

Theory of non-hermitian localization in one dimension: Localization length and eigenenergies

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Abstract. We recall some basic aspects of the pinning of flux lines in a superconducting cylindrical shell subjected to a depinning magnetic field, as well as its description by the quantum mechanics of a disordered ring with an imaginary vector potential proportional to the depinning field (N Hatano and D R Nelson, *Phys. Rev.* **B56**, 8651 (1997)). We then discuss our recent analysis of the pinning-depinning transition in terms of an explicit solution for the inverse localization length of the eigenstates of the non-hermitian quantum system for weak disorder. Our results as to the nature of the non hermitian quantum states, differ qualitatively from earlier studies which did not examine the detailed properties of the localization length. Nevertheless we obtain a well-defined simple picture for the pinning-depinning transition of flux lines. We discuss furthermore a new exact calculation of localized state eigenenergies for weak disorder, which we compare with previous analytic and numerical results.

Keywords. Pinning of flux lines; localization; non-hermitian quantum mechanics.

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1. Introduction

The subject of my talk is the study of the localization of the eigenstates of a 1D disordered ring subjected to a constant imaginary vector potential (IVP). The theory of localization is central to much of the research done by Prof. N Kumar, be it his work on localization in random Bethe lattices and on spectral diffusion problems in the 1970s, or his pioneering studies of distributions of resistance and conductance in the framework of the scaling theory of localization and of the metal-insulator transition in the 1980s.

The plan of the talk is defined by the topics in the following sections. In §2A, I briefly describe the physical situation which underlies the non-hermitian 1D quantum hamiltonian, namely the study of pinning of flux lines by random columnar defects in a superconducting cylindrical shell subjected to a magnetic depinning field [1,2]. This depinning field leads to the IVP in the Schrödinger equation for the corresponding quantum system derived by Hatano and Nelson [2] (hereafter referred to as HN), which is discussed in 2B. I will focus on the nature of the localization-delocalization transition found by HN in this system, as corresponding to depinning of flux lines. Finally, 2C contains some critical remarks on the analysis of HN which have partly motivated our work on this problem and this is discussed in §§3 and 4. In §3, I present the derivation of a simple exact expression for the inverse localization length (or Lyapunov exponent) in a non-hermitian ring for weak

disorder (eq. (30) and its explicit form (34)). The properties of the inverse localization length and their significance for the nature of the depinning transition of flux lines, which is quite different from the transition found by HN, are also discussed in §3. In §4 I include a similar exact calculation of the eigenvalues of the localized states in the ring. This is of particular interest in view of the availability of extensive numerical results for eigenvalues of the 1D non-hermitian random system, which I am able to explain quantitatively in the case of complex eigenvalues.

2. Flux lines and non-hermitian quantum mechanics

A. Flux line pinning

The pinning of flux lines by random columnar defects in a superconducting cylindrical shell of height L_τ in the τ -direction and perimeter L_x in the x -direction is depicted in figure 1 [1,2]. A flux line represented by the wavy line is induced by the magnetic field \vec{H}_\parallel and is pinned to a columnar defect parallel to the cylinder axis. A current \vec{I} threading the ring creates a transverse depinning field \vec{H}_\perp . For a weak \vec{H}_\perp the flux line is depinned near the top and the bottom of the cylindrical surface while remaining pinned by the defect in the central part. This localization of the flux line must be viewed as a result of the competition between the effect of the disorder of the columnar defects and the effect of the finite depinning field \vec{H}_\perp . The degree of depinning is measured by the averaged relative curvilinear displacement of the top and bottom ends of the flux line $\langle x \rangle_{L_\tau} - \langle x \rangle_0$, which we refer to in the following as the transverse displacement of the line. In the small field, pinning regime this displacement is smaller than the perimeter, L_x , of the cylindrical surface. For large fields, such that the transverse displacement is larger than L_x , the

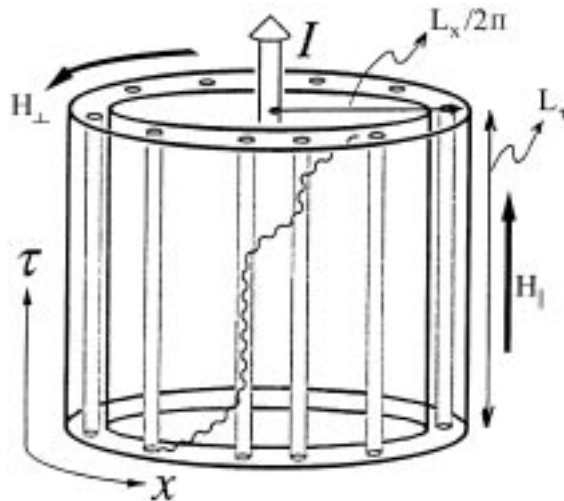


Figure 1. Flux line induced by the field H_\parallel in a superconducting cylindrical shell with random columnar defects. The transverse depinning field H_\perp is generated by the current I threading the ring.

flux line is depinned [2]. The depinning- or delocalization-edge is defined by the threshold depinning field at which the averaged transverse displacement of a flux line is of the order of the perimeter L_x .

B. Random non-hermitian Schrödinger equation

Hatano and Nelson have mapped the flux line system in figure 1 onto a non-hermitian 1D quantum system described by the hamiltonian

$$H = \frac{1}{2m}(p_x - ig)^2 + V(x), \quad (1)$$

which is defined on a ring of radius $L_x/2\pi$ in a plane parallel to the bases of the cylinder of figure 1, with periodic boundary conditions for the corresponding Schrödinger equation

$$H \varphi(x) = E \varphi(x). \quad (2)$$

Here ig is a constant IVP proportional to the depinning field $|\vec{H}_\perp|$ and $V(x)$ corresponds to the random potential of the columnar defects. The correspondence between the 2D flux-line system and the 1D quantum system (1-2) implies [1,2], in particular, that the τ -coordinate axis of the former system is identified with an imaginary time direction for the quantum system. Also, since H is non-hermitian, HN define left and right eigenvectors, $\varphi_L(x)$ and $\varphi_R(x)$, which, in their mapping, correspond to the probability distributions of a flux line at $\tau = 0$ and $\tau = L_\tau$, respectively [2].

The distinctive feature of (1) is, of course, that it may have complex eigenvalues. Now, HN have shown that the Schrödinger equation (2) leads to a localization-delocalization transition in response to a sufficiently large IVP. This is in contrast to what is known in the hermitian case ($g = 0$) for an open chain with free-space boundary conditions, namely that all states are localized for any disorder and correspond to real eigenenergies. The discovery of this delocalization transition associated with a pinning-depinning transition for the flux line system, has generated considerable interest in non-hermitian localization in the last few years [3–15].

We now briefly recall the argument of HN for the existence of a localization-delocalization transition in the non-hermitian system (1-2) [2]. By choosing the solutions of (1-2) in the form

$$\varphi(x) = e^{-gx/\hbar} \psi(x), \quad (3)$$

which one may look upon as an imaginary gauge transformation, (2) transforms into

$$\left(\frac{p_x^2}{2m} + V(x) \right) \psi(x) = E \psi(x), \quad (4)$$

from which the IVP is eliminated. On the basis of this transformation, HN assume that, for small g , the eigenvalues of (2) coincide with the real eigenvalues $E \rightarrow E_n$ of (4), and the functions $\psi(x)$ in (3) with the corresponding eigenfunctions $\psi(x) \rightarrow \psi_n(x)$, provided that the resulting solutions of (3), $\varphi(x) \rightarrow \varphi_n(x)$, are normalizable. Assuming then that eigenstates in a ring for $g = 0$ are localized about a localization centre x_0 , as in an open

linear chain, HN obtain the following expressions for $\varphi_n(x)$ (with ξ_n the zero-field, energy-dependent localization length):

$$\varphi_n(x) \sim e^{\pm \frac{g}{\hbar}(x-x_0) - \frac{|x-x_0|}{\xi_n}}, \quad (5)$$

where the \pm signs correspond to the left and right eigenfunctions, respectively [2]. Observing then that (5) is normalizable if

$$\frac{\hbar}{\xi_n} > |g|, \quad (6)$$

HN predict the existence of real energy exponentially localized eigenstates at low fields such that (6) is obeyed.

On the other hand, for $|g| > \hbar/\xi_n$ the wavefunctions (5) are non-normalizable and thus unacceptable. In fact (5) indicates that e.g. the localization length of the left eigenfunctions, $(\xi_n^{-1} - |g|/\hbar)^{-1}$, in the direction of \vec{g} diverges for $\hbar/\xi_n = |g|$, which implies that the latter is non-zero for $|x| \rightarrow \infty$ i.e. it is delocalized. This suggests that the point $\hbar/\xi_n = |g|$ is the delocalization point for the state φ_n beyond which one would expect this state to take the form of the extended Bloch waves. This is, in fact, clear for $|g| \rightarrow \infty$ where the random potential is negligible. In this case the solution of (2) are indeed the extended plane waves

$$\varphi(x) = e^{iqx}, \quad (7)$$

with real $q = 2\pi k/L_x$, $k = 0, \pm 1, \pm 2, \dots$, as shown by the periodic boundary conditions for the wavefunctions of the ring [2],

$$\varphi(x + L_x) = \varphi(x). \quad (8)$$

On the other hand, the eigenvalues of these wavefunctions are complex:

$$E \equiv E_q = \frac{\hbar^2(q - ig)^2}{2m}. \quad (9)$$

In fact HN have argued that complex eigenvalues are a distinctive specific feature of the delocalized eigenstates. According to the above discussion, at a given $|g|$ -value, those zero-field localized states ψ_m with localization lengths $\xi_m > \hbar/|g|$ have been replaced by the corresponding delocalized states with complex eigenvalues. This leads to the HN picture of a localization-delocalization transition in a non-hermitian field where localized states with real, field-independent eigenenergies, and delocalized states with complex eigenenergies coexist in contiguous non-overlapping domains of the $\text{Re}E$ -axis.

C. Critique of the Hatano–Nelson delocalization transition

An obvious objection to the real energy localized eigenstate solutions (5) is that they ignore the boundary condition

$$\psi(x + L_x) = e^{\frac{gL_x}{\hbar}} \psi(x), \quad (10)$$

which follows from (8) and (3). This boundary condition is the analog of the familiar twisted boundary condition [16,17] in the case where a real vector potential is applied to a ring system. The condition (10) generally implies that the eigenvalues of (2) are complex. It shows, in fact, that the g -dependence of $\psi(x)$ and hence that of the eigenvalues of (4) will be negligible (which amounts to assuming that $\psi(x+L_x) = \psi(x)$) only if $|g|L_x/\hbar \ll 1$. This condition is not met for typical values of $|g|c/\hbar$ and a ring of size $L_x = 10^3 c$ considered in numerical calculations of HN [2]. Another consequence of the g -dependence of $\psi(x)$ is the breakdown of the additivity property of the effects of the random potential and of the non-hermitian field in the inverse localization length of the left and the right eigenstates in the direction of g [2],

$$\frac{1}{\xi_n^{l,r}} = \frac{1}{\xi_n} \mp \frac{|g|}{\hbar}, \quad (11)$$

which is expected to be invalid indeed when $|g| \sim \hbar/\xi_n$.

3. Localization length in a non-hermitian ring

In view of the doubts raised in §2 with regard to the existence of real eigenvalue localized states in a 1D non hermitian ring, we have recently re-examined the nature of the eigenstates in terms of a first principles analytic calculation of the inverse localization length for weak disorder [18]. We recall that the correspondence between the probability distribution of a flux line in a superconducting ring and the eigenvectors of a non-hermitian quantum system [2] shows a direct relationship between the localization length of the quantum states and the degree of pinning of a flux line: if the localization length is smaller than the ring perimeter the flux line remains localized while being partially depinned at its extremities; for larger localization lengths the flux line is depinned.

Instead of the continuous system described by (1-2) we shall analyse an equivalent tight-binding model.

A. Tight-binding Schrödinger equation

A discrete form of the Schrödinger equation (2) may be obtained by rewriting (1-2) as

$$\left(\frac{p_x^2}{2m} + V(x) \right) e^{gx/\hbar} \varphi(x) = E e^{gx/\hbar} \varphi(x), \quad (12)$$

using (3) and (4). This equation is discretized by defining N equidistant sites of spacing c on the ring of perimeter $L_x = (N-1)c \simeq Nc$ and replacing x by site positions nc . The wavefunctions $\varphi(x)$ and the random potential $V(x)$ are then replaced by amplitudes $\varphi(nc) \equiv \varphi_n$ and potentials $V(nc) \equiv V_n$ at lattice sites n on the ring. Moreover, to make the Schrödinger equation at a lattice site n look as simple as possible we define the reduced energy parameters $(-2m c^2/\hbar^2) V_n \equiv \varepsilon_n$, $(-2m c^2/\hbar^2) E \equiv E$ and absorb a constant energy term proportional to φ_n , which comes from discretization of the kinetic energy term, in a redefinition of the zero of energy. Finally we define $h = g/\hbar$ as our new parameter proportional to the depinning field H_{\perp} . Then the Schrödinger equation at $x = nc$ is

$$e^{hc} \varphi_{n+1} + e^{-hc} \varphi_{n-1} + \varepsilon_n \varphi_n = E \varphi_n. \quad (13)$$

This discretized form of (12) is equivalent to the Schrödinger equation of an Anderson tight-binding model in a non-hermitian field h , where the discrete sites n , at which the wavefunction has amplitudes φ_n , are occupied by atoms with a random energy level ε_n . The first two terms in (13) describe hopping of a particle between site n and its neighbours $n \pm 1$. The fact that the hopping rate for hopping to the right of n is different from the rate for hopping to the left shows that the IVP breaks the symmetry between right- and left-moving particles. For this reason, the localization in such a system is often referred to as *directed localization*. Thus (13) applies for all sites n on a ring having first neighbours labelled $n \pm 1$, namely the sites $n = 2, 3, \dots, N - 1$. On the other hand, the sites $n = 1$ and $n = N$ have their neighbours labelled $2, N$ and $1, N - 1$, respectively. The analogous equations for $n = 1$ and $n = N$ are thus

$$e^{hc} \varphi_2 + e^{-hc} \varphi_N + \varepsilon_1 \varphi_1 = E \varphi_1, \quad (13a)$$

$$e^{hc} \varphi_1 + e^{-hc} \varphi_{N-1} + \varepsilon_N \varphi_N = E \varphi_N, \quad (13b)$$

which act as periodic boundary conditions modified by the IVP. Indeed by performing an imaginary gauge transformation

$$\varphi_n = e^{-hcn} \psi_n \quad (14)$$

one easily verifies that the site wavefunctions obey

$$\psi_{n+1} + \psi_{n-1} + \varepsilon_n \psi_n = E \psi_n, \quad n = 1, 2, \dots, N, \quad (15)$$

with the field modified periodic boundary conditions

$$\psi_{N+1} = e^{Nhc} \psi_1, \quad (16a)$$

$$\psi_0 = e^{Nhc} \psi_N, \quad (16b)$$

for the added amplitudes ψ_{N+1} and ψ_0 .

B. Localization length for weak disorder

We shall now derive an explicit expression for the Lyapunov exponent (LE) or the inverse localization length from the Schrödinger equations (13) and (13a,b) for the non-hermitian ring, for weak disorder. The same general treatment will also allow us to obtain the Lyapunov exponent for the case of an *open disordered chain*, for comparison.

In order to obtain the wavefunction amplitudes φ_n at sites n we solve (13)–(13a,b) recursively starting from some arbitrary value φ_1 , at site $n = 1$. We focus, as usual, on the LE λ at an energy E , which gives the rate of the exponential variation (either of growth or of decay) of the modulus of φ_n at asymptotically large distances from the initial site $n = 1$. Assuming the number of sites on the ring to be large we define

$$\lambda = \lim_{(n \leq N) \rightarrow \infty} \frac{1}{nc} \ln |\varphi_n|, \quad (17)$$

and, by assuming furthermore that φ_n self-averages to a central limit value for $n \rightarrow \infty$ this expression yields

$$\lambda = \lim_{n \rightarrow \infty} \frac{1}{nc} \langle \ln |\varphi_n| \rangle, \quad (17a)$$

where the angular bracket denotes the average over the random site energies ε_i , which we assume to be independent gaussian variables with mean $\langle \varepsilon_i \rangle = 0$, and correlation

$$\langle \varepsilon_i \varepsilon_j \rangle = \varepsilon_0^2 \delta_{i,j}. \quad (18)$$

It is convenient to determine the amplitudes φ_n from the so-called Riccati ratios

$$R_n = \frac{\varphi_n}{\varphi_{n-1}}, \quad n = 2, 3, \dots, N, \quad (19)$$

which yield

$$\varphi_n = \prod_{m=2}^n R_m \varphi_1, \quad (20)$$

and in terms of which (13)–(13a,b) become

$$e^{hc} R_{n+1} + e^{-hc} R_n^{-1} = E - \varepsilon_n, \quad n = 2, 3, \dots, N-1, \quad (21)$$

$$e^{hc} R_2 + e^{-hc} Q_n^{-1} = E - \varepsilon_1, \quad Q_1 = \frac{\varphi_1}{\varphi_N}, \quad (21a)$$

$$e^{hc} Q_1 + e^{-hc} R_N^{-1} = E - \varepsilon_N, \quad (21b)$$

which determine R_n , $n = 2, 3, \dots, N$, and Q_1 as functions of an arbitrary energy E . The justification for studying the amplitudes φ_n as a function of an arbitrary energy E rather than at the eigenvalues [19] themselves is provided by the standard Mott–Twose–Borland [20] conjecture. This says that when E is close to an eigenvalue the exponential rate of variation of the amplitudes φ_n at far away sites tends to the (largest) exponential rate of localization of the corresponding eigenstate of the ring about a fixed localization centre.

In solving eqs (21) and (21a, b) we assume the disorder to be weak and expand the Riccati variables in the form of successive order contributions in the site energies,

$$R_n = R_n^{(0)} + R_n^{(1)} + R_n^{(2)} + \dots, \quad Q_1 = Q_1^{(0)} + Q_1^{(1)} + Q_1^{(2)} + \dots, \quad (22)$$

where $R_n^{(1)}$ is linear in the site energies, $R_n^{(2)}$ is quadratic in these energies and so on. The zeroth order solutions are of the form

$$R_n^{(0)} = Q_1^{(0)} = e^{iq}, \quad (23)$$

where the wavenumber q may be real or complex and defines an arbitrary complex energy

$$E = 2 \cos(q - ihc). \quad (24)$$

For real q this expression leads to the equation

$$\frac{(\operatorname{Re} E)^2}{\cosh^2 hc} + \frac{(\operatorname{Im} E)^2}{\sinh^2 hc} = 4, \quad (25)$$

which shows that energies lie on an ellipse in the complex energy plane. For real q , the expressions (23) and (24) reflect the fact that in the absence of a disorder, the wavefunctions are plane waves associated with complex eigenvalues as discussed in §2 in the continuous case. The eigenvalues are complex because of the presence of the IVP.

Now the first and second order corrections $R_n^{(1)}$ and $R_n^{(2)}$ in the Riccati parameters are determined by the equations obtained by substituting (22) in (21) and (21a, b) and equating contributions at a given order on both sides. The final explicit solutions for $R_n^{(1)}$ and $R_n^{(2)}$ obtained by solving the corresponding first and second order equations by straightforward recursions, starting from the equations for $n = 2$ and iterating to higher and higher values of n are [18]:

$$R_n^{(1)} = \tilde{a}^{n-2} R_2^{(1)} - e^{-hc} \sum_{m=2}^{n-1} \tilde{a}^{n-m-1} \varepsilon_m, \quad n = 3, 4, \dots, N, \quad (26a)$$

$$R_2^{(1)} = \frac{e^{-hc}}{\tilde{a}^N - 1} \left(\varepsilon_1 + \tilde{a} \varepsilon_N + \tilde{a} \sum_{m=2}^{N-1} \tilde{a}^{N-m} \varepsilon_m \right), \quad (26b)$$

$$R_n^{(2)} = \tilde{a}^{n-2} R_2^{(2)} - e^{hc} \sqrt{\tilde{a}} \sum_{m=2}^{n-1} \tilde{a}^{n-m} \left(R_m^{(1)} \right)^2, \quad n = 3, 4, \dots, N, \quad (27a)$$

$$R_2^{(2)} = \frac{e^{hc} \sqrt{\tilde{a}}}{\tilde{a}^N - 1} \left[\frac{1}{\tilde{a}} \left(e^{-hc} \varepsilon_1 + R_2^{(1)} \right)^2 + \sum_{m=2}^N \tilde{a}^{N-m+2} \left(R_m^{(1)} \right)^2 \right], \quad (27b)$$

where

$$\tilde{a} = e^{-2i(q-ihc)} \quad (28)$$

and q defines energies (24). It should be noted that the expressions for $R_2^{(1)}$ and $R_2^{(2)}$ are invalid at zero field where they diverge at the eigenenergies of the perfect ring, $E = \cos q$, $q = 2\pi k/N$, $k = 0, \pm 1, \pm 2, \dots$. In fact, as is well-known, perturbation theory for eigenstate energies in a weakly disordered ring also diverges for $h = 0$ [21]. This strongly suggests that the Lyapunov exponents of the zero field eigenstates in the ring may not be discussed by perturbation theory.

The next step is now to calculate the LE exponent (17a) to second order in the site energies using (26a, b) and (27a, b). By expressing (17a) in terms of the Riccati parameters R_p and expanding $\ln |R_p|$ to second order we obtain (with $\langle R_n^{(1)} \rangle = 0$)

$$\lambda = \frac{1}{2} \lim_{n \rightarrow \infty} \frac{1}{nc} \sum_{p=2}^n \left[e^{-iq} \left(\langle R_p^{(2)} \rangle - \frac{1}{2} e^{-iq} \langle R_p^{(1)2} \rangle \right) + \text{c.c.} \right]. \quad (29)$$

A remarkable simplification occurs in (29) when calculating the averages $\langle R_p^{(2)} \rangle$ and $\langle R_p^{(1)2} \rangle$. It turns out that the quantities $\langle R_p^{(2)} \rangle$ for all p , from $p = 2$ to $p = N$, and $\langle Q_1^{(2)} \rangle$ reduce to a common value, and so do the quantities $\langle R_p^{(1)2} \rangle$, $p = 2, 3, \dots, N$, and $\langle Q_1^{(1)2} \rangle$ [18].

This reflects the independence of these quantities of the initial site(s) for the recursive solution of the tight-binding equations, as expected. With these results the final expression for the LE for the ring, to second order in the site energies, reduces to the remarkably simple form

$$\lambda = -\frac{\varepsilon_0^2}{4} \frac{\tilde{a}}{(1-\tilde{a})^2} \left(\frac{1+\tilde{a}^N}{1-\tilde{a}^N} \right) + \text{c.c.}, \quad (30)$$

which is valid for $h \neq 0$, as discussed above. Note that the explicit central limit form of λ will appear in the detailed discussion below.

Before discussing (30) further, we now derive the corresponding expression for the LE for a weakly disordered *linear chain* connected at both ends to perfect leads. In this case the solutions for the first- and second order corrections in the Riccati ratios are given by (26a) and (27a) regarded as expressing $R_n^{(1)}$ and $R_n^{(2)}$ in terms of arbitrary initial constants, $R_2^{(1)}$ and $R_2^{(2)}$, for the iteration of the corresponding recursion relations. In this case, the final result obtained for λ from (29) is [18]

$$\lambda = \frac{\varepsilon_0^2}{4} \left(\frac{1}{4-E^2} + \text{c.c.} \right), \quad E = 2 \cos(q - ihc), \quad (31)$$

which reduces, for $h = 0$, to the well-known Thouless formula for the inverse localization length

$$\frac{1}{\xi} \equiv \lambda = \frac{\varepsilon_0^2}{2(4-E^2)}, \quad (32)$$

as expected. In this case all eigenstates with eigenvalues within the band $E \equiv 2 \cos q$ are exponentially localized.

The recovery of the Thouless formula from the general perturbation solutions (26a) and (27a), as well as the expected site-independence of the relevant averages obtained from these solutions for the ring lattice strongly support the validity of our treatment of the non-hermitian localization length in a ring for weak disorder.

C. Properties of the localization length in the ring lattice

From (30) it follows that λ is an antisymmetric function of h :

$$\lambda(-h) = -\lambda(h). \quad (33)$$

This property is easily understood. We expect the Lyapunov exponent in (30) to describe a symmetric exponentially localized state: this means that since $\lambda(h)$ determines the exponential growth (if $\lambda(h) > 0$) of an initial amplitude φ_1 with increasing $n = 1, 2, \dots$, the corresponding exponent describing the growth of the localized state with decreasing $n = N, N-1, N-2, \dots$ must be $-\lambda(h)$. This may be verified directly by studying the growth of the initial amplitude φ_1 , at $n = 1$ as a function of decreasing $n = N, N-1, \dots$, following the scheme discussed above. This yields the result:

$$\lambda = -\frac{\varepsilon_0^2}{4} \frac{\tilde{a}}{(1-\tilde{a})^2} \left(\frac{1+\tilde{a}^{-N}}{1-\tilde{a}^{-N}} \right), \quad (30a)$$

which coincides indeed with minus the Lyapunov exponent (30) along the direction of increasing n . On the other hand, the fact that the rates of hopping from a given site to its left- and right-neighbours are interchanged when h is replaced by $-h$ implies that the rates of exponential variation of the initial wavefunction, $\pm\lambda(h)$, are also interchanged. This leads to (33). We recall that the Mott–Twose–Borland conjecture implies that the two exponentially growing (for $\lambda(h) > 0$) or decaying (for $\lambda(h) < 0$) amplitudes obtained by iterating from an initial value in the two directions of the ring, respectively, match at a localization centre n_0 when E coincides with a (complex) eigenenergy.

In the limit $N|h|c \gg 1$ (30) reduces to the central limit form

$$\lambda = \frac{\epsilon_0^2}{4} \left(\frac{1}{4-E^2} + \text{c.c.} \right) \text{sign } h, \quad E = 2 \cos(q - ihc), \quad (34)$$

where the factor $\text{sign } h$ ensures that $\lambda(h)$ is an odd function of h , as shown above. Note that (34) coincides in magnitude with the result (31) for a linear chain. This shows the insensitiveness of the non-hermitian localization length to the boundary conditions. For definiteness we shall assume $h > 0$ from now on. Then λ in (34) has the following properties. It is positive in the $\text{Re}E$ -interval

$$-E_d < \text{Re } E < E_d, \quad (35)$$

where

$$E_d = \frac{2 \cosh hc}{\sqrt{1 + \tanh^2 hc}}. \quad (36)$$

It possesses a maximum of magnitude

$$\lambda = \lambda_0 = \frac{\epsilon_0^2}{8} \frac{1}{1 + \sinh^2 hc}, \quad (37)$$

at the band centre, $\text{Re } E = 0$, and vanishes at the energies

$$E = \pm(E_d \pm i E'_d), \quad (38)$$

with

$$E'_d = \frac{2 \sinh^2 hc}{\sqrt{1 + 2 \sinh^2 hc}}, \quad (39)$$

which correspond to symmetrically placed points on the ellipse (25) in the complex energy plane. Finally λ is negative in the domains extending from E_d to the upper band edge, $2 \cosh hc$, and from $-E_d$ to the lower band edge, $-2 \cosh hc$, respectively; it takes a minimum negative value

$$\lambda = \lambda_e = -\frac{\epsilon_0^2}{8 \sinh^2 hc}, \quad (40)$$

at the band edges. Thus, as a function of $\text{Re } E$ alone obtained using (25), λ is a symmetric function which decreases monotonically from a maximum, λ_0 , at the origin down to a

negative minimum value, λ_e , at the band edges $\pm 2 \cosh hc$. The value $|\lambda_e| > \lambda_0$ corresponds in fact to the smallest localization length in the energy band. Our analysis thus shows that all eigenstates in the complex energy domain (24) are localized except for four possible eigenstates having energies coinciding with (38).

These results are at variance with the suggestion of HN that all eigenstates with complex eigenvalues are delocalized and thus correspond to depinned flux lines. However, we believe that while complex eigenvalues are a necessary feature of the delocalized states, as discussed by HN and in §2, a complex eigenvalue is not sufficient for a state to be delocalized. In fact, we find that there exists a natural interpretation of localized complex eigenvalue states in terms of flux lines which are pinned or depinned on the scale of the *finite* length of the ring. To see this, we recall that the progressive depinning of a flux line with increasing depinning field H_\perp is described by the transverse displacement $\langle x \rangle_{L_\tau} - \langle x \rangle_0$ defined above. Now HN [2] have shown that this displacement is determined by an imaginary current which is induced by the IVP proportional to H_\perp . The imaginary current carried by an eigenstate n of energy E_n of the quantum system is defined by

$$J_n = -i \frac{\partial E_n}{\partial h}, \quad (41)$$

which follows from the formal analogy between the imaginary momentum, ih , associated with such an eigenstate and the real momentum of e.g. a Bloch wave state. This shows that a complex energy eigenstate corresponds to a flux line with non zero transverse displacement, $\langle x \rangle_{L_\tau} - \langle x \rangle_0$. As recalled above, if the localization length of a complex energy state is smaller than the perimeter, $L_x \simeq Nc$, of the ring the corresponding flux line remains pinned by a defect, the finite value of $\langle x \rangle_{L_\tau} - \langle x \rangle_0$ reflecting the depinning of the extremities of the line. On the other hand, if the eigenstate is delocalized on the scale of the length of the ring then the corresponding flux line is depinned. Now, we observe that there exist energy domains around the special values $\pm (E_d \pm iE'_d)$ (at which the localization length diverges) in which the localization length exceeds the perimeter of the ring. These domains thus correspond to the depinned flux lines in the superconductor. Furthermore, one easily sees that the widths of these domains grow with increasing h , indicating that more and more eigenstates of the ring correspond to depinned flux lines. In fact when h reaches a threshold h_0 such that

$$|\lambda_e(h_0)| = \frac{1}{Nc}, \quad (42)$$

all eigenstates in the energy band have localization lengths equal to or larger than the ring circumference so that the whole system of flux lines is depinned. We also note that in the opposite limit, that is for the lowest fields compatible with the condition $N|h|c \gg 1$ the width of the Re E -domain for such delocalized states goes to zero. This reflects the existence of a low field region where there are no depinned flux lines.

4. Localized state eigenvalues in a non-hermitian ring

There are several reasons for performing an analytic study of the eigenvalues of the non-hermitian tight-binding ring described by the Schrödinger equation (13) and (13a, b). On the one hand, the localization lengths of the eigenstates of this model studied in §3 are

explicitly fixed only after obtaining the corresponding eigenvalues. On the other hand, the extensive numerical results for eigenvalues in this model [2,4] might be explained by such a complementary analytic study. Furthermore, the recent exact analysis of eigenvalues for weak disorder [21] discussed below leads to important improvements of the earlier approximate treatment by Brouwer, Silvestrov and Beenakker [4] (hereafter referred to as BSB).

A. Transfer matrix for eigenvalues for weak disorder

In order to determine the eigenvalues of the system (13)–(13a, b) we define transfer matrices in the space of the atomic sites. The transfer matrix associated with the n th site is

$$\hat{P}_n = \begin{pmatrix} E - \varepsilon_n & -e^{-hc} \\ e^{hc} & 0 \end{pmatrix}, \quad n = 1, 2, \dots, N. \quad (43)$$

It allows us to rewrite (13)–(13a, b) as

$$\begin{pmatrix} \varphi_{n+1} \\ \varphi_n \end{pmatrix} = e^{-hc} \hat{P}_n \begin{pmatrix} \varphi_n \\ \varphi_{n-1} \end{pmatrix}, \quad n = 2, 3, \dots, N-1, \quad (44)$$

$$\begin{pmatrix} \varphi_2 \\ \varphi_1 \end{pmatrix} = e^{-hc} \hat{P}_1 \begin{pmatrix} \varphi_1 \\ \varphi_N \end{pmatrix}, \quad (45a)$$

$$\begin{pmatrix} \varphi_1 \\ \varphi_N \end{pmatrix} = e^{-hc} \hat{P}_N \begin{pmatrix} \varphi_N \\ \varphi_{N-1} \end{pmatrix}. \quad (45b)$$

These equations may be combined to yield a 2×2 determinantal equation for the eigenvalues, which reduces to

$$\text{tr} \left(\prod_{n=1}^N \hat{P}_n \right) = 2 \cosh Nhc. \quad (46)$$

Here we discuss the analytic solution of this equation for weak disorder, to second order in the site energies ε_j [21]. For this purpose we write \hat{P}_n as the sum of the unperturbed and perturbation matrices

$$\hat{P}_n = \hat{P} + \hat{V}_n, \quad \hat{P} = \begin{pmatrix} E_0 & -e^{-hc} \\ e^{hc} & 0 \end{pmatrix}, \quad \hat{V}_n = \begin{pmatrix} \Delta E - \varepsilon_n & 0 \\ 0 & 0 \end{pmatrix}, \quad (47)$$

where E_0 is an eigenvalue of the perfect system and

$$\begin{aligned} \Delta E &= E - E_0 \\ &= E^{(1)} + E^{(2)} \dots, \end{aligned} \quad (48)$$

is the correction due to the disorder, which is written as a sum of successive order contributions in the site energies: $E^{(1)}$ is a linear functional in the the site energies, and $E^{(2)}$ is a

quadratic functional of site energies, etc. We then expand the matrix product in (46) up to second order in the perturbation matrices \widehat{V}_n in the form

$$\begin{aligned} \prod_{n=1}^N \widehat{P}_n &= \widehat{P}^N + \sum_{m=1}^N \widehat{P}^{m-1} \widehat{V}_m \widehat{P}^{N-m} \\ &+ \sum_{n=2}^N \sum_{m=1}^{n-1} \widehat{P}^{m-1} \widehat{V}_m \widehat{P}^{n-m-1} \widehat{V}_n \widehat{P}^{N-n}, \end{aligned} \quad (49)$$

which we substitute in (46) to determine the energies E_0 , $E^{(1)}$ and $E^{(2)}$ by equating terms of the same order in the equation. The powers of \widehat{P} which enter (49) are evaluated in a standard way by using the similarity transformation which diagonalizes this matrix. This leads to

$$\widehat{P}^m = \frac{1}{\sin s} \begin{pmatrix} \sin(m+1)s & -e^{-hc} \sin ms \\ e^{hc} \sin ms & -\sin(m-1)s \end{pmatrix}, \quad (50)$$

where we have put

$$E_0 = 2 \cos s. \quad (51)$$

From (46) and (49) we then obtain the following expressions for the first and second order corrections in the eigenenergies $E = E_0 + E^{(1)} + E^{(2)} + \dots$:

$$E^{(1)} = \frac{1}{N} \sum_{n=1}^N \varepsilon_n, \quad (52)$$

$$\begin{aligned} E^{(2)} &= -\frac{1}{N \sin Ns \sin s} \sum_{n=2}^N \sum_{m=1}^{n-1} \\ & (E^{(1)} - \varepsilon_n)(E^{(1)} - \varepsilon_m) \sin(n-m)s \sin(N-n+m)s. \end{aligned} \quad (53)$$

The values of s are determined by the form of (46) for the perfect system, which reduces to

$$\cos Ns = \cosh Nhc, \quad (54)$$

and yields

$$s = q - ihc, \quad q = \frac{2\pi k}{N}, \quad k = 0, \pm 1, \dots \quad (55)$$

For further discussions, we now average these expressions over the random site energies, using (18). One has $\langle (E^{(1)}) \rangle = 0$ and after performing the geometric sums left after averaging we obtain

$$\langle E^{(2)} \rangle = -\frac{\varepsilon_0^2}{4 \sin(q - ihc)} \left[\cot N(q - ihc) - \frac{1}{N} \cot(q - ihc) \right], \quad (56)$$

where the term proportional to $1/N$ may be dropped in first approximation. The divergence of the second order corrections (53) and (56) for $h = 0$ signals the invalidity of perturbation theory for studying the eigenenergies of a disordered ring in zero field for weak disorder.

Following the suggestion of HN [2] that localized eigenstates of a non-hermitian system correspond to real eigenvalues while extended states are associated with complex eigenvalues many studies have been devoted to analysing eigenvalues [3–7,13,15] and, more specifically, the threshold energy for the disappearance of complex eigenvalues in the non-hermitian system. We now wish to determine this threshold from exact analytical results (51), (55) and (56) for the average eigenvalues for weak disorder, and examine their applicability for explaining the numerical thresholds obtained by HN for a wide range of non-hermitian fields.

B. Properties of averaged eigenvalues and comparison with previous work

For finite h and large N such that $N|h|c \gg 1$ we get

$$\text{Re}\langle E \rangle = (2 \cosh hc - A \sinh hc) \cos q, \tag{57a}$$

$$\text{Im}\langle E \rangle = (2 \sinh hc + A \cosh hc) \sin q, \tag{57b}$$

$$A = -\frac{\epsilon_0^2}{4} \frac{\text{sign } h}{\sin^2 q + \sinh^2 hc}, \quad q = \frac{2\pi k}{N}, \quad k = 0, \pm 1, \pm 2, \dots \tag{58}$$

For $\epsilon_0 = 0$ the averaged support of the complex eigenvalues is the ellipse centered at the origin in the complex energy plane defined by (25). The effect of the weak disorder is to deform this ellipse and to make the semi-axes energy-dependent, as shown by the second term on the r.h.s. of (57a, b). Now we find that the complex eigenvalues disappear, i.e. $\text{Im}\langle E \rangle = 0$, for two distinct values, E_c and E'_c , of $|\text{Re}\langle E \rangle|$. The value

$$E_c = 2 \cosh hc + \frac{\epsilon_0^2}{4 \sinh |h|c} \tag{59}$$

is obtained for $q = 0, \pi$ and corresponds to the semi-axis of the perfect system shifted by the effect of the disorder. The other value

$$E'_c = 2 \cosh hc (1 + \tanh^2 hc) \sqrt{\cosh^2 hc - \frac{\epsilon_0^2}{8 \tanh |h|c}}, \tag{60}$$

is the energy at which the effect of the disorder compensates exactly the zeroth order term in (57b). Beyond this energy, the perturbation expansion for weak disorder is, of course, invalid. In particular, for E_c to be an acceptable threshold for the disappearance of complex eigenvalues we require

$$E'_c > E_c, \tag{61}$$

which is obeyed for fields $h > h_1$, where h_1 is given by

$$\frac{\sinh^3 h_1 c}{\cosh h_1 c} = \frac{\varepsilon_0^2}{8}. \quad (62)$$

On the other hand, for E'_c to be a real zero of $\text{Im}\langle E \rangle$ in (57b) the square root in (60) must be real, which requires

$$h > h_2, \quad (63)$$

where $h_2 < h_1$ is defined by

$$\sinh h_2 c \cosh h_2 c = \frac{\varepsilon_0^2}{8}. \quad (64)$$

To summarize, our analysis identifies three distinct domains for the non-hermitian field h . In the high-field domain, $h > h_1$, the threshold for disappearance of complex eigenvalues (or $\text{Re}\langle E \rangle$ edge of the complex energy band) is accurately given by (59). This is the domain where the perturbation theory works best. In the intermediate field domain, $h_2 < h < h_1$, where $E_c > E'_c$, the threshold value E_c must be rejected because it is incompatible with the perturbation theory. In this case therefore, the only estimate of the threshold for the disappearance of complex eigenvalues in perturbation theory is provided by the zero E'_c of $\text{Im}\langle E \rangle$. This estimate is, however, rather poor since the value $\text{Re}\langle E \rangle = E'_c$ lies at the borderline for the validity of perturbation theory. Finally, for $h < h_2$ perturbation theory is invalid despite the fact that E'_c does not exist in this case. Indeed one verifies that for $h < h_2$ perturbation theory breaks down on the imaginary axis (i.e. for $\text{Re}\langle E \rangle = 0$). This shows that complex eigenvalues do not appear for field strengths less than h_2 .

We now compare our analytical results for threshold energies for the disappearance of complex eigenvalues with numerical simulation results by HN [2] and by BSB [4]. The numerical values of ε_0 associated with the simulation results are $\varepsilon_0^{\text{HN}} = 2/\sqrt{3} \simeq 1.155$ for the HN data and $\varepsilon_0^{\text{BSB}} = 0.3\sqrt{8} \simeq 0.848$ for those of BSB [21]. For medium fields $|h|c$, these values verify the condition

$$|\varepsilon_0| \ll 2 \cosh hc, \quad (65)$$

of smallness of the variance of the site energies compared to the squared half width of the real energy band, which is required for the applicability of perturbation theory. In table 1 we compare the numerical values of $E_c/2$ and $E'_c/2$ with the numerical result for the threshold obtained by BSB for $|h|c = 0.1$ (first line) and those obtained by HN, for a series of field values. For $\varepsilon_0 \equiv \varepsilon_0^{\text{HN}}$ we find $h_1^{\text{HN}}c \simeq 0.55$ and $h_2^{\text{HN}}c \simeq 0.163$ and for $\varepsilon_0 \equiv \varepsilon_0^{\text{BSB}}$, $h_1^{\text{BSB}}c \simeq 0.45$, $h_2^{\text{BSB}}c \simeq 0.0895$. Table 1 shows that for the first field values $hc > h_1^{\text{HN}}c$ the numerical results of HN agree well with the corresponding values of E_c , as expected. In contrast, for the intermediate values, $|h|c = 0.1, 0.3$ and 0.5 , E'_c agrees more closely with the numerical results, in agreement with our analytical discussion.

In table 2 we show the comparison of our theoretical eigenvalues with numerical values on the imaginary axis, as a typical example of the behaviour in another spectral region. For $\text{Re}\langle E \rangle = 0$ we obtain from (57b)

$$|\text{Im}\langle E \rangle| \equiv E_c = 2 \sinh |h|c - \frac{\varepsilon_0^2}{4 \cosh hc}. \quad (66)$$

Table 1. Theoretical values of thresholds for the disappearance of complex eigenvalues at high fields (E_c) and at low fields (E'_c), for comparison with numerical results (E_c^{numer}) of ref. [4] (first line) and of ref. [2] (rest of the table) (see main text).

ε_0^2	$ h c$	$E_c/2$	$E'_c/2$	$E_c^{\text{numer}}/2$
0.72	0.1	1.90	0.33	0.82
1.333	0.3	1.32	0.82	0.86
1.333	0.5	1.45	1.31	1.3
1.333	0.7	1.47	1.95	1.48
1.333	0.9	1.60	2.92	1.65
1.333	1	1.69	3.58	1.73
1.333	1.1	1.79	4.4	1.82

Table 2. Comparison of purely imaginary averaged eigenvalues with numerical results of ref. [4] (first line) and of ref. [2] (rest of the table) (see main text).

ε_0^2	$ h c$	$E_c/2$	$E_c^{\text{numer}}/2$
0.72	0.1	0.011	0.056
1.333	0.3	0.145	0.151
1.333	0.5	0.373	0.393
1.333	0.7	0.626	0.639
1.333	0.9	0.910	0.918
1.333	1.0	1.067	1.082
1.333	1.1	1.236	1.272

Again we find a remarkable agreement between the theoretical results and the numerical values of HN [2]. We also observe that the absence of complex eigenvalues at low fields, $|h|c < h_2c$ which we have established theoretically, is consistent with the numerical results of HN which do not show complex eigenvalues in a range of very low-fields.

We now briefly comment on the existence of real eigenvalues in the presence of a non-hermitian field as suggested by HN [2]. We recall that besides the complex eigenvalues, which they associate with delocalized states, HN obtain real eigenvalues for $h \neq 0$, which correspond to localized states. According to them these real eigenvalues coincide with localized state eigenvalues for an open chain with free-space boundary conditions. The latter are independent of h and equal to their zero-field values because in this case the gauge transformation, (3) or (14), by which the effect of the field is eliminated, does leave the boundary conditions unchanged. In the HN picture, the real eigenvalues exist only outside the central $\text{Re } E$ domain where complex eigenvalues appear for $h \neq 0$. This is because, as discussed in §2, the occurrence of any delocalized complex eigenvalue state at finite h , at a given $\text{Re } E$ -value, results from the disappearance of a corresponding localized state of energy $\varepsilon_m \sim \text{Re}E$, which existed for $h = 0$. Now our exact perturbation theory for weak disorder does not lead to real eigenvalues for $h \neq 0$, apart from the thresholds $\text{Re } \langle E \rangle = \pm E_c$, for $|h| > h_1$, and $\text{Re } \langle E \rangle = \pm E'_c$ for $h_2 < |h| < h_1$, where $\text{Im } \langle E \rangle = 0$. In fact, the possibility of real eigenvalues for weak disorder is excluded since the zeroth order eigenvalues given by (51) and (55) are complex. Nevertheless we find that the existence of real eigenval-

ues at moderate fields in the numerical results of HN [2] are not incompatible with our analysis of complex eigenvalues. Firstly one observes that HN obtain no real eigenvalues for $hc \gtrsim 0.9$. This is, in fact, consistent with our perturbation theory which, as we have seen, works best for h sufficiently far above the threshold h_1c ($h_1^{\text{HN}}c = 0.55$) and yields only complex eigenvalues. Secondly, as in the numerical results [2], we find no complex eigenvalues at very low fields, for $0 < h < h_2$ ($h_2^{\text{HN}}c \simeq 0.16$), because of the breakdown of perturbation theory. In this case we may then expect the existence of real eigenvalues, just like in the zero-field case, where real eigenvalues which are not obtainable by perturbation theory exist. Similarly, in the intermediate domain $h_2 < h < h_1$, where perturbation theory is barely valid in the vicinity of the Re E -band edge of the complex spectrum, we may expect an improved treatment near the band edge energy to yield some additional real eigenvalues, as found in the numerical calculations [2]. Finally, we emphasize again that our discussion of the pinning-depinning transition of flux lines in §2 requires neither the existence of real eigenvalue localized states nor the existence of strictly delocalized complex eigenvalue states.

Finally, we compare our analysis of complex eigenvalues of a ring with the interesting but approximate analytic study for weak disorder and weak non-hermitian field of BSB [4]. They have derived a relation between the real and the imaginary parts of the averaged eigenvalues for weak disorder, which coincides with the leading form for $hc \rightarrow 0$ obtained from (57a, b) and (58) [21]. This relation leads to the leading terms for E'_c and h_2 obtained by expanding (60) and (64) for $hc \rightarrow 0$. However, it does not incorporate the high field threshold E_c for disappearance (or appearance) of complex eigenvalues, nor the threshold field h_1 separating high- and low-field domains. In their analysis [4] BSB have adapted an expression for the localization length for a light wave propagating in a disordered amplifying medium. It is interesting to note that this expression is itself based on well-known results on the distribution of the reflection coefficient for light scattered by such a medium obtained by Prof. N Kumar (with Dr P Pradhan).

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