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Infinite range correlations of intensity in random media

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Abstract. We study a new type of long-range correlations for waves propagating in a random medium. These correlations originate from scattering events which take place close to a point source. The scattered waves propagate by diffusion to distant regions. In this way long range correlations, between any pair of distant points, are established.

Keywords. Infinite range correlation; four-point function.

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1. Introduction

A wave propagating in a random medium undergoes multiple scattering. As a result, a highly irregular distribution of intensity in the sample is produced. It is usually referred to as a speckle pattern and it should be described in statistical terms. One of its important characteristics is the correlation function $\langle I(\vec{r}_1)I(\vec{r}_2)\rangle$, where $I(\vec{r})$ is the wave intensity at point $\vec{r}$ and the angular brackets designate the average over the ensemble of macroscopically identical random samples. This correlation function exhibits a short-range part, which determines the typical size of a speckle, i.e. of a 'bright' or 'dark' spot in the sample. In addition, however, there are also weak long-range correlations, which are due to a diffusive propagation of the locally large intensity fluctuations to distant regions in space. A comprehensive review of the subject is given in refs [1,2].

Recently a new type of long-range correlations in intensity has been proposed [3]. This correlation is due to scattering events in the immediate vicinity of the radiation source, which is assumed to be a point source embedded in the infinite three-dimensional disordered medium. If for instance, the concentration of scatterers near the source is larger than the average value, then a build-up of intensity near the source would occur. This intensity enhancement will propagate, by diffusion, and will be 'felt' anywhere in the sample. This is the mechanism for the long-range (essentially, infinite) correlation discussed in [3] and designated there as C_0. This type of correlation has been further studied in [4], where its non-universal character, i.e. dependence on the local properties of disorder, was emphasized. The authors of [4] have considered the dependence of the C_0-term on the size of the source and on the correlation length of the disorder, and have suggested to use the non-universality of the C_0 correlations for probing the local properties of the disorder.

In this paper we extend the calculation of [3] in a different direction and study the contribution of the C_0-term to the general four-point function, defined as

$$\frac{\langle \Delta I_\omega(\vec{r},\vec{R})\Delta I_\omega(\vec{r}',\vec{R}')\rangle}{\langle I_\omega(\vec{r},\vec{R})\rangle\langle I_\omega(\vec{r}',\vec{R}')\rangle} \equiv C(\vec{r},\vec{r}';\vec{R},\vec{R}'). \tag{1}$$

Here $I_\omega(\vec{r},\vec{R})$ is the intensity produced at point $\vec{r}$ by a point source, at frequency ω, located at $\vec{R}$. ΔI is the deviation of intensity from its average value. Such four-point function has been recently studied in [5] , both experimentally and theoretically, and its structure with respect to the two arguments, $|r-r'|$ and $|R-R'|$, has been identified. The contribution of the C_0 term, however, has not been discussed. This four-point function can be further generalized by assigning different frequencies, ω and ω', to the two sources. We shall use the standard diagram technique (refs [1,2]) which is appropriate for weak disorder, when the mean free path ℓ is much greater than the radiation wavelength $\lambda = \frac{2\pi}{k_0} = \frac{2\pi c}{\omega}$ (c is the speed of propagation in the average medium). In §2 we briefly discuss the average field and intensity. Our results for the four-point function are presented in §3 and the conclusions are summarized in §4.

2. Average field and intensity

For scalar waves, we have the following equation for the field (the Green function) at point $\vec{r}$, due to a monochromatic source at point $\vec{R}$:

$$\left\{\nabla^2 + k_0^2[1+\mu(\vec{r})] + i\eta\right\} G_\omega(\vec{r},\vec{R}) = \delta(\vec{r}-\vec{R}), \tag{2}$$

where $\mu(\vec{r})$ represents the fractional fluctuation in the dielectric constant and η is a positive infinitesimal. We assume white noise Gaussian statistics for $\mu(\vec{r})$:

$$\langle \mu(\vec{r})\rangle = 0, \quad \langle \mu(\vec{r})\mu(\vec{r}')\rangle = u\delta(\vec{r}-\vec{r}') \tag{3}$$

and higher order correlations are factorized into all possible pairs. The constant u characterizes the strength of the disorder. The Green function $G_\omega(\vec{r},\vec{R})$ can be expanded in the Born series and averaged, term by term (figure 1). A partial summation of the resulting series is done with the help of the Dyson equation. In the leading order, with respect to the self-energy, one obtains the following expression for the averaged field:

$$\langle G_\omega(\vec{r},\vec{R})\rangle = -\frac{1}{4\pi\,|\,\vec{r}-\vec{R}\,|}\exp\left[\left(ik_0 - \frac{1}{2\ell}\right)|\vec{r}-\vec{R}|\right], \tag{4}$$

where $\ell = 4\pi/uk_0^4$ and the parameter $\frac{1}{k_0\ell}$ is assumed to be small (weak disorder).

The intensity at point $\vec{r}$ is defined as $I_\omega(\vec{r}) = |G_\omega(\vec{r},\vec{R})|^2$. An example of a diagram for the intensity before averaging is given in figure 2. Upon averaging, the leading contribution to the intensity comes from the ladder diagram (figure 3). All solid lines in figure 3 correspond to averaged Green functions. The diffusion ladder (the shaded box in figure 3) is equal to

$$L(\vec{r},\vec{R}) = \frac{3}{\ell^3|\vec{r}-\vec{R}|}. \tag{5}$$

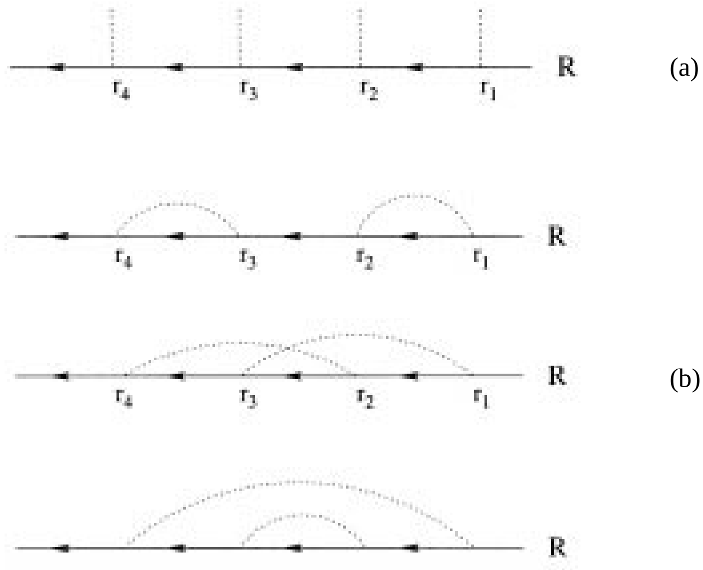

Figure 1. (**a**) A diagram in the Born series before averaging. (**b**) The three diagrams, obtained from (a) upon averaging.

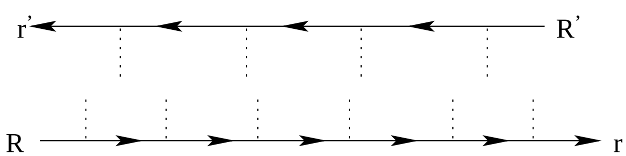

Figure 2. An example of a diagram for $G^*_\omega(\vec r,\vec R)G_\omega(\vec r',\vec R')$.

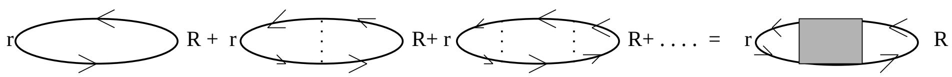

Figure 3. The ladder diagram.

Since the average Green functions are rapidly decaying in space, at a distance of a few ℓ, the vertices in figure 3, connecting the ladder to the source ($\vec R$) and the observation point ($\vec r$), are short-range objects. To each vertex corresponds a number

$$\int \mathrm{d}^3 r_1 \langle G_\omega(\vec r,\vec r_1)\rangle\langle G^*_\omega(\vec r,\vec r_1)\rangle = \frac{\ell}{4\pi}. \tag{6}$$

Thus, the average intensity at $\vec r$ created by a source at $\vec R$ is

$$\langle I(r,R)\rangle = \left(\frac{\ell}{4\pi}\right)^2 L(\vec r,\vec R) = \frac{3}{16\pi^2\ell|\vec r-\vec R|}. \tag{7}$$

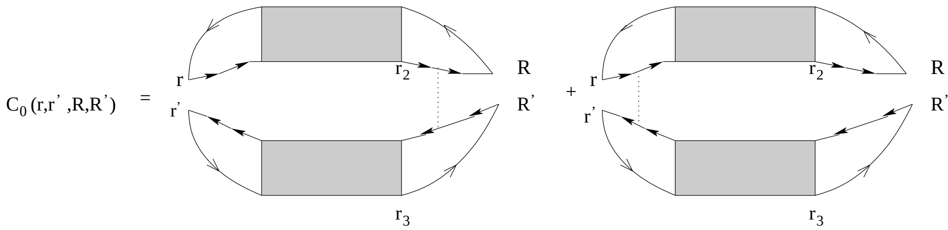

Figure 4. Diagrams for C_0. Each diagram should be assigned a combinatorial factor 2 since the dashed line could connect the two external lines, instead of the internal ones.

3. Four-point function

The four-point function $C(\vec{r},\vec{r}',\vec{R},\vec{R}')$ was defined in eq. (1). In particular, for $\vec{R}'=\vec{R}$, this function describes correlations of intensity at points $\vec{r}$ and $\vec{r}'$, for a source at $\vec{R}$. On the other hand, for $\vec{r}'=\vec{r}$, it describes the auto correlation of intensity at point $\vec{r}$, when the source is moved from $\vec{R}$ to $\vec{R}'$.

The C_0-term of the four-point function is given by the two diagrams in figure 4. The first diagram describes correlation due to scattering of the wave near the source, with subsequent diffusion to detectors at $\vec{r}$ and $\vec{r}'$. This diagram becomes negligible when $\Delta R \equiv |\vec{R}-\vec{R}'|$ is much larger than the mean free path ℓ. The second diagram corresponds to a different process: first, the wave propagates diffusively towards the detectors, whereas the interference process (leading to intensity correlations) occurs close to the observation points. This diagram is negligible if $|\vec{r}-\vec{r}'| \equiv \Delta r \gg \ell$.

The evaluation of the diagrams is straightforward. The ladders are given by eq. (5) and it remains to compute the vertex shown separately in figure 5. This is a short-range object, given by

$$V(\Delta R) = \frac{4\pi}{\ell}\int \mathrm{d}^3r_0\mathrm{d}^3r_2\mathrm{d}^3r_3\langle G_\omega(\vec{r}_2-\vec{R})\rangle\langle G^*_\omega(\vec{r}_2-\vec{r}_0)\rangle\langle G^*_\omega(\vec{r}_0-\vec{R})\rangle \cdot\langle G_\omega(\vec{r}_3-\vec{r}_0)\rangle\langle G_\omega(\vec{r}_0-\vec{R}')\rangle\langle G^*_\omega(\vec{r}_3-\vec{R}')\rangle. \tag{8}$$

Integration over r_2 yields

$$\int \mathrm{d}^3r_2\langle G_\omega(\vec{r}_2-\vec{R})\rangle\langle G^*_\omega(\vec{r}_2-\vec{r}_0)\rangle = \frac{\ell}{4\pi}\frac{\sin k_0\rho}{k_0\rho}\mathrm{e}^{-\rho/2\ell}, \tag{9}$$

where $\rho = |\vec{r}_0-\vec{R}|$. Similarly, integration over r_3 gives the expression in eq. (9) but with ρ replaced by $\rho' = |\vec{r}_0-\vec{R}'|$. The resulting expression for V is

$$V(\Delta R) = \frac{\ell}{4\pi}\frac{1}{16\pi^2}\int \frac{\sin(k_0\rho)\sin(k_0\rho')}{k_0^2\rho^2\rho'^2}\mathrm{e}^{-ik_0(\rho-\rho')}\mathrm{e}^{-\rho+\rho'/\ell}\,\mathrm{d}^3r_0. \tag{10}$$

Since the main contribution to the integral comes from the region of several wavelengths near $\vec{R}_1,\vec{R}_2$ the second exponent can be replaced by 1, and therefore

$$V(\Delta R) = \frac{\ell}{4\pi}\frac{1}{16\pi^2}\int \frac{\sin(k_0\rho)\sin(k_0\rho')}{k_0^2\rho^2\rho'^2}\mathrm{e}^{-ik_0(\rho-\rho')}\,\mathrm{d}^3r_0. \tag{11}$$

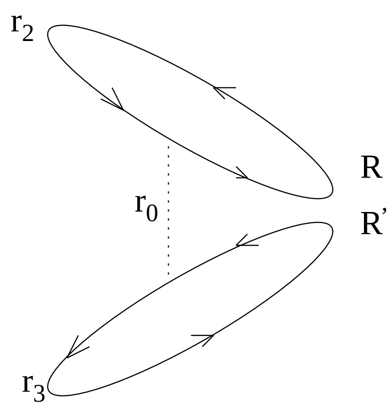

Figure 5. The vertex.

The analytic expression corresponding to the first diagram in figure 4 is

$$f(\Delta R) = -\frac{1}{8\ell k_0^2}\left(-\frac{4\pi}{\Delta R} si(2k_0\Delta R) - \frac{\pi^2}{\Delta R} + \frac{2}{\Delta R}\int_{-1}^{1}\cos(2k_0\Delta R\nu)\frac{1}{\nu}\ln\left(\frac{1-\nu}{1+\nu}\right)\mathrm{d}\nu\right), \tag{12}$$

where *si* is the integral sine.

Note that this expression does not depend on the distance between r and r', for the following reason: The dependence on r and r', for the first diagram in figure 4, enters only via the two diffusion ladders propagating to the points r and r'. This contributes the product of two average intensities, which cancel the denominator in the definition of the four-point function, eq. (1). Thus, indeed, the expression in eq. (12) describes infinite range correlation with respect to two observation points, r and r'. The reason for this correlation is that the information on the configuration of the disorder near the source is carried, by diffusion, to infinity. Let us emphasize that this expression holds only as long as $\Delta R \ll l$, and that in the opposite case it becomes exponentially small. The function $f(x)$, defined in eq. (12), reaches its maximal value at zero, $f(0) = \frac{\pi}{k_0 l}$. This function, normalized to its maximal value, is shown in figure 6.

The second diagram in figure 4 is computed in complete analogy with the first one, the only difference being that ΔR is replaced by Δr. Thus, the resulting expression for the correlation function $C_0(r,r';R,R')$ is

$$C_0(r,r';R,R') = f(\Delta R) + f(\Delta r). \tag{13}$$

In particular, for a single source ($\Delta R = 0$) we obtain

$$C_0(\Delta r) = f(0)\left(1 + \frac{f(x)}{f(0)}\right). \tag{14}$$

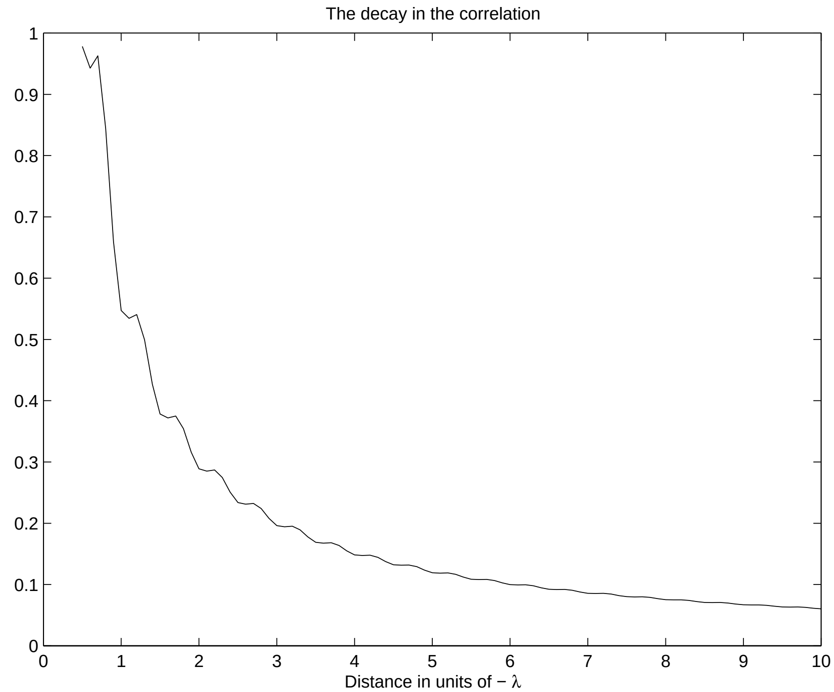

Figure 6. The function $f(x)/\frac{\pi}{k_0\ell}$. x is measured in units of the wavelength λ.

The maximal correlation $C_0(0) = 2f(0) = \frac{2\pi}{k_0 l}$ is achieved for $\Delta r = 0$ and it gradually drops with increase in Δr. For Δr larger than the mean free path ℓ, it saturates at half of its maximal value. Note that $C_0(0)$ describes a (small) correction, $\frac{2\pi}{k_0 l}$, to the leading value of the second moment of the normalized intensity $\tilde{I}(\vec{r}) \equiv \frac{I(\vec{r})}{\langle I(\vec{r})\rangle}$. The leading value is equal to 2 and it corresponds to the Rayleigh distribution $P_0(I) = \frac{1}{\langle I\rangle}\mathrm{e}^{-I/\langle I\rangle}$. The knowledge of the correction enables one to compute deviations from the Rayleigh distribution [3,6].

4. Conclusions

We have studied infinite range correlations, the C_0-term, in the four-point function. This function, in its general form, describes correlations for two sources and two detectors. The C_0 contribution has a very simple structure, being a sum of two identical functions, one depending only on the distance between the sources and the other only on the distance between the detectors. For a single source ($\Delta R = 0$) the correlation function, $C_0(\Delta r)$ achieves its maximal value, $\frac{2\pi}{k_0 l}$, for $\Delta r = 0$ and gradually drops, with oscillations, to half of this value when Δr approaches infinity.

The C_0-term, unlike other types of long-range correlations (C_2 and C_3, [1,2]), is dominated by the configuration of the disorder near the source. In the considered, three-dimensional geometry the C_0-term dominates over the other types of long-range correlations (it can also be distinguished by its insensitivity to changes in the source frequency). The C_0-term is also present in the lower dimensionality. For instance, in the quasi-one-dimensional geometry, which is often used in experiments [5,7], the relative importance of the C_0-term will depend on the length of the tube [3].

Finally, let us emphasize that this paper consideres a wave propagating in an open system. In a closed system one is usually interested in the statistics of eigenmodes and eigenfrequencies. Correlations, of the same origin as the C_0-term, also exist in that case [8].

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