, in the case of a static metric, there exists some resemblance in the expressions for the γ-function and that of ν_0 as given in eq. (32).

A comparison of the metric coefficients of the line elements (1) and (15) gives

$$f = -g_{11}, \tag{38}$$

$$\omega = -(2g_{11})^{-1}(g_{12}+g_{21}), \tag{39}$$

$$e^{2\gamma} = -g_{11}e^{\nu}. \tag{40}$$

For diagonal metric $(\Omega_0)_{12} = (\Omega_0)_{21}$ one obtains, $g_{12} = g_{21}$. We, therefore, have

$$\omega = -g_{12}(g_{11})^{-1}. \tag{41}$$

The metric coefficient g_{22} is found to be

$$g_{22} = K^2 f^{-1}(x^2-1)(1-y^2) - f\omega^2. \tag{42}$$

Under the coordinate transformation (4) and with the help of eqs (38)–(41), it is easy to show that

$$\det g = -r^2. \tag{43}$$

This satisfies the supplementary condition (22).

In Belinskii–Zakharov's technique the eigenfunction, satisfying eq. (20), is determined by the values of F_k (see eq. (35)). The function F_k in turn, can be evaluated from eq. (37) for a known Laplace's solution. Since in soliton technique, the solutions are required along the pole trajectories $\lambda = \mu_k$, it appears from eq. (37) that F_k depends on the values of μ_k. For two-soliton solutions μ_k has two values μ_1 and μ_2 and accordingly F_k can take values F_1 and F_2.

In prolate spheroidal coordinates (x,y), the poles μ_1 and μ_2 are obtained as

$$\begin{aligned} \mu_1 &= K(x+1)(1-y), \\ \mu_2 &= K(x-1)(1-y). \end{aligned} \tag{44}$$

In the same coordinate system, it is found that F_1 and F_2 satisfy the following relations [9]:

$$\begin{aligned} F_{1,x} &= \frac{(1+y)}{2(x-y)}[(x-1)\psi_x + (1-y)\psi_y], \\ F_{2,x} &= \frac{(1+y)}{2(x+y)}[(x+1)\psi_x + (1-y)\psi_y], \\ F_{1,y} &= \frac{(x-1)}{2(x-y)}[(1+y)\psi_y - (x+1)\psi_x], \\ F_{2,y} &= \frac{(x+1)}{2(x+y)}[(1+y)\psi_y - (x-1)\psi_x], \end{aligned} \tag{45}$$

where the subscripts x and y denote partial differentiations. We are now in a position to compare the generating functions of the metric coefficients of both the methods. Comparing eqs (7) and (45), one obtains,

$$a = -\eta \exp[\beta(2F_1 - \psi)], \tag{46}$$
$$b = \eta \exp[-\beta(2F_2 - \psi)], \tag{47}$$

where η and β are two arbitrary constants.

Equation (46) and (47) are combined to form a single equation

$$a_k = -\varepsilon_k \eta \exp(\varepsilon_k \beta \chi_k), \tag{48}$$

where, k can take values 1, 2;

$$a_1 = a, \;\; a_2 = b, \; \varepsilon_k = 1 \text{ for } k = 1, \; \varepsilon_k = -1 \text{ for } k = 2, \;\text{ and }\; \chi_k = 2F_k - \psi. \tag{49}$$

The parameters a and b defined in GM method (eq. (5)) can thus be derived from the function F_k and the Laplace's solution ψ.

The relation presented in eq. (48) is an important one, because by this transformation one can directly switch over to the solutions of Gutsunaev–Manko method from the soliton solutions of Belinskii–Zakharov.

A, B and C given in eqs (12)–(14), are expressed in terms of F_1 and F_2 as

$$A = (x^2-1)[1-\eta^2 e^{2\beta(F_1-F_2)}]^2 + (1-y^2)\eta^2 e^{2\beta}(e^{\chi_1} + e^{-\chi_2})^2, \tag{50}$$

$$B = \left[\{1+\eta^2 e^{2\beta(F_1-F_2)}\} + x\{1-\eta^2 e^{2\beta(F_1-F_2)}\}\right]^2 + \eta^2 e^{2\beta}[y(e^{\chi_1} + e^{-\chi_2}) + (e^{\chi_1} - e^{-\chi_2})]^2, \tag{51}$$

$$C = (x^2-1)\eta e^{\beta}\left[\{1-\eta^2 e^{2\beta(F_1-F_2)}\}\{(e^{\chi_1} + e^{-\chi_2}) + y(e^{\chi_1} - e^{-\chi_2})\}\right] + (1-y^2)\eta e^{\beta}(e^{\chi_1} + e^{-\chi_2})\left[(1+x) - \eta^2(1-x)e^{2\beta(F_1-F_2)}\right]. \tag{52}$$

The metric coefficients f, ω and γ given by eqs (8)–(10) can then be calculated in terms of F_1, F_2 and Ψ.

As an illustration of the above transformation technique (48), let us take a Laplace's solution,

$$\psi = \alpha_0 (x-y)^{-1}, \tag{53}$$

where α_0 is a constant. From eqs (7), one obtains,

$$a = -\alpha \exp[2\alpha_0(xy-1)(x-y)^{-2}], \tag{54}$$
$$b = \alpha \exp[-2\alpha_0 y(x-y)^{-1}], \tag{55}$$

where α is an arbitrary constant.

We now use the same Ψ as in eq. (53) to calculate F_1 and F_2 from eqs (45). The result is

$$F_1 = \left(\frac{\alpha_0}{2}\right)(x-1)(1+y)(x-y)^{-2}, \tag{56}$$
$$F_2 = \left(\frac{\alpha_0}{2}\right)(1+y)(x-y)^{-1}. \tag{57}$$

With these values of F_1 and F_2, if one now substitutes $\eta = \alpha$ and $\beta = 2$ in (48), one obtains the results of (54) and (55).

Acknowledgement

The author wishes to thank UGC, India for financial support (Research Project No. F.PSW-1/99(ERO)) and IUCAA, Pune for its hospitality during his visit.

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