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Wave scattering through classically chaotic cavities in the presence of absorption: A maximum-entropy model

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Abstract. We present a maximum-entropy model for the transport of waves through a classically chaotic cavity in the presence of absorption. The entropy of the S-matrix statistical distribution is maximized, with the constraint $\langle \mathrm{Tr}SS^{\dagger}\rangle = \alpha n$: n is the dimensionality of S, and $0 \leq \alpha \leq 1$. For $\alpha = 1$ the S-matrix distribution concentrates on the unitarity sphere and we have no absorption; for $\alpha = 0$ the distribution becomes a delta function at the origin and we have complete absorption. For strong absorption our result agrees with a number of analytical calculations already given in the literature. In that limit, the distribution of the individual (angular) transmission and reflection coefficients becomes exponential – Rayleigh statistics – even for $n = 1$. For $n \gg 1$ Rayleigh statistics is attained even with no absorption; here we extend the study to $\alpha < 1$. The model is compared with random-matrix-theory numerical simulations: it describes the problem very well for strong absorption, but fails for moderate and weak absorptions. The success of the model for strong absorption is understood in the light of a central-limit theorem. For weak absorption, some important physical constraint is missing in the construction of the model.

Keywords. Chaotic systems; wave propagation.

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1. Introduction

The interference arising from multiple scattering of classical waves (sound, microwaves, or light) can be affected as a result of absorption. In addition to its theoretical interest, this problem is important from an experimental point of view, because absorption is always present. This is in contrast with electron systems, where absorption is absent.

For diffusive transport the problem has been studied experimentally [1] as well as theoretically [2–4]. The phase-coherent reflection of light by a disordered medium which amplifies radiation has been analysed in refs [5,6] and the relation between absorption and dephasing in refs [4,7]. The statistical distribution of the reflection-matrix for a semi-infinite disordered waveguide has been obtained analytically for arbitrary absorption in ref. [5], and that of the S-matrix distribution for a chaotic cavity with absorption and one propagating mode in each of the two waveguides, in ref. [7].

In what follows we consider again, in the presence of absorption, the problem of the propagation of scalar waves through a cavity – whose classical dynamics would be chaotic

– connected to the outside through a number of waveguides supporting an arbitrary number of propagating modes.

Maximum-entropy models [8] have been successful in the description of chaotic scattering through cavities in the absence of absorption. These models rely on the idea of doing statistics directly on the S matrix of the system, on the basis of the information which is physically relevant for the problem in question.

We propose below an extension of such models to study the effect of absorption, in the belief that the present approach would complement the analytical derivations mentioned above.

2. The S matrix of the problem

The scattering of waves through a cavity can be described by an S matrix that relates incoming and outgoing amplitudes. For two N-channel waveguides the S matrix has dimensionality $n = 2N$, with the structure

$$S = \begin{bmatrix} r & t' \\ t & r' \end{bmatrix}, \tag{1}$$

where r, r' and t, t' are the N-dimensional matrices of reflection and transmission amplitudes with incidence from either waveguide.

In quantum-mechanics, the *universality classes* for S matrices were introduced by Dyson [9,10]. In the absence of any symmetries, the only restriction on S is unitarity, $SS^\dagger = I$, due to flux conservation (the *unitary* or $\beta = 2$ case). In the *orthogonal* case ($\beta = 1$), S is symmetric because of either time-reversal invariance (TRI) and integral spin, or TRI, half-integral spin and rotational symmetry. In the *symplectic* case ($\beta = 4$), S is self-dual because of TRI with half-integral spin and no rotational symmetry. The intuitive idea of *equal-a-priori probabilities* is expressed mathematically by the *invariant measure* on the matrix space under the symmetry operation for the class in question, giving the *circular* orthogonal, unitary and symplectic ensembles (COE, CUE, CSE).

As a simple example of the orthogonal case, consider the one-dimensional scattering of a spinless particle by the delta potential $u_0\delta(x)$, u_0 being a real number. The two-dimensional S matrix for this case is

$$S = \begin{bmatrix} \frac{u_0/(2ik)}{1-u_0/(2ik)} & \frac{1}{1-u_0/(2ik)} \\ \frac{1}{1-u_0/(2ik)} & \frac{u_0/(2ik)}{1-u_0/(2ik)} \end{bmatrix}, \tag{2}$$

which is clearly a unitary and symmetric matrix. Suppose we analytically continue u_0 to complex values: $u_0 = u_0' - iu_0''$, with $u_0'' > 0$ to ensure absorption. The sum of the reflection and transmission coefficients becomes

$$|r|^2 + |t|^2 = \frac{1 + \frac{|u_0|^2}{4k^2}}{1 + \frac{|u_0|^2}{4k^2} + \frac{u_0''}{k}}, \tag{3}$$

which is indeed < 1 for the absorptive case, $u_0'' > 0$: because of loss of flux, the resulting S is now *sub-unitary*, meaning that the eigenvalues of the Hermitian matrix $h = SS^\dagger$ lie in the interval between zero and one. However, the *symmetry* property, $S = S^T$, is not altered. These properties can be generalized to more complex problems, in which we still speak of the orthogonal case. In the scattering problem of scalar classical waves, the orthogonal case is the physically relevant one and is the one we deal with in the next section.

3. The invariant measure and the maximum-entropy model

In the analysis of ref. [8], the starting point to construct a random S-matrix theory in the absence of absorption is the definition of equal-*a-priori*-probabilities in the space where the S matrix for the problem is defined. Similarly, in the present case where absorption is present we introduce a *uniform weight in the space of sub-unitary matrices*. Following ref. [11] (pp. 63, 64), we define the measure as

$$\mathrm{d}\mu_{\mathrm{sub}}(S) = C\theta(I - SS^{\dagger}) \prod_{a \le b} \mathrm{d}x_{ab}\mathrm{d}y_{ab}, \tag{4}$$

where $S_{ab} = x_{ab} + iy_{ab}$ and the step function $\theta(H)$ (for a Hermitian matrix H) is nonzero for $H > 0$ (i.e. for H positive definite, so that all its eigenvalues are positive definite) and thus selects *sub-unitary* matrices.

A complex $n \times n$ sub-unitary matrix can be written in the *polar representation* (refs [11], pp. 63, 64 and [7]) as

$$S = UDU^T, \tag{5a}$$

where the index T indicates 'transpose'. The matrix D is diagonal, with the structure

$$D_{ab} = \rho_a \delta_{ab}, \quad a = 1, \ldots, n, \tag{5b}$$

with $0 \le \rho_a \le 1$.

The explicit expression of the above measure (4) in terms of the independent parameters of the polar representation (5) is (ref. [11], pp. 63, 64)

$$\mathrm{d}\mu_{\mathrm{sub}}(S) = \prod_{a<b}^{n} |\rho_a^2 - \rho_b^2| \prod_c \rho_c \mathrm{d}\rho_c \mathrm{d}\mu(U), \tag{6}$$

where $\mathrm{d}\mu(U)$ is the invariant measure for the unitary group in n dimensions.

More general statistical distributions of sub-unitary matrices carrying more information than the equal-*a-priori* probability distribution (4) can now be constructed using $\mathrm{d}\mu_{\mathrm{sub}}(S)$ as a starting point and writing

$$\mathrm{d}P(S) = p(S)\mathrm{d}\mu_{\mathrm{sub}}(S). \tag{7}$$

In what follows we propose an information-theoretic criterion to choose $p(S)$. The *information-theoretic entropy* $\mathscr{S}$ of the S-matrix distribution is defined as [12]

$$\mathscr{S}[p(S)] = -\int p(S) \ln p(S) \mathrm{d}\mu_{\mathrm{sub}}(S). \tag{8}$$

We propose, as a model, *the distribution that maximizes the information entropy* (8), subject to the *constraint that the absorption has a given average strength*. Mathematically, this is expressed by the average departure from unitarity of our S matrices; we thus write the constraint as

$$\left\langle \mathrm{Tr}SS^{\dagger} \right\rangle = \alpha n, \quad 0 \leq \alpha \leq 1. \tag{9}$$

Thus $\alpha = 0$ corresponds to complete absorption and $\alpha = 1$ to lack of absorption. We find

$$\mathrm{d}P(S) = C\mathrm{e}^{-\nu \mathrm{Tr}SS^{\dagger}} \mathrm{d}\mu_{\mathrm{sub}}(S), \tag{10a}$$

where the constant C and the Lagrange multiplier ν ensure normalization and the fulfillment of the constraint (9). The limit of no absorption, $\alpha = 1$, is attained when the Lagrange multiplier $\nu \to -\infty$ and the distribution concentrates on the unitarity sphere. The limit of complete absorption, $\alpha \to 0$, is attained when $\nu \to +\infty$: the distribution then becomes a δ-function at the origin and there is no exit signal. Using eqs (5a), (5b) and (6), we write the distribution (10a) as

$$\mathrm{d}P(S) = C \exp\left[-\nu \sum_a \rho_a^2\right] \prod_{a<b}^{n} |\rho_a^2 - \rho_b^2| \prod_c \rho_c \mathrm{d}\rho_c \mathrm{d}\mu(U). \tag{10b}$$

The result of eq. (10b) – Laguerre ensemble for the variables ρ_a^2 – coincides, for strong absorption, with that obtained in ref. [5] for diffusive waveguides and in ref. [7] for cavities with $N = 1$.

The 'ansatz' (10) entails a number of properties and restrictions. First, the average of S under the distribution (10) is

$$\left\langle S_{ab} \right\rangle = C \sum_d \int U_{ad} U_{bd} \rho_d \exp\left[-\nu \sum_a \rho_a^2\right] \prod_{a<b}^{n} |\rho_a^2 - \rho_b^2| \prod_c \rho_c \mathrm{d}\rho_c \mathrm{d}\mu(U). \tag{11}$$

From the properties of $\mathrm{d}\mu(U)$ [13] one verifies that

$$\int U_{ad} U_{bd} \mathrm{d}\mu(U) = 0, \tag{12}$$

and thus $\left\langle S_{ab} \right\rangle = 0$. Therefore, applications of the model (10) should be restricted to cases where *prompt processes* are absent and so $\langle S \rangle = 0$ [8,12,14]. Second, we comment on the so-called analyticity–ergodicity requirements

$$\left\langle S_{a_1 b_1}^{n_1} \cdots S_{a_p b_p}^{n_p} \right\rangle = \left\langle S_{a_1 b_1} \right\rangle^{n_1} \cdots \left\langle S_{a_p b_p} \right\rangle^{n_p} \tag{13}$$

which are discussed in ref. [12] for unitary matrices. As we shall see in the next section, the S matrix for uniform volume absorption (in quantum mechanics, a constant potential $-iW$ throughout the cavity) can be obtained from that without absorption evaluating it at the complex energy $E + iW$, i.e. $S(E + iW)$. It follows that the argument of ref. [12] leading to the analyticity–ergodicity requirements for unitary matrices applies here as well. That eq. (13) is satisfied for the distribution (10) can be seen writing the averages in question as in eq. (11).

4. Results

Some of the predictions of the present model have been compared with RMT numerical simulations, in which the S matrices are constructed as

$$S(E) = -\frac{I_n + iK(E)}{I_n - iK(E)}, \tag{14}$$

with

$$K_{ab}(E) = \sum_{\lambda} \frac{\gamma_{\lambda a}\gamma_{\lambda b}}{E_{\lambda} - E}. \tag{15}$$

The matrix I_n above is the $n \times n$ unit matrix. The E_{λ}'s are generated from an 'unfolded' zero-centered GOE [15] with average spacing Δ. The $\gamma_{\lambda a}$'s are statistically independent, real, zero-centered Gaussian random variables. At $E = 0$,

$$\langle S_{ab}\rangle = -\frac{1 - \pi\left\langle \gamma_{\lambda a}^2 \right\rangle/\Delta}{1 + \pi\left\langle \gamma_{\lambda a}^2 \right\rangle/\Delta}\delta_{ab}, \tag{16}$$

and we require $\langle S\rangle = 0$. In the quantum case, addition of a constant imaginary potential $-\,iW$ inside the cavity makes the E_{λ}'s complex and equal to $E_{\lambda} - iW$ (see also ref. [7]). This is equivalent to evaluating the above expressions at the complex energy $E + iW$, which makes $S(E + iW)$ subunitary.

The simplest case is that of a cavity with one waveguide supporting only one open channel ($n = 1$): S is thus the reflection amplitude back to the only channel we have. Equation (5) for S in the polar representation reduces to $S = \rho \exp i\theta$. Thus ρ^2 represents the reflection coefficient R. The uniform weight (6) and the distribution (10) reduce to

$$\mathrm{d}\mu_{\mathrm{sub}}(S) = \rho\,\mathrm{d}\rho\,\mathrm{d}\theta, \quad \mathrm{d}P(S) = C\mathrm{e}^{-\nu\rho^2}\rho\,\mathrm{d}\rho\,\mathrm{d}\theta. \tag{17}$$

The R-probability density is

$$w(R) = D\mathrm{e}^{-\nu R}, \qquad 0 \le R \le 1 \tag{18}$$

D and ν being given by

$$D = \frac{\nu}{1 - \mathrm{e}^{-\nu}}, \quad \langle R\rangle = \frac{1}{\nu} - \frac{1}{\mathrm{e}^{\nu} - 1} = \alpha. \tag{19}$$

For weak absorption, $\alpha \approx 1$, $\nu \to -\infty$ and the distribution (18) becomes strongly peaked around $R = 1$, i.e. the unitarity circle, reducing to the one-sided delta function $\delta(1 - R)$ as $\alpha \to 1$. In the other extreme of *strong absorption*, $\nu \to +\infty$, $\alpha \approx 1/\nu$, $D \approx 1/\alpha$ and

$$w(R) \approx \langle R\rangle^{-1}\,\mathrm{e}^{-R/\langle R\rangle}, \tag{20}$$

which is *Rayleigh's distribution* with the average $\langle R\rangle = \alpha$.

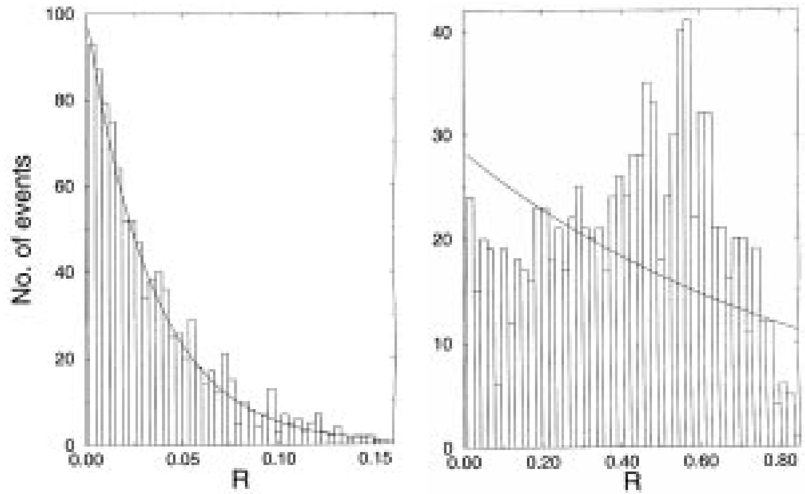

Figure 1. The distribution of the reflection coefficient R for a cavity with one wave guide, supporting one channel: (left) in the presence of strong absorption ($\langle R \rangle = \alpha = 0.034$); (right) for moderate absorption ($\langle R \rangle = \alpha = 0.410$).

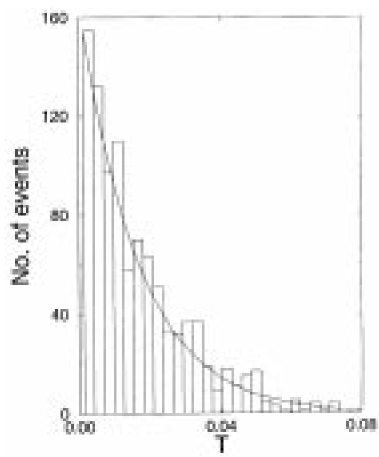

Figure 2. The distribution of the transmission coefficient T for a cavity with two waveguides, each supporting one open channel, in the presence of strong absorption ($\alpha = 0.049$).

Figure 1 shows for $n = 1$ the results of the RMT numerical simulation described above (as histograms), compared with the present model for the corresponding value of α (continuous curves). For strong absorption the model works very well, the agreement with Rayleigh's law being excellent, while for moderate and weak absorptions the model fails.

Figure 2 shows the distribution of the transmission coefficient T obtained from a RMT simulation for $n = 2$ and strong absorption. The agreement with the Rayleigh distribution with the centroid $\langle T \rangle = \alpha/3$ predicted by the present model is excellent; $\langle R \rangle$ was also checked and found to agree with the theoretical result $2\alpha/3$.

That individual transmission and reflection coefficients attain a Rayleigh distribution for strong absorption can be understood as follows: $S_{ab}(E + iW)$ coincides [8,12] with the energy average of $S_{ab}(E)$ evaluated with a Lorentzian weighting function of half-width W. If $\Gamma^{\rm corr}$ is the correlation energy, W can be thought of as containing approximately $m = W/\Gamma^{\rm corr}$ approximately independent intervals. If $m \gg 1$, by the central-limit theorem the real and imaginary parts of S_{ab} attain a Gaussian distribution, and $\left|S_{ab}\right|^2$ an exponential distribution. This seems to be the situation captured by the maximum-entropy approach.

Summarizing, the results presented in this paper indicate that wave scattering through classically chaotic cavities in the presence of strong absorption can be described in terms of a maximum-entropy model, while for weak absorption some relevant physical information is missed in our model.

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