

## Percolation systems away from the critical point

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**Abstract.** This article reviews some effects of disorder in percolation systems away from the critical density  $p_c$ . For densities below  $p_c$ , the statistics of large clusters defines the animals problem. Its relation to the directed animals problem and the Lee–Yang edge singularity problem is described. Rare compact clusters give rise to Griffiths singularities in the free energy of diluted ferromagnets, and lead to a very slow relaxation of magnetization. In biased diffusion on percolation clusters, trapping in dead-end branches leads to asymptotic drift velocity becoming zero for strong bias, and very slow relaxation of velocity near the critical bias field.

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### 1. Introduction

In this paper, I will review some examples of interesting effects caused by disorder in a classical percolation system. The aim is mainly pedagogical, and I do not attempt to be exhaustive. The selection of topics was determined primarily by my familiarity rather than any other reasons. I will only outline the main results. The interested reader will have to go back to the cited literature for details.

Consider a random mixture of equal-sized conducting and insulating beads in a box. It is easy to see that as the fraction of conducting balls is varied from zero to one, the bulk mixture undergoes a transition from insulating to conducting. This is the basic percolation transition, and has been studied a lot as a simple geometrical model of phase transitions. As in the case of thermal critical phase transitions, various physical quantities have singular behavior near the phase transition, characterized by ‘critical exponents’. Scaling theory, simulations and theoretical techniques such as renormalization group etc. have been used very successfully to understand the behavior of percolation systems at its critical point. In particular, in two dimensions, all the critical exponents of percolation are known from conformal field theory. Several good reviews on this subject are now available [1].

There is more to percolation theory than the critical exponents. Of course, an experimental disordered system may often be modelled by site- or bond-percolation. In general, one is much more likely to find it not near its percolation threshold. Critical exponents of the percolation theory are not of much use in describing these systems. It turns out that percolation systems, such as the classical models of disordered media show many interest-

ing features, even away from the critical point. I will discuss some examples of these in the following sections.

## 2. Off-critical exponents in percolation

Let  $P_n(p)$  be a probability that the cluster of connected sites containing the origin has exactly  $n$  sites. Then, for all  $p < p_c$ , for large  $n$ ,

$$P_n(p) \sim An^{-\theta} \exp[-B(p)n], \quad (1)$$

where  $B(p)$  is a  $p$ -dependent function that goes to zero as  $p$  tends to  $p_c$ . The exponent  $\theta$  is independent of  $p$ , and depends only on the dimension of space. This law is valid for  $n$  much greater than the typical cluster size  $n^*(p)$ . For  $n \ll n^*(p)$ , one gets a different exponent  $P_n(p) \sim n^{-\tau}$ , with  $\tau \neq \theta$ . As  $p$  tends to  $p_c$ ,  $n^*(p)$  diverges. Here  $\tau$  is a critical exponent of percolation theory, but  $\theta$  is an *off-critical* exponent.

In the limit  $p \rightarrow 0$ , all clusters of  $n$  sites have equal weight ( $p^n$ ). Let  $A_n$  be the number of different clusters of  $n$  sites possible that contain the origin. (These are called rooted animals: same cluster with different positions of origin are counted as distinct [2].) Then one expects that for large  $n$ ,

$$A_n \simeq K\lambda^n n^{-\theta}. \quad (2)$$

The exponent  $\theta$  in this equation is the same as that defined by eq. (1), because for small  $p$ ,  $P_n(p) \sim A_n p^n$ . One can also define the average linear size of an animal of  $N$  sites. This grows as  $n^\nu$ , where the exponent  $\nu$  is related to the exponent  $\theta$  defined above by the relation

$$\theta = (d - 2)\nu, \quad (3)$$

where  $d$  is the dimension of space. The above equation is valid for  $1 \leq d \leq 8$ . For  $d \geq 8$ , the exponents  $\theta$  and  $\nu$  stick to their mean-field values  $3/2$  and  $1/4$  respectively. Equation (3) has the form of a hyperscaling relation except that  $(d - 2)$  appears here instead of  $d$ . This is understood as being due to a hidden supersymmetry in the problem [3], which makes the problem of determining the number of animals in  $d$  dimensions related to the problem of Lee–Yang edge singularity in  $(d - 2)$  dimension.

The Lee–Yang description of the mathematical mechanism of phase transitions is well-known [4]. For a hard-core lattice gas in a finite volume, with possible additional attractive short-ranged interactions, the grand-canonical partition functions as a finite degree polynomial with positive coefficients in the chemical activity  $z$ . The zeroes of this polynomial often (not always) lie on lines in the complex  $z$  plane. As the temperature is varied, the coefficients of the polynomial change, and the zeroes move. If at some temperature, the zeroes come arbitrarily close to the real axis as the size of the system is increased, the free energy per site becomes a non-analytic function of  $z$ , signalling the onset of a phase transition.

The density of zeroes along such a line of zeroes near its end point shows a power-law dependence on the distance from the end point. We define the Lee–Yang edge singularity exponent  $\sigma$  by the relation that the density varies as  $\varepsilon^\sigma$  at a distance  $\varepsilon$  from the end point. It turns out that  $\sigma$  is independent of temperature for all temperatures  $T$  above the critical

temperature  $T_c$ , and depends only on the dimension of the system. It was shown by Parisi and Sourlas [3] that the animal exponent  $\theta(d)$  in  $d$  dimensions is related to the Lee–Yang edge singularity exponent in  $d - 2$  dimensions

$$\theta(d) = \sigma(d - 2) + 1. \quad (4)$$

If we allow only neighbors in the ‘forward direction’, (say along the direction of increasing coordinates on a hypercubical lattice), we get animals with a directional constraint. It turns out [5] that critical exponents of directed animals in  $d$  dimensions is related to those of undirected animals in  $(d + 1)$  dimensions.

$$\theta_{\text{dir}}(d) = \theta(d + 1) \quad (5)$$

and the transverse size exponent  $\nu_{\perp, \text{dir}}$  for directed animals in  $d$  dimensions is the same as the (only one) size exponent  $\nu$  for undirected animals in  $(d + 1)$  dimensions

$$\nu_{\perp, \text{dir}}(d) = \nu(d + 1). \quad (6)$$

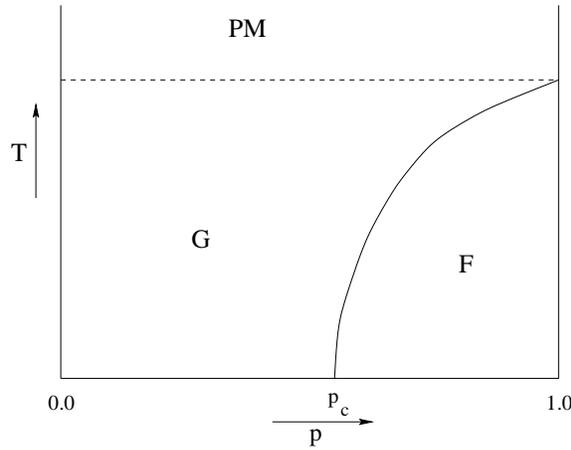
The exponent  $\sigma$  is easily shown to take the values  $\sigma = -1$  for  $d = 0$  (a point), and  $\sigma = -1/2$  for  $d = 1$ . For  $d = 2$ , one can use the exact solution of hard hexagon lattice gas by Baxter to show that  $\sigma = -1/6$  for  $d = 2$  [6]. This then shows that the exact values of the exponent  $\theta$  for undirected animal in dimensions 1, 2, 3, 4 are  $-1, 0, \frac{1}{2}, \frac{5}{6}$  respectively. The corresponding values of the size exponent  $\nu$  are  $1, \frac{1}{2}, \frac{5}{12}$  in dimensions 1, 3 and 4 respectively. In two dimensions, the exponent  $\nu$  is not determined by the scaling relations given above. The upper critical dimension for the animals problem is 8, and for all  $d \geq 8$ , we get  $\theta = \frac{3}{2}$ , and  $\nu = \frac{1}{4}$ . The exponents for directed animals are easily determined from the scaling relations given above.

We have seen that there is a fairly good understanding of the off-critical ‘below  $p_c$ ’ exponents of percolation. One can also define off-critical exponents in the super-critical regime of percolation theory. It was shown by Kunz and Souillard [7] that for all  $p > p_c$ , the probability that the origin belongs to a finite cluster of  $n$  sites varies as  $\exp(-b(p)n^{(d-1)/d})$ , for sufficiently large  $n$ . This is easy to understand: to get a finite cluster of  $n$  sites, we need to disconnect it from the infinite cluster. This needs perimeter bonds of order  $(n^{(d-1)/d})$  bonds, which gives the result. More accurately, there is a power-law prefactor multiplying the exponential term, and the probability for a finite cluster of  $n$  sites varies as

$$P_n(p) \sim Kn^{-\theta'} \exp(-b(p)n^{(d-1)/d}), \quad (7)$$

where  $b(p)$  is a function of  $p$ , and  $K$  is a constant. The exponent  $\theta'$  can be calculated exactly [8] using the fact that such clusters are roughly compact, with linear size varying as  $n^{1/d}$ , and the fluctuations in the  $(d - 1)$ -dimensional roughly spherical surface can be described in terms of normal modes of vibration of the surface. Lubensky and McKane using field theory techniques showed that the exponent  $\theta'$  takes the values  $\frac{5}{4}, -\frac{1}{9}, \frac{1}{8}, -\frac{449}{450}, -\frac{11}{12}$  for  $d = 2, 3, 4, 5$  and 6 respectively. The non-monotonic behavior of  $\theta'$  as a function of  $d$  comes from the fact that the integrals over normal modes in the odd and even dimensions coming in the theory have different behaviors.

Note that this is one of the few cases where non-trivial values of exponents can be exactly calculated in many dimensions greater than 2.



**Figure 1.** The phase diagram of a diluted magnet. The ferromagnetic, paramagnetic and Griffiths phases are denoted by PM, F, and G respectively.

### 3. Relaxation in disordered ferromagnets

Consider now the case when the atom at each occupied site in a percolation network carries a magnetic moment, and there is a nearest neighbor ferromagnetic interaction  $J$  between the magnetic atoms. The ‘unoccupied’ sites may be vacancies, or occupied by non-magnetic atoms. Let  $p$  denote the concentration of magnetic atoms. Then if  $p < p_c$ , the percolation threshold, then the system breaks up into mutually-disconnected clusters of magnetically coupled spins. In such a system no long-range spontaneous magnetization is possible at any non-zero temperature. For  $p > p_c$ , there is an infinite connected cluster, and at sufficiently low temperatures, spontaneous magnetization exists. The transition temperature  $T_{\text{curie}}(p)$  depends on  $p$ , and goes to zero, as  $p$  is decreased from 1 to  $p_c$ . The ferromagnetic phase is denoted by  $F$  in figure 1.

For  $p < p_c$ , there is no spontaneous magnetization. Let  $f(T, h)$  be the disorder-averaged free energy per site of this system at a temperature  $T$  in a magnetic field  $h$ . However, it was shown by Griffiths [9] that  $f(T, h)$  is a non-analytic function of the magnetic field at  $h = 0$  for all  $T < T_{\text{curie}}(p = 1)$ . While the partial derivatives  $\frac{\partial^n}{\partial h^n} f(T, h)$  exist for all positive integers  $n$ , and are finite, the Taylor series for  $f(T, h)$  in powers of  $h$  does not converge for any  $T$  below the  $T_{\text{curie}}(p = 1)$ , the Curie-temperature of the ‘pure’ system. Thus, in the entire region marked G (for the Griffiths phase) in figure 1, there is no spontaneous magnetization, but the free energy per site  $f(T, h)$  is a non-analytic function of  $h$ .

While the non-analyticity in  $f(T, h)$  as a function of  $h$  is of the nature of an essential singularity, and is difficult to verify in experimental systems, the rare large clusters responsible for it have a much more pronounced effect on the rate of relaxation to equilibrium in disordered systems. Consider the decay of magnetization in such a system. We assume that the system is coupled to heat bath at temperature  $T < T_{\text{curie}}(p = 1)$ , and relaxes by single-spin-flip Glauber dynamics. At time  $t = 0$ , the system is prepared in a state with all spins up. We monitor the bulk magnetization  $M(t)$  at large times  $t$ . Since the spins in different clusters do not interact with each other, we can write  $M(t)$  as a weighted sum over different cluster configurations  $\mathcal{C}$ .

$$M(t) = \sum_{\mathcal{C}} \text{Prob}(\mathcal{C}) \langle S(t) \rangle_{\mathcal{C}}. \quad (8)$$

For a given finite cluster  $\mathcal{C}$  of  $n$ -sites, one can determine the average magnetization at a site in the cluster by explicit diagonalization of the matrix of size  $2^n \times 2^n$ . This is non-trivial, except for very small  $n$ . Fortunately, the behavior of  $M(t)$  for large  $t$  can be determined by simple qualitative arguments.

In the summation eq. (8), the leading behavior of each term is exponential in time, with the decay rate  $\gamma_{\mathcal{C}}$  depending on the cluster  $\mathcal{C}$ . Thus, we may write

$$\langle S(t) \rangle_{\mathcal{C}} \simeq \exp(-\gamma_{\mathcal{C}} t) \quad (9)$$

and hence

$$M(t) \simeq \sum_{\mathcal{C}} \text{Prob}(\mathcal{C}) \exp(-\gamma_{\mathcal{C}} t). \quad (10)$$

At large times, clusters with smallest decay rates contribute the most. The slowest relaxing clusters are those where all sites within a disc of radius  $R$  are occupied. The density of such clusters varies as  $\exp(-AR^d)$ , where the constant  $A$  depends on  $p$ . If the magnetization of such a cluster has to flip, it would need to create a domain wall of energy  $\simeq \sigma R^{d-1}$ . The rate of such an activated process will decrease for large  $R$  as  $\exp(-\sigma R^{d-1}/T)$ . It is then straightforward to put these estimates in eq. (8), and deduce that

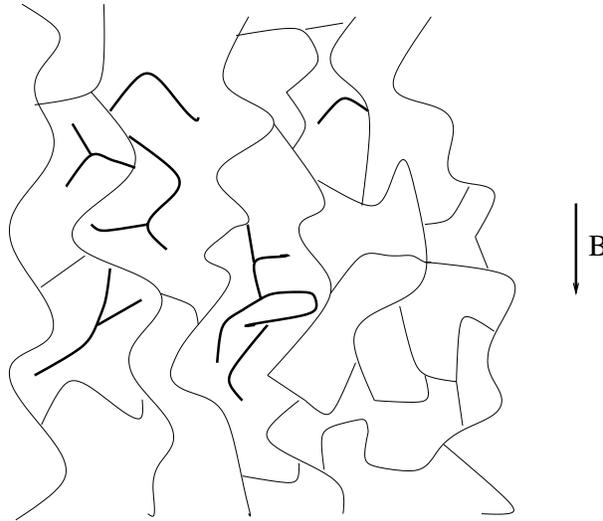
$$M(t) \sim \exp(-K(\log t)^{d/(d-1)}), \quad \text{for large } t. \quad (11)$$

For  $p > p_c$ , in the ferromagnetic phase (F in figure 1) the probability of a large finite cluster of radius  $R$ , varies as  $\exp(-aR^{d-1})$ , and not as  $\exp(-R^d)$ . Then the steepest descent calculation shows that in the ferromagnetic phase F, the magnetization at long times decays as a power law  $M(t) \sim t^{-c}$ , where the exponent  $c$  depends on both  $p$  and  $T$ .

The argument outlined above was first presented in [10]. The argument has been refined [11,12]. Unfortunately, neither actual experimental data on disordered ferromagnets, nor the results of numerical simulations [13] show a clear evidence of such a  $\exp[-(\log t)^x]$  behavior. Presumably the time scale beyond which the contribution of rare clusters will dominate is larger than experimentally accessible time scales. The experimental data seems to fit better a stretched exponential  $\exp(-t^x)$ . It seems that a more careful argument, that gives not only the correct asymptotic behavior at longest times, but also at intermediate times, is needed.

#### 4. Biased diffusion on percolation networks

Consider the motion of a single diffusing particle on a percolation network, say in two dimensions, with density of occupied sites being  $p$ . We assume that the particle can move only on the occupied sites of the lattice (figure 2). Then if  $p$  is less than the critical probability  $p_c$ , the particle is localized. For  $p > p_c$ , if the particle starts on the infinite cluster, its mean square deviation from the initial position grows linearly with time,  $\langle R^2 \rangle \sim D(p)t$ , where the diffusion constant  $D(p)$  depends on  $p$  and tends to zero as  $p$  tends to  $p_c$ . This problem of (unbiased) diffusion on percolation clusters has been studied much [14].



**Figure 2.** Schematic representation of the percolation cluster with density above the critical threshold. The heavy lines denote the dead-end branches.

If there is a larger probability of displacement in some direction, due to an imposed field, we have biased diffusion. We shall model it by assuming that at any time the diffusing particle attempts to take a step in the up, right, down or left directions with probabilities  $(1 - B)/4, 1/4, (1 + B)/4$  and  $1/4$  respectively. The step is actually taken if the intended destination site is occupied. If the biasing field  $B$  is small, we have a non-zero value of average displacement per step, and this gives rise to mean displacement in time  $t$  in the direction of the field growing linearly with  $t$ , and the mean velocity in time  $t$  tends to a constant

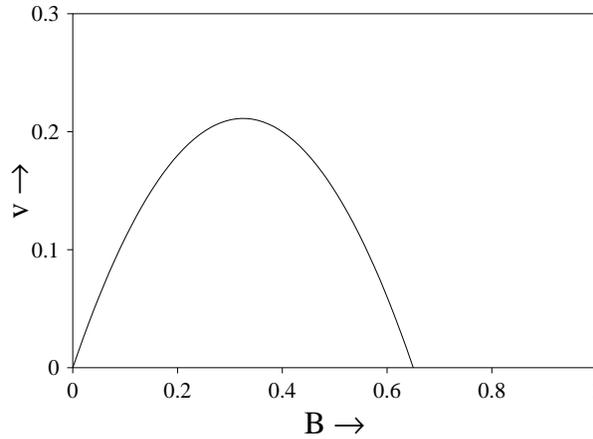
$$\vec{v}_\infty = \lim_{t \rightarrow \infty} \langle \vec{R}_t \rangle / t. \quad (12)$$

This asymptotic velocity  $\vec{v}_\infty$  is proportional to  $B$  for small  $B$ .

If  $p$  is near 1, most of the sites are occupied, and at large length scales, the medium looks homogeneous. One then expects that so long as  $p > p_c$ , we expect the same behavior as in the system without disorder ( $p = 1$ ) (figure 2).

Bottger and Bryksin [15] realized that this is not so. They argued that the mean velocity must tend to zero as  $B$  tends to 1, because of the possibility of trapping in dead-ends. We argued in [16], that for any  $p < 1$ , there exists a critical bias  $B_c$  such that the asymptotic drift velocity  $\vec{v}$  is exactly zero for all  $B > B_c$ . This is easily seen as during its motion, the particle may get trapped in dead-end branches for long times, as it has to move against the field to get out of the trap. For a trap of depth  $\ell$  the potential barrier to cross increases with  $\ell$ , and the trapping time varies as  $(\frac{1+B}{1-B})^\ell$ . The density  $\rho(\ell)$  of traps of depth  $\ell$  varies as  $\exp(-\ell/\xi)$ . Hence the average trapping time per step along the backbone is

$$\sum_{\ell=1}^{\infty} \rho(\ell) \left( \frac{1+B}{1-B} \right)^\ell, \quad (13)$$



**Figure 3.** The average velocity  $v$  as a function of the biasing field  $B$ .

where  $\xi$  is the  $p$ -dependent percolation correlation length of the system. This summation converges only for  $B < B_c = \tanh(\frac{1}{2\xi})$ . For  $B = B_c - \epsilon$ , with  $\epsilon > 0$ , this summation varies as  $1/\epsilon$ , and the mean velocity, which varies inversely as the mean trapping time varies proportional to  $\epsilon$  (figure 3).

For  $B > B_c$ , the asymptotic velocity  $\vec{V}_\infty$  is zero, but the mean displacement of the particle  $\langle \vec{R}_t \rangle$  increases as  $t^a$  with  $a < 1$ . The exponent  $a$  depends continuously on  $B$ , and is easily obtained by using the condition that the density of traps with trapping time greater than or equal to  $t$  varies as  $t^{-a}$ , where

$$a = \left( \frac{1}{\xi} \right) \log \left( \frac{1+B}{1-B} \right). \tag{14}$$

In a time  $t$ , the particle can, on the average, only travel a distance  $t^a$  before it encounters a trap with trapping time bigger than  $t$ , and gets stuck there. Eventually, it will exit from this trap, only to get stuck in other traps, some with even larger trapping times. Thus, if we examine all particles at some large time  $t$ , typically they would be stuck in, or just emerging from a trap with trapping time of order  $t$ .

For  $B = B_c$ , the average trapping time per step encountered by the walker is given by (13), except that summation over  $\ell$  is cutoff at a value  $\ell_{\max}$ , where  $\ell_{\max}$  is the typical value of the trapping time of the deepest trap encountered by a walker up to time  $t$ . It is easy to see that  $\ell_{\max}$  varies as  $\log t$  for large  $t$ , and hence the average velocity of the particle up to time  $t$  varies as  $\frac{1}{\log t}$ . Thus the average velocity decreases very slowly to zero at  $B = B_c$ . This very slow relaxation has been checked in large-scale simulations of this problem [17].

If  $B = B_c - \epsilon$ , then this slow decay of the velocity continues from its initial value of order 1, to the final value which is of order  $\epsilon$ . As the initial decay of velocity would be nearly the same as that for  $B = B_c$ , we see that typical relaxation time  $\tau(B)$  for the average velocity in an ensemble of non-interacting particles to reach the steady state value varies as

$$\tau(B) \sim \exp\left(\frac{A}{B_c - B}\right), \quad (15)$$

where  $A$  is a constant. Note that this relaxation is the overall relaxation time for a macroscopic observable (average current–density for an ensemble of non-interacting particles). We are not discussing the largest relaxation time, as that is infinite for all  $B$ , however small.

Thus, biased diffusion of non-interacting particles on a percolation network provides a very simple model where a fast rise of relaxation time near a dynamical phase-transition (‘the Vogel–Fulcher law’ of glassy dynamics) can be seen.

We see that disorder affects both the static and dynamical properties of the system in a very significant way. The affect on non-equilibrium properties like response functions is much more pronounced. I hope that further work will lead to a better understanding of these systems.

## References

- [1] See, for example, D Stauffer, *Phys. Rep.* **54**, 1 (1979)  
D Stauffer and A Aharony, *Introduction to percolation theory* (Taylor and Francis, London, 1991)  
G Grimmett, *Percolation theory* (Springer-Verlag, NY, 1999)
- [2] This definition differs from the definition often used in the literature, where  $\theta$  is usually defined in terms of the number of *unrooted* animals. As the number of unrooted clusters of  $n$  sites is just  $A_n/n$ , this definition gives a value 1 larger than the one used by us.  
For a recent treatment of this result, see D C Brydges and J Z Imbrie, preprint math-ph/0107005.
- [3] G Parisi and N Sourlas, *Phys. Rev. Lett.* **46**, 871 (1981)
- [4] C N Yang and T D Lee, *Phys. Rev.* **87**, 404 (1952)  
T D Lee and C N Yang, *Phys. Rev.* **87**, 410 (1952)
- [5] A R Day and T C Lubensky, *J. Phys.* **A15**, L285 (1982)  
J L Cardy, *J. Phys.* **A15**, 593 (1982)  
N Breuer and H K Janssen, *Z. Phys.* **B48**, 347 (1982)
- [6] D Dhar, *Phys. Rev. Lett.* **51**, 853 (1983)
- [7] H Kunz and B Souillard, *Phys. Rev. Lett.* **40**, 133 (1978)
- [8] T C Lubensky and A J McKane, *J. Phys.* **A14**, L157 (1981)
- [9] R B Griffiths, *Phys. Rev. Lett.* **23**, 17 (1969)
- [10] Stochastic evolution in Ising models, in *Stochastic processes: Formalism and applications*, edited by G S Agarwal and S Dattagupta, *Lecture notes in physics* (Springer, Berlin, 1983) vol. 184, p. 300
- [11] D Dhar, M Randeria and J P Sethna, *Europhys. Lett.* **5**, 485 (1988)
- [12] A J Bray, *Phys. Rev. Lett.* **60**, 720 (1988)
- [13] H Takano and S Miyashita, *J. Phys. Soc. Jpn.* **64**, 3688 (1995)
- [14] P G de Gennes, *La Recherche* **7**, 916 (1976)  
C D Mitescu, H Ottavi and J Roussenoq, *AIP Conf. Proc.* **40**, 377 (1978)  
S Havlin and D ben Avraham, *Adv. Phys.* **36**, 395 (1987)
- [15] H Bottger and V V Bryksin, *Phys. Status Solidi.* **B113**, 9 (1982)
- [16] M Barma and D Dhar, *J. Phys.* **C16**, 1451 (1982)
- [17] D Dhar and D Stauffer, *Int. J. Mod. Phys.* **C9**, 349 (1998)