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Quantum cosmology in Ashtekar variables with non-minimally coupled scalar–tensor theory

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Abstract. Using non-minimally coupled scalar–tensor theory in homogeneous and isotropic cosmological model, quantum cosmology has been developed for Ashtekar variables. The wave function has been evaluated by solving the Wheeler–Dewitt (WD) equation and also using path integral formulation. Semi-classical limit using WKB approximation has also been discussed. Finally, the quantum Bohmian trajectories has been studied in detail.

Keywords. Quantum cosmology; non-minimally coupled; Bohmian trajectories.

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1. Introduction

In modern physics, the unification of the fundamental forces in nature is one of the most challenging problems even today. Attempts for unification have been diversified in different ways, namely, higher dimensional Kaluza–Klein model, supergravity, superstring theory etc. These theories need non-minimal couplings between the geometry of space-time and a scalar field. In order to get some qualitative features of the quantization of these theories, it is interesting to pursue non-perturbative methods.

So far there exist two major approaches to construct a quantum theory of gravity [1], namely canonical quantization and path integral formalism. In canonical quantization, the wave function of the universe is obtained as the solution of the WD equation, which is a second order functional differential equation, complicated in form. So it is very difficult to obtain the wave function even in simple minisuperspace models. Moreover, there are factor ordering problem and difficulties to know the initial conditions of the Universe [2].

On the other hand, the path integral formulation in quantum cosmology has some advantages, as there are definite proposals for the sum over histories, namely by Hartle and Hawking [3,4] and by Vilenkin [5,6]. After reducing to a single integration overlap function, they have usually been evaluated by the method of steepest descent [7].

The canonical quantization procedure has been put one step forward by the introduction of Ashtekar variables [8]. In his formulation, the Einstein equation (both evolution and constraint) can be expressed in simple polynomial form with the local SL (2.C) invariant structure. Thus the full set of constraint equations has been solved in the loop-space representation [9,10] and are classified using topological knot and link invariants [11]. It is very difficult to find a direct connection to quantum cosmology with this abstract loop space results.

Though a lot of work has been done and some interesting results are obtained, yet we are far from the goal due to some problems which still persist, like the issue of time, of how to interpret the wave function of the Universe, the elimination of singularities by quantum effects or the conditions for the classical limit. In this paper we try to address these issues considering gravity with non-minimally coupled scalar field for isotropic and homogeneous space-time model using Ashtekar variables. The paper is organized as follows: The basic equations are constructed and classical solutions are presented in §2. In §3, the wave functions have been evaluated by path integral formulation while those by solving WD equation are presented in §4. Also semi-classical limit using WKB approximation has been given in this section. The Bohmian trajectories have been studied in detail in §5. The paper ends with a short discussion in §6.

2. The basic equations

The general form of the Lagrangian for scalar–tensor theories (without ordinary matter) is

$$L=\sqrt{-g}\left(f(\phi)\cdot R-w(\phi)\cdot\frac{\phi_{,\mu}\cdot\phi^{\cdot\mu}}{\phi}+V(\phi)\right), \tag{1}$$

where $f(\phi)$ and $w(\phi)$ are arbitrary functions of the scalar field ϕ and $V(\phi)$ is some adhoc potential term. (It is to be noted that $f(\phi)=\phi, w(\phi)=w$, a constant and $V(\phi)=0$ gives BD-theory [12]. In the low-energy limit of string theory and in the reduction of Kaluza–Klein theories to four dimension the effective Lagrangian is in this form, depending on the nature of compactification and on the original multidimensional theories. As in cosmology, we generally concentrate on the bosonic section with no gauge fields in four dimensions and so the above effective Lagrangian reduces to

$$L=\sqrt{-g}\left(\phi\cdot R-w\cdot\frac{\phi_{,\mu}\cdot\phi^{\cdot\mu}}{\phi}-\frac{\psi_{,\upsilon}\cdot\psi^{\cdot\upsilon}}{\phi}\right). \tag{2}$$

In BD-theory this Lagrangian shows another scalar field ψ coupled non-trivially with the BD-field ϕ. Also in string cosmology this can be interpreted as that of an axion field $(w=-1)$ while in higher dimensional theory, gravity is coupled to external gauge field with space-time dimension $n=4+d(w=-(d-1)/d)$ [13].

The metric ansatz for the homogeneous and isotropic model (FRW Universe) is

$$\mathrm{d}s^2=N^2\mathrm{d}t^2-a^2(t)\mathrm{d}\pi_3^2. \tag{3}$$

So the above Lagrangian becomes

$$L=\frac{1}{n}\left(3\frac{\dot{u}^2}{u^2}+6\frac{\dot{u}}{u}\cdot\frac{\dot{\phi}}{\phi}-2w\frac{\dot{\phi}^2}{\phi^2}-\frac{2\dot{\psi}^2}{\phi^2}\right)-12nu^2\phi^2, \tag{4}$$

where $u = a^2$ is the triad variable (Ashtekar variable) and n is the rescaled lapse variable.

This can further be simplified to

$$L = \frac{1}{n}\left(3\dot{x}^2 - (2w+3)\dot{y}^2 - 2\dot{\psi}^2 \mathrm{e}^{-2y}\right) - 12n\mathrm{e}^{2x}, \tag{5}$$

by the transformation

$$x = \ln(u \cdot \phi), \quad y = \ln\phi. \tag{6}$$

In this gauge $\dot{n} = 0$, the field equations can be solved easily to obtain

$$\begin{aligned} \mathrm{e}^x &= \frac{\sqrt{c_1}}{2}\,\mathrm{sech}(\sqrt{c_1}nt) \\ \mathrm{e}^y &= \frac{\sqrt{c_2(2w+3)}}{\sqrt{2}\psi_0}\mathrm{sech}(\sqrt{c_2}nt), \\ \psi &= \sqrt{c_2}\,\frac{(2w+3)}{2\psi_0}\,\tanh(\sqrt{c_2}nt), \end{aligned} \tag{7}$$

where ψ_0, c_1 and c_2 are integration constants. The action at the classical solution has the expression

$$I_0 = (nAt) - 6\sqrt{c_1}\,\tanh\theta_1, \tag{8}$$

with

$$A = 3c_1 - (2w+3)c_2, \quad \theta_1 = \sqrt{c_1}nt.$$

3. Path integral formulation

In path integral formalism, we shall evaluate the quantum propagator (which is a more general concept than the wave function) between fixed values x, y and ψ as

$$G = \int \mathrm{d}n \int DxDyD\psi \cdot \exp[-I(x,y,\psi)]. \tag{9}$$

It is to be noted that, usually the lapse function n is unrestricted and so G is a solution of the WD equation. But if N is restricted to half of an infinite range then G is Green's function to the WD operator. As the action I is not positive definite, complex contours are needed for the convergence of the path integral. Moreover, the above classical solutions represent saddle points of the path integral and I_0 is the action at the saddle point. For evaluation of the path integral we shall make a shift of the integration variables

$$\begin{aligned} x(t) &= x_0(t) + X(t), \\ y(t) &= y_0(t) + Y(t), \\ \psi(t) &= \bar{\psi}(t) + \underline{\bar{\psi}}(t), \end{aligned} \tag{10}$$

where x_0, y_0, and $\bar{\psi}$ are the classical solutions represented by eq. (7) and X, Y, and $\underline{\bar{\psi}}$ vanish at the two end points. Hence the classical action can be written as

$$I = I_0 + I_2(X(t), Y(t), \underline{\bar{\psi}}(t)), \tag{11}$$

where I_0 is the action given by eq. (8) and I_2 is the part of the action containing square powers of X, Y, and $\underline{\psi}$ and their derivatives.

Hence the above path integral becomes

$$G = \int \mathrm{d}n \, \exp(-I_0) \int DXDYD\underline{\bar{\psi}} \, \exp(-I_2). \tag{12}$$

As the functional integrals over X, Y, and $\underline{\bar{\psi}}$ are Gaussian in form, using the standard result one obtains [2,14] (except for irrelevant factors)

$$G = \int \frac{\mathrm{d}n \, \exp(-I_0)}{(\coth nA)^3}. \tag{13}$$

We shall now evaluate this single ordinary integration over the lapse by performing a steepest-descent analysis. The saddle points are given by $(\partial I_0/\partial n) = 0$ and are expressed through

$$\cosh(\sqrt{c_1} n) = \sqrt{\frac{6c_1}{A}} = \alpha(\text{say}). \tag{14}$$

So there are three possibilities for which the explicit expressions for saddle points are

$$\sqrt{c}n = \begin{cases} 2k\Pi i \pm \cosh^{-1}\alpha, & \text{for } 3c_1 + (2w+3)c_2 > 0 \\ 2k\Pi i \pm i\cos^{-1}\alpha, & \text{for } 3c_1 + (2w+3)c_2 < 0 \\ 2k\Pi i, & \text{for } 3c_1 + (2w+3)c_2 = 0 \end{cases},$$

where

$$k = 0, \pm 1, \pm 2. \tag{15}$$

The corresponding action takes the form

$$\bar{I_0} = \pm\left(\frac{A}{\sqrt{c_1}}\cosh^{-1}\sqrt{\frac{6c_1}{A}} - \sqrt{6\{3c_1 + (2w+3)c_2\}}\right)$$

or

$$= \pm i\left(\frac{A}{\sqrt{c_1}}\cosh^{-1}\sqrt{\frac{6c_1}{A}} - \sqrt{6|3c_1 + (2w+3)c_2|}\right), \tag{16}$$

or

$$= \pm i\frac{A}{\sqrt{c_1}}(2k\Pi)$$

in the three consecutive cases respectively.

We note that the actions I_0 at the saddle points will satisfy the time-independent H–J equation. To apply the saddle point approximation to solve the lapse integral, the paths of steepest descent are characterized by $I_m(I_0) = \text{constant}$. Here $nA = i\left(k+\frac{1}{2}\right)\Pi$ are the essential singularities for these contours. The saddle points are all dominating saddle points. So, using polynomial corrections at any of these saddle points, the value of the lapse integral will be

$$\exp(-\bar{I_0})\left[d_0 + d_1\cdot\rho^{1/2} + d_2\cdot\rho + \cdots + d_m\cdot\rho^{m/2} + \cdots\right],$$

where the coefficients $d_0, d_1, \ldots$ are determined by higher derivatives of the action at the saddle point and ρ is some suitable small parameter in the asymptotic expansion. Lastly, it is to be mentioned that the above result changes for any other saddle point only by an overall phase factor.

4. Wheeler–Dewitt equation: The canonical approach

The ADM Lagrangian (4) can be written as

$$L = p_\mu\cdot u + p_\phi\cdot\phi + p_\psi\cdot\psi - nH,$$

where p_μ, p_ϕ and p_ψ are the ADM momentum conjugate to the triad variable u, and the scalar fields ϕ and ψ. Also the momentum p_μ is related to the Ashtekar momentum by

$$P_A = p_\mu \pm i/2.$$

Now, using the transformation (6) of the configuration variables the Hamiltonian of the system becomes

$$\begin{aligned} H &= n\left(\frac{p_x^2}{12} - \frac{p_y^2}{4(2w+3)} - p_\psi^2\cdot\frac{\mathrm{e}^{2y}}{8} + 12\mathrm{e}^{2x}\right) \\ &= nH. \end{aligned} \tag{17}$$

Since n is a Lagrange multiplier, we have the constraint $H \approx 0$. Hence in Dirac quantization procedure, the quantum states must be annihilated by the operator version of H, resulting $\tilde{H}\cdot\Phi = 0$, i.e.

$$\left(-\frac{1}{12}\frac{\partial^2}{\partial x^2} + \frac{1}{4(2w+3)}\frac{\partial^2}{\partial y^2} + \frac{\mathrm{e}^{2y}}{8}\cdot\frac{\partial^2}{\partial\psi^2} + 12\mathrm{e}^{2x}\right)\phi(x,y,\psi) = 0 \tag{18a}$$

with Φ as the wave function of the Universe. In order to solve this partial differential equation we write Φ in the separable form as

$$\Phi(x,y,\psi) = U(x)\cdot V(y)\cdot W(\psi), \tag{18b}$$

and the resulting solutions for U, V and W are

$$U(x) = A_u\cdot I_{\nu_1}(2\sqrt{3}\,\mathrm{e}^x) + B_u\cdot K_{\upsilon_1}(2\sqrt{3}\,\mathrm{e}^y)$$

and

$$V(y) = A_v J_{\upsilon_2}(2\sqrt{(2w+3)\lambda_2}\ \mathrm{e}^y) + B_v \cdot Y_{\upsilon_2}(2\sqrt{(2w+3)\lambda_2}\ \mathrm{e}^y),$$

$$W(\psi) = A_w \cdot \exp(i\sqrt{8\lambda_2}\ \psi) + B_w \cdot \exp(-i\sqrt{8\lambda_2}\ \psi), \tag{19a}$$

or

$$V(y) = A_v I_{\upsilon_2}(2\sqrt{(2w+3)|\lambda_2|}\ \mathrm{e}^y) B_v \cdot k_{\upsilon_2}(2\sqrt{(2w+3)|\lambda_2|}\ \mathrm{e}^y),$$

$$W(\psi) = A_w \cdot \exp(i\sqrt{8|\lambda_2|}\psi) + B_w \cdot \exp(-i\sqrt{8|\lambda_2|}\psi), \tag{19b}$$

depending on $\lambda_2 >$ or < 0. Here λ_1, λ_2 are separation constants and $\upsilon_1 = 2\sqrt{3\lambda_1}$ and $\upsilon_2 = 2\sqrt{(2w+3)\lambda_1}$. Hence the general solution of the W–D equation can be written as the linear superposition

$$\Phi = \int \rho(\lambda_1, \lambda_2) U_{\lambda_1}(x) V_{\lambda_1,\lambda_2}(y) W_{\lambda_2}(\psi) \mathrm{d}\lambda_1 \mathrm{d}\lambda_2. \tag{20}$$

Now, we shall discuss the WKB approximation to determine the semi-classical limit [13,15]. We write

$$\Phi = \exp\left(\frac{i}{h} S\right),$$

where the phase $S(x,y,\psi)$ satisfies the classical H–J equation. If we now substitute this form of ϕ in the W–D equation and expand S in order of $\hbar$ as

$$S = S_0 + \hbar S_1 + \hbar^2 S_2 + \cdots$$

then the classical solution can be recovered from the wave packet

$$\Phi = \int A(\vec{k}) \exp\left(\frac{i}{\hbar} S_0\right) \mathrm{d}\vec{k}, \tag{21}$$

with $\vec{k} = (k_1, k_2)$ as some arbitrary parameters (i.e. separation constants). So up to zeroth order in $\hbar$ the different equation for S_0 will be

$$-\frac{1}{12}\left(\frac{\partial S_0}{\partial x}\right)^2 + \frac{1}{4(2w+3)}\left(\frac{\partial S_0}{\partial y}\right)^2 + \frac{\mathrm{e}^{2y}}{8}\left(\frac{\partial S_0}{\partial \psi}\right)^2 + 12\mathrm{e}^{2x} = 0. \tag{22}$$

The separable ansatz

$$S_0(x,y,\psi) = S_1(x) + S_2(y) + S_3(\psi)$$

satisfies the above partial differential equation with

$$S_1(x) = \pm\left(\sqrt{3k_1}\ \text{arc cosech}\ \left(4\sqrt{\frac{3}{k_1}}\mathrm{e}^x\right) - \sqrt{3(k_1 + 48\mathrm{e}^{2x})}\right) + A_0,$$

$$S_2(y) = \pm\sqrt{2w+3}\left(-\sqrt{k_1}\ \ln\left\{\frac{\sqrt{k_1} + \sqrt{k_1 - k_2\mathrm{e}^{2y}}}{\sqrt{k_2\mathrm{e}^v}}\right\} + \sqrt{k_1 - k_2\mathrm{e}^{2y}}\right) + B_0, \tag{23}$$

$$S_3(\psi) = \pm\sqrt{2K_2}\psi + c_0.$$

Here (A_0, B_0, C_0) are integration constants and (k_1, k_2) are separation constants (mentioned earlier).

Hence the wave packet gives

$$\phi(x,y,\psi) = \int\int \rho(k_1,k_2) \exp\frac{i}{h}\cdot S_1(x,k_1)\cdot S_2(y,k_1,k_2)\cdot S_3(\psi,k_2)\mathrm{d}k_1\,\mathrm{d}k_2, \tag{24}$$

where $\rho(k_1,k_2)$ has a Gaussian distribution having $\bar{k}_1 > 0$ and $\bar{k}_2 > 0$ and standard deviation σ_1 and σ_2 respectively. For $k_1 > 0$ the wave function will oscillate very rapidly if $x, y \to -\infty$ i.e. $\phi \to 0$. So constructive interference is possible provided

$$\left(\left.\frac{\partial S_0(x,y,\psi)}{\partial k_1}\right|_{k_1=\bar{k}_1}\right)^2 + \left(\left.\frac{\partial S_0}{\partial k_2}\right|_{k_2=\bar{k}_2}\right)^2 = 0. \tag{25}$$

This is supported by the classical solution (7). Thus, classical limit will be obtained when $\phi \to 0$ and there is no restriction on u.

5. Causal interpretation: Bohmian trajectories

We shall first give a general description of causal interpretation. In case of homogeneous minisuperspace models the vector constraint (i.e. supermomentum constraint) vanishes identically. So we are left with only the Hamiltonian constraint, which in operator version gives the WD-equation

$$H\cdot[\hat{p}^{\alpha}(t),\hat{q}_{\alpha}(t)]\Phi(q_{\alpha}(t)) = 0.$$

Here $p^{\alpha}(t)$ and $q_{\alpha}(t)$ represent the homogeneous degrees of freedom, obtained from the three metric h_{ij} and the conjugate momenta Π^{ij}.

Similar to WKB ansatz let us write

$$\Phi = R(q)\ \exp\left(\frac{i}{h}S(q)\right).$$

So the above WD-equation becomes

$$\frac{1}{2}f_{\alpha\beta}(q_{\mu})\cdot\frac{\partial s}{\partial q_{\alpha}}\cdot\frac{\partial s}{\partial q_{\beta}} + U(q_{\mu}) + Q(q_{\mu}) = 0, \tag{26}$$

where

$$Q(q_{\mu}) = -\frac{1}{R}f_{\alpha\beta}\cdot\frac{\partial^2 R}{\partial q_{\alpha}\partial q_{\beta}}, \tag{27}$$

is the quantum potential. Here $f_{\alpha\beta}$ is the reduction of supermetric G_{ijkl} [16] to the given minisuperspace and $U(q_{\mu})$ is the particularization of the scalar curvature density $(-h^{1/2(3)}R)$ of the space-like hypersurfaces. Now according to causal interpretation in quantum cosmology, the trajectories $q_\alpha(t)$ must be real and independent of any observations. As a consequence, eq. (26) represents H–J equation for these trajectories with quantum correction given by eq. (27). Thus we identify

$$P^\alpha = \frac{\partial s}{\partial q_\alpha}$$

with the usual momentum–velocity relation

$$p^\alpha = f^{\alpha\beta}\cdot\frac{1}{N}\cdot\frac{\partial q_\beta}{\partial t}. \tag{28}$$

Hence the quantum trajectories (also as Bohmian trajectories) by the first order differential equation

$$\frac{\partial S(q_\alpha)}{\partial q_\alpha} = f^{\alpha\beta}\cdot\frac{1}{N}\cdot\frac{\partial q_\beta}{\partial t}, \tag{29}$$

which are invariant under time reparametrization [15]. So there is no problem of time in the causal interpretation of minisuperspace quantum cosmology and in particular we choose the gauge $\dot{n} = 0$.

In the present problem the Bohmian trajectories are characterized by

$$\frac{\partial S}{\partial x}(x,y,\psi) = 6\dot{x}, \tag{30a}$$

$$\frac{\partial S}{\partial y} = -2(2w+3)\dot{y}, \tag{30b}$$

$$\frac{\partial S}{\partial \psi} = -4\mathrm{e}^{-2x}\cdot\dot{\psi}. \tag{30c}$$

The H–J equation with quantum correction reads

$$-\frac{1}{12}\left(\frac{\partial s}{\partial x}\right)^2 + \frac{1}{4(2w+3)}\left(\frac{\partial s}{\partial y}\right)^2 + \frac{\mathrm{e}^{2y}}{8}\left(\frac{\partial s}{\partial \psi}\right)^2 + Q + 12\mathrm{e}^{2x} = 0, \tag{31}$$

where

$$Q = \frac{1}{R}\left(\frac{1}{12}\,\frac{\partial^2 R}{\partial x^2} - \frac{1}{4(2w+3)}\,\frac{\partial^2 R}{\partial y^2} - \frac{\mathrm{e}^{2y}}{8}\,\frac{\partial^2 R}{\partial \psi^2}\right), \tag{32}$$

and

$$V = 12\mathrm{e}^{2x}$$

are termed as quantum and classical potentials. For simplicity, let us take

$$\Phi = U_{\lambda_1}(x)V_{\lambda_1,\lambda_2}(y)W_{\lambda_2}(\psi)$$

(see eq. (18a)), then the WKB ansatz will be

$$\Phi = R_1(x)R_2(y)R_3(\psi)\cdot \mathrm{e}^{i}[S_1(x)+S_2(y)+S_3(\psi)], \tag{33}$$

with

$$S(x,y,\psi) = S_1(x)+S_2(y)+S_3(\psi). \tag{34}$$

This implies that the differential equations (characterized by Bohmian trajectories) i.e. eqs (30a) and (30b) are independent of (y,ψ) and (x,ψ) respectively and consequently the quantum potential can be written as

$$Q(x,y,\psi) = Q_1(x)+Q_2(y)+Q_3(\psi,y).$$

To study the behavior of classical singularity in the quantum trajectories we must take the tetrad variable u to be small, i.e. x has infinitely large negative value. In this approximation we consider only the first term of the series expansion

$$I_{\upsilon_1}(x) = \sum_{m=0}^{\infty}\frac{1}{m!\,\overline{|(\upsilon_1+m+1)}}\cdot\left(\frac{x}{2}\right)^{\upsilon_1+2m}.$$

Now we consider two cases

Case I. υ_1 *is real*

Then $I_{\upsilon 1}(x)$ is real, so the phase $U_{\lambda 1}(x)$, i.e. $S_1 = 0$. Hence x is constant (see eq. (30a)) and U cannot tend to zero. Thus it is a non-singular quantum solution but not of much physical interest as the only triad variable becomes constant.

Case II. υ_1 *is pure imaginary*

Let $\upsilon_1 = in_1$, then

$$U_{\lambda_1}(x) = u_0\cdot \mathrm{e}^{in_1x}.$$

Hence

$$S_1(x) = n_1x,\quad R_1(x) = U_0,\quad n_1^2 = -12\lambda_1.$$

So from eq. (30a)

$$x = \frac{n_1}{6}t.$$

Further, we note that υ_2 is also imaginary whenever υ_1 is imaginary. Suppose $\upsilon_2 = in_2$ then

$$\upsilon_{\lambda_1,\lambda_2}(y) = U_0(y)\mathrm{e}^{in_2y}$$

i.e.

$$S_2(y) = n_2y,\quad R_2(y) = U_0(y).$$

Hence from eq. (30b)

$$y = -\frac{n_2}{2(2w+3)}t, \quad n_2^2 = -4(2w+3)\lambda_1^2.$$

As

$$W(\psi) = A_w \exp(i\sqrt{8\lambda_2}\psi),$$

so

$$S_3(\psi) = \sqrt{8\lambda_2}\psi, \quad R_3 = A_w.$$

Thus on integration, eq. (30c) gives

$$\psi = \frac{(2w+3)}{n_2}\sqrt{\frac{\lambda_2}{2}}\mathrm{e}^{-(n_2 t/(2w+3))}.$$

Therefore, the behavior of quantum trajectory is in accordance with the classical one discussed in the previous section and the classical singularity persists in quantum domain. This is not surprising because as R_1, R and R_3 are constants, the quantum potential has no effect and in a sense describes the classical trajectories. Moreover, the expansion or contraction of the Universe depends on whether n_1 is positive or negative.

Finally, we consider the situation when there is a large expanding Universe, i.e., the triad variable u is large (so x is large). Using the asymptotic form for Bessel functions (also the associated Bessel functions) we have

$$U(x) \sim \bar{U}_0 \exp(2\sqrt{3}\,\mathrm{e}^x)/\mathrm{e}^{x/2},$$

$$V(y) \sim \bar{V}_0 \exp[i2\sqrt{(2w+3)\lambda_2}\,\mathrm{e}^x]/\mathrm{e}^{y/2}$$

and

$$W(\psi) = \bar{W}_0 \cdot \exp(i\sqrt{8\lambda_2}\psi),$$

where $\bar{U}_0, \bar{V}_0$ are complex constants and $\bar{W}_0$ is real. So the corresponding phases and the norms are

$$S_1(x) = \text{constant}, \quad R_1(x) = |U_0|\mathrm{e}^{[(x/2)+2\sqrt{3}\mathrm{e}^x]},$$

$$S_2(y) = 2\sqrt{(2w+3)\lambda_2}\,\mathrm{e}^x, \quad R_2(y) = |\bar{V}_0|\mathrm{e}^{-y/2},$$

$$S_3(\psi) = \sqrt{8\lambda_2}\,\psi, \quad R_3(\psi) = \bar{W}_0.$$

So from the quantum trajectory we get

$$\mathrm{e}^x = \text{constant},$$

$$\mathrm{e}^y = \sqrt{2w+3}/t,$$

$$\psi = \left(\sqrt{\frac{\lambda_2}{2}}\right)(2w+3)/t.$$

This solution is not in accordance with classical solution for large volume. This is expected as in this case the quantum potential term has non-zero effect and consequently gives rise to a deviation from classical path.

It is to be noted that if we take solution (19a) instead of (19b) then the asymptotic behavior of V will change. In that case S_2 will be constant, i.e. ϕ is constant at a later time. This behavior also does not agree with classical trajectory in this limit. Finally, it is to be noted that if we take υ_1 to be complex then there will be no qualitative change in the results in comparison to those for imaginary υ_1.

6. Conclusions

In this paper we have studied classical and quantum minisuperspace models with non-minimally coupled scalar–tensor theory. In quantum cosmology, important issues like avoidance of classical singularities using quantum effect and the predication of the classical behavior of the Universe, has been studied using causal interpretation (i.e. quantum Bohmian trajectories). For some exact solutions of the WD-equation, the quantum trajectories indicate that the Universe is classical when the triad variable is small (of the order of h) and singularities are unavoidable. The non-classical behavior has been speculated for large values of the triad variable. We have studied Gaussian superpositions of WKB wave functions to investigate if they correspond to quasiclassical states and have found that the wave functions are peaked around classical trajectories, only for small value of the scalar field ϕ. We have also evaluated wave function by path integral formulation using the method of steepest descent, but it is not possible to make a comparative study of these wave functions with those by solving the WD equation. So for future work it will be interesting to find appropriate measure in these abstract spaces so that a comparative study is possible.

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