PRAMANA — journal of physics
© Indian Academy of Sciences
Vol. 58, No. 1
January 2002
pp. 21–26

Star products from commutative string theory

SUNIL MUKHI
Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400 005, India

MS received 16 August 2001

Abstract. A boundary-state computation is performed to obtain derivative corrections to the Chern–Simons coupling between a p-brane and the RR gauge potential C^{p-3}. We work to quadratic order in the gauge field strength F, but all orders in derivatives. In a certain limit, which requires the presence of a constant B-field background, it is found that these corrections neatly sum up into the $*_2$ product of (commutative) gauge fields. The result is in agreement with a recent prediction using noncommutativity.

Keywords. String theory; effective action; noncommutativity; field theory.

PACS Nos 11.25.Db; 11.25.Hf; 02.10.Pk

1. Introduction

In a recent paper [1] it was shown that the noncommutative formulation of open-string theory can actually give detailed information about ordinary commutative string theory. Once open Wilson lines are included in the noncommutative action, one has exact equality of commutative and noncommutative actions including all α' corrections on both sides. As a result, a lot of information about α' corrections on the commutative side is encoded in the lowest-order term (Chern–Simons or DBI) on the noncommutative side, and can be extracted explicitly.

The predictions of ref. [1] were tested against several boundary-state computations in commutative open-string theory performed in [2], and an impressive agreement was found. The latter calculations were restricted to low-derivative orders, largely because the boundary-state computation becomes rather tedious when we go to high derivative order. However, in some specific cases, particularly when focusing on Chern–Simons couplings in the Seiberg–Witten limit [3], the predictions from noncommutativity in [1] are simple and elegant to all derivative orders as long as we work with weak field strengths (quadratic order in F). This suggests that the boundary state computation can be performed for these special cases, and in the given limits, to all derivative orders.

In this short paper, we perform precisely such a calculation using techniques and formulae already established in [2]. It turns out that the derivative corrections neatly sum up and give rise to a $*_2$ product [4] between a pair of commutative field strengths:

$$\langle F_{ij}(x), F_{kl}(x)\rangle_{*_2} \equiv F_{ij}(x)\frac{\sin(\frac{1}{2}\overleftarrow{\partial_p}\,\theta^{pq}\,\overrightarrow{\partial_q})}{\frac{1}{2}\overleftarrow{\partial_p}\,\theta^{pq}\,\overrightarrow{\partial_q}}F_{kl}(x). \tag{1}$$

The expression obtained in this way for the derivative corrections agrees perfectly with a prediction from noncommutativity that was made in [1].

Besides verifying this prediction, the calculation described here suggests that derivative corrections to brane actions in commutative string theory, even away from the Seiberg–Witten limit, might have a novel underlying mathematical structure. We will comment on this at the end.

2. Chern–Simons corrections: RR 6-form

In this section, we compute the corrections to the term

$$S_{\mathrm{CS}} = \frac{1}{2}\int C^{(6)}_{\mathrm{RR}} \wedge F \wedge F \tag{2}$$

on a Euclidean D9-brane of type IIB string theory with noncommutativity along all 10 directions. Here C^6 is the Ramond–Ramond 6-form potential. The computation is performed to all orders in the derivative expansion, but keeping only terms of order (F^2). The use of D9-branes is purely a convenience, the same calculation can be trivially applied to Dp-branes and their coupling to the RR form $C^{(p-3)}$.

The computation of corrections will be done in the boundary-state formalism. Useful background on how to compute derivative corrections in this formalism may be found in [2]. The formalism itself was developed in [5], and has been reviewed recently in [6]. Earlier work on derivative corrections can be found in [7].

Let us denote the sum of all derivative corrections to S_{CS} as ΔS_{CS}. Our starting point is the expression

$$S_{\mathrm{CS}} + \Delta S_{\mathrm{CS}} = \langle C|\mathrm{e}^{-\frac{i}{2\pi\alpha'}\int \mathrm{d}\sigma\mathrm{d}\theta D\phi^\mu A_\mu(\phi)}|B\rangle_{\mathrm{R}}, \tag{3}$$

where $|C\rangle$ represents the RR field and $|B\rangle_{\mathrm{R}}$ is the Ramond-sector boundary state for zero field strength. We are using superspace notation, for example $\phi^\mu = X^\mu + \theta\psi^\mu$ and D is the supercovariant derivative.

Combining eqs (2.3), (2.6), (2.13) of [2], we can rewrite this as

$$S_{\mathrm{CS}} + \Delta S_{\mathrm{CS}} = \langle C|\mathrm{e}^{\frac{i}{2\pi\alpha'}\int \mathrm{d}\sigma\mathrm{d}\theta\sum_{k=0}^{\infty}\frac{1}{(k+1)!}\frac{k+1}{k+2}D\tilde{\phi}^\nu\tilde{\phi}^\mu\tilde{\phi}^{\lambda_1}\cdots\tilde{\phi}^{\lambda_k}\partial_{\lambda_1}\cdots\partial_{\lambda_k}F_{\mu\nu}(x)}$$
$$\times\quad \mathrm{e}^{\frac{i}{2\pi\alpha'}\int \mathrm{d}\sigma[\tilde{\Psi}^\mu\psi_0^\nu + \psi_0^\mu\psi_0^\nu]\sum_{k=0}^{\infty}\frac{1}{k!}\tilde{X}^{\lambda_1}\cdots\tilde{X}^{\lambda_k}\partial_{\lambda_1}\cdots\partial_{\lambda_k}F_{\mu\nu}(x)}|B\rangle_{\mathrm{R}}, \tag{4}$$

where nonzero modes have a tilde on them, while the zero modes are explicitly indicated.

Since we are looking for couplings to the RR 6-form $C^{(6)}$, and working to order F^2, we only need terms with the structure $\partial\ldots\partial F\wedge\partial\ldots\partial F$. For such terms, two F's and 4 ψ_0's must be retained. Thus we can drop the first exponential factor in eq. (4) above as well as the first fermion bilinear $\tilde{\Psi}^\mu\psi_0^\nu$ in the second exponential. Then, expanding the exponential to second order, we get

$$S_{\rm CS}+\Delta S_{\rm CS}=\frac{1}{2}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\left(\frac{i}{2\pi\alpha'}\right)^2\int_0^{2\pi}\mathrm{d}\sigma_1\int_0^{2\pi}\mathrm{d}\sigma_2\,\langle C|\left(\frac{1}{2}\psi_0^\mu\psi_0^\nu\right)\left(\frac{1}{2}\psi_0^\alpha\psi_0^\beta\right)$$
$$\times\frac{1}{n!}\widetilde{X}^{\lambda_1}(\sigma_1)\cdots\widetilde{X}^{\lambda_n}(\sigma_1)\frac{1}{p!}\widetilde{X}^{\rho_1}(\sigma_2)\cdots\widetilde{X}^{\rho_p}(\sigma_2)$$
$$\times\partial_{\lambda_1}\ldots\partial_{\lambda_n}F_{\mu\nu}(x)\,\partial_{\rho_1}\ldots\partial_{\rho_p}F_{\alpha\beta}(x)|B\rangle_{\rm R}. \tag{5}$$

Now we need to evaluate the 2-point functions of the $\widetilde{X}$. The relevant contributions have nonlogarithmic finite parts [2] and come from propagators for which there is no self-contraction. This requires that $n=p$. Then we get a combinatorial factor of $n!$ from the number of such contractions in $\left\langle\left(\widetilde{X}(\sigma_1)\right)^n\left(\widetilde{X}(\sigma_2)\right)^n\right\rangle$. The result is

$$S_{\rm CS}+\Delta S_{\rm CS}=\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{i}{2\pi\alpha'}\right)^2\int_0^{2\pi}\mathrm{d}\sigma_1\int_0^{2\pi}\mathrm{d}\sigma_2\,D^{\lambda_1\rho_1}(\sigma_1-\sigma_2)\cdots D^{\lambda_n\rho_n}(\sigma_1-\sigma_2)$$
$$\times\partial_{\lambda_1}\cdots\partial_{\lambda_n}F_{\mu\nu}(x)\,\partial_{\rho_1}\cdots\partial_{\rho_n}F_{\alpha\beta}(x)\,\langle C|\left(\frac{1}{2}\psi_0^\mu\psi_0^\nu\right)\left(\frac{1}{2}\psi_0^\alpha\psi_0^\beta\right)|B\rangle_{\rm R}. \tag{6}$$

The fermion zero mode expectation values are evaluated using the recipe

$$\frac{1}{2}\psi_0^\mu\psi_0^\nu F_{\mu\nu}\to(-i\alpha')F, \tag{7}$$

where the F on the right hand side is a differential 2-form. The justification for this can be found in eq. (B3) of [2]. Thus we are led to

$$S_{\rm CS}+\Delta S_{\rm CS}=T^{\lambda_1\cdots\lambda_n;\rho_1\cdots\rho_n}\partial_{\lambda_1}\ldots\partial_{\lambda_n}F\wedge\partial_{\rho_1}\ldots\partial_{\rho_n}F, \tag{8}$$

where

$$T^{\lambda_1\cdots\lambda_n;\rho_1\cdots\rho_n}\equiv\frac{1}{2}\frac{1}{n!}\left(\frac{i}{2\pi\alpha'}\right)^2(-i\alpha')^2$$
$$\times\int_0^{2\pi}\mathrm{d}\sigma_1\int_0^{2\pi}\mathrm{d}\sigma_2 D^{\lambda_1\rho_1}(\sigma_1-\sigma_2)\cdots D^{\lambda_n\rho_n}(\sigma_1-\sigma_2). \tag{9}$$

Now we insert the expression for the propagator

$$D^{\mu\nu}(\sigma_1-\sigma_2)=\alpha'\sum_{m=1}^{\infty}\frac{\mathrm{e}^{-\varepsilon m}}{m}\left(h^{\mu\nu}\mathrm{e}^{im(\sigma_2-\sigma_1)}+h^{\nu\mu}\mathrm{e}^{-im(\sigma_2-\sigma_1)}\right), \tag{10}$$

where ε is a regulator, and

$$h^{\mu\nu}\equiv\frac{1}{g+2\pi\alpha'(B+F)}. \tag{11}$$

As is well-known, the propagator is no longer symmetric when a B-field background is turned on. We now find that

$$T^{\lambda_1\cdots\lambda_n;\rho_1\cdots\rho_n} = \frac{1}{2}\frac{1}{n!}(\alpha')^n \sum_{m_1=1}^{\infty}\cdots\sum_{m_n=1}^{\infty} \frac{\mathrm{e}^{-\varepsilon(m_1+\cdots+m_n)}}{m_1\ldots m_n}$$
$$\times \int_0^{2\pi}\frac{\mathrm{d}\sigma_1}{2\pi}\int_0^{2\pi}\frac{\mathrm{d}\sigma_2}{2\pi}\prod_{i=1}^{n}\left(h^{\lambda_i\rho_i}\mathrm{e}^{im_i(\sigma_2-\sigma_1)} + h^{\rho_i\lambda_i}\mathrm{e}^{-im_i(\sigma_2-\sigma_1)}\right). \quad (12)$$

It is convenient to define

$$(h^+)^{\mu\nu} \equiv h^{\mu\nu}, \quad (h^-)^{\mu\nu} \equiv h^{\nu\mu}$$

which allows us to write

$$\left(h^{\mu\nu}\mathrm{e}^{im(\sigma_2-\sigma_1)} + h^{\nu\mu}\mathrm{e}^{-im(\sigma_2-\sigma_1)}\right) = \sum_{\pm}(h^{\pm})^{\mu\nu}\mathrm{e}^{\pm im(\sigma_2-\sigma_1)}$$

and we find that

$$T^{\lambda_1\cdots\lambda_n;\rho_1\cdots\rho_n} = \frac{1}{2}\frac{1}{n!}(\alpha')^n\int_0^{2\pi}\frac{\mathrm{d}\sigma}{2\pi}\prod_{i=1}^{n}\left(\sum_{\pm}(h^{\pm})^{\lambda_i\rho_i}\sum_{m=1}^{\infty}\frac{\mathrm{e}^{-\varepsilon m}}{m}\mathrm{e}^{\pm im\sigma}\right). \quad (13)$$

After evaluating the sum over m, the result depending on the regulator ε, is

$$T^{\lambda_1\cdots\lambda_n;\rho_1\cdots\rho_n} = \frac{1}{2}\frac{1}{n!}(\alpha')^n\int_0^{2\pi}\frac{\mathrm{d}\sigma}{2\pi}\prod_{i=1}^{n}\left(-\sum_{\pm}(h^{\pm})^{\lambda_i\rho_i}\ \ln(1-\mathrm{e}^{-\varepsilon\pm i\sigma})\right). \quad (14)$$

At this point it is difficult to proceed further without introducing some simplification. The integral above, for general $h^{\mu\nu}$, can only be performed explicitly for $n=2$, as has in fact been done in [2]. However, if we take a limit where

$$g_{\mu\nu}\sim\delta, \quad B_{\mu\nu}\sim \text{fixed}, \quad \alpha'\sim\sqrt{\delta} \quad (15)$$

with $\delta\to 0$, a simplification occurs. This is indeed just the Seiberg–Witten limit [3]. In this limit, the ‘metric’ $h_{\mu\nu}$ becomes antisymmetric

$$h^{\mu\nu}\to\frac{\theta^{\mu\nu}}{2\pi\alpha'}, \quad (16)$$

where

$$\theta^{\mu\nu}\equiv\left(\frac{1}{B}\right)^{\mu\nu} \quad (17)$$

and hence we find

$$\sum_{\pm}(h^{\pm})^{\lambda_i\rho_i}\ \ln(1-\mathrm{e}^{-\varepsilon\pm i\sigma}) = \frac{1}{2\pi\alpha'}\theta^{\lambda_i\rho_i}\ \ln\left(\frac{1-\mathrm{e}^{-\varepsilon+i\sigma}}{1-\mathrm{e}^{-\varepsilon-i\sigma}}\right)$$
$$= \frac{1}{2\pi\alpha'}\,i\,(\sigma-\pi)\,\theta^{\lambda_i\rho_i}. \quad (18)$$

The integrand has simplified considerably and the integral can now be done. Also, we have now taken the regulator ε to 0, as it is no longer needed. It follows that

$$T^{\lambda_1\cdots\lambda_n;\rho_1\cdots\rho_n} = \frac{1}{2}\frac{1}{n!}\left(-\frac{i}{2\pi}\right)^n \theta^{\lambda_1\rho_1}\dots\theta^{\lambda_n\rho_n}\int_0^{2\pi}\frac{\mathrm{d}\sigma}{2\pi}(\sigma-\pi)^n$$
$$= \begin{cases} \frac{1}{2}\frac{1}{n!}\left(-\frac{i}{2\pi}\right)^n \frac{\pi^n}{n+1}\ \theta^{\lambda_1\rho_1}\dots\theta^{\lambda_n\rho_n} \quad (\text{even } n) \\ 0 \quad (\text{odd } n) \end{cases}. \tag{19}$$

Inserting this in eq. (8), it follows that keeping all derivative orders, but restricting to quadratic order in F, and in the Seiberg–Witten limit

$$S_{\mathrm{CS}} + \Delta S_{\mathrm{CS}} = \frac{1}{2}\int C^{(6)} \wedge \sum_{j=0}^{\infty}(-1)^j \frac{1}{2^{2j}(2j+1)!}$$
$$\times\theta^{\lambda_1\rho_1}\dots\theta^{\lambda_{2j}\rho_{2j}}\,\partial_{\lambda_1}\dots\partial_{\lambda_{2j}}F\wedge\partial_{\rho_1}\dots\partial_{\rho_{2j}}F$$
$$= \frac{1}{2}\int C^{(6)}\wedge\langle F\wedge F\rangle_{*_2}, \tag{20}$$

where the product $*_2$ is defined in eq. (1).

This agrees with a prediction from noncommutativity made in eq. (4.13) of ref. [1]. In that sense, the result is not surprising. However, it is amusing that using the boundary-state formalism in ordinary (commutative) string theory, we were explicitly able to obtain the $*_2$ product without invoking noncommutativity in any form.

3. Conclusions

It should be reasonably straightforward to repeat the calculation above to compute derivative corrections to $\int C^{(10-2n)}\wedge(F)^n$ for $n = 3,4,5$ restricting to corrections of order F^n. In the Seiberg–Witten limit, one should find the $*_n$ product in this way for these values of n. The analogous calculation for the DBI action will perhaps be more difficult.

One of the most interesting questions raised by this calculation and the work in ref. [1] is, what is the full expression for the derivative corrections, away from the Seiberg–Witten limit. We know that general string amplitudes depend on transcendental numbers, for example ζ-functions of odd argument. As noted in [1], the Seiberg–Witten limit causes these to go away in all the cases examined, leading to much simpler results which can then be recovered using noncommutativity or, as in this paper, explicit boundary state calculation. Clearly these simpler results place a strong constraint on the form of the full derivative corrections, away from the Seiberg–Witten limit. The question is then whether this constraint can be combined with other inputs, such as boundary-state computations, gauge invariance and background-independence [3,8], to recover the full corrections. This could have important consequences in understanding string theory beyond the derivative expansion.

Acknowledgements

I am grateful to Sumit Das, Rajesh Gopakumar, Gautam Mandal, Shiraz Minwalla, Ashoke Sen and Nemani Suryanarayana for helpful discussions.

References

[1] S R Das, S Mukhi and N V Suryanarayana, *J. High Energy Phys.* **0108**, 039 (2001), hep-th/0106024
[2] N Wyllard, *Nucl. Phys.* **B598**, 247 (2001), hep-th/0008125
[3] N Seiberg and E Witten, *J. High Energy Phys.* **9909**, 032 (1999), hep-th/9908142
[4] M Garousi, *Nucl. Phys.* **B579**, 209 (2000), hep-th/9909214
[5] C G Callan, C Lovelace, C R Nappi and S A Yost, *Nucl. Phys.* **B288**, 525 (1987)
C G Callan, C Lovelace, C R Nappi and S A Yost, *Nucl. Phys.* **B293**, 83 (1987)
C G Callan, C Lovelace, C R Nappi and S A Yost, *Nucl. Phys.* **B308**, 221 (1988)
[6] P Di Vecchia and A Liccardo, hep-th/9912161
P Di Vecchia and A Liccardo, hep-th/9912275
[7] J H Schwarz, *Phys. Rep.* **89**, 223 (1982)
A A Tseytlin, *Nucl. Phys.* **B276**, 391 (1986)
O D Andreev and A A Tseytlin, *Nucl. Phys.* **B311**, 205 (1988)
K Hashimoto, *Phys. Rev.* **D61**, 106002 (2000), hep-th/9909027
K Hashimoto, *J. High Energy Phys.* **0004**, 023 (2000), hep-th/9909095
[8] N Seiberg, *J. High Energy Phys.* **0009**, 003 (2000), hep-th/0008013