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Stable complex solitary waves of Sasa Satsuma equation

SASANKA GHOSH
Physics Department, Indian Institute of Technology, North Guwahati, Guwahati 781 039, India
Email: sasanka@iitg.ernet.in

Abstract. Existence of a new class of complex solitary waves is shown for Sasa Satsuma equation. These solitary waves are found to be stable in a certain domain of the parameter and become chaotic if the parameter exceeds the value 2.4. Significantly, the complex solitary waves propagate at higher bit rate over the most stable solitons under the same conditions of the input parameters.

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The importance of nonlinear Schrödinger equation [1–4] and its higher order generalisation, namely the Sasa Satsuma equation [8] in the context of optical communication through the nonlinear fibre is well-known [5–7]. Solitons play a key role for lossless propagation through the nonlinear optical fibres. The lossless propagation is achieved primarily because of the presence of the stimulated Raman the scattering term in the dynamical equation itself of the Sasa Satsuma equation. The explicit form of the Sasa Satsuma equation [8] is given by

$$\partial_z E = i[\partial_{\tau\tau} E + 2|E|^2 E] + \partial_{\tau\tau\tau} E + 6\partial_\tau(|E|^2 E) + 3\partial_\tau(|E|^2)E \tag{1}$$

where, E is envelope of the electric field propagating in z direction at a time τ. The terms on the RHS respectively represent group velocity dispersion (GD), Kerr effect, third order dispersion (TOD), self-steeping (SS) related to Kerr effect and the self frequency shifting via stimulated Raman scattering. It is the last term, which plays an important role in the propagation of distortionless optical pulses over a long distance. But the contribution from the last three terms becomes appreciable only for the very short optical pulses, typically of the femtosecond order. On the other hand, the nonlinear Schrödinger equation successfully describes the dynamics of the optical pulses of picosecond order.

Realizing the importance of the Sasa Satsuma equation in describing the propagation of ultra short pulses through the fibre, several attempts have been made to find soliton solutions of the Sasa Satsuma eq. (1) [8–12] by various methods like Hirota's bilinear method, Painlevé analysis, Backlund transformation and inverse scattering technique. It has been observed recently that the dispersion managed optical pulses for the nonlinear Schrödinger equation are more useful if the pulses are made in the form of a power series of one soliton solution. These complex pulses are not only stable in a certain domain of the parameter, but also propagate at a higher bit rate than the most stable soliton under the same input condition.

Motivated by these observations, we look for a class of complex solitary wave solutions for the Sasa Satsuma equation. Nonetheless, these pulses will be shown to be stable in a certain domain of the parameter, controlling the rate of propagation. In this context, it is important to mention that the nonlinear Schrödinger equation also exhibits similar kind of solitary wave solutions [13], which are found in the form of a series of localised stationery pulses. These solitary waves are found to propagate at a much higher rate (say 50 GB/s) than the usual NLS solitons, which can propagate at a rate of 10 GB/s for a typical soliton transmission system.

The organization of this paper is as follows. We discuss the existence of a simple localised solution of sech type. As a byproduct we obtain an interesting relationship between the input power and the characteristic properties of the fibre for the Sasa Satsuma equation. Next we obtain a class of solutions as a power series of one soliton. We also study the stability of the solutions and show that onset of deterministic chaos occurs when the value of the parameter controlling the bit rate exceeds 2.4. Moreover, we obtain a relationship of one of the series solutions with the group of nonlinear transformations.

To begin with let us assume a simple trial solution of (1) in the form

$$E(z,\tau) = E_0 \mathrm{e}^{iaz+ib\tau} y(\gamma\tau - \beta\gamma x) \tag{2}$$

where a and b are real parameters and $y(\gamma\tau - \beta\gamma x)$ is chosen as a real function. a and b in the above equation may be interpreted as frequency and wave vector respectively, whereas γ^{-1} and β^{-1} may be related respectively to the width and group velocity of the envelope wave. Substituting the trial solution (2) in (1), we obtain a set of two differential equations in y:

$$(-a - b^2 - b^3)y + (1+3b)y'' + (2+6b)E_0^2 y^3 = 0 \tag{3}$$

$$y_x + (-2b - 3b^2)y' + y''' + 4E_0^2(y^3)' = 0 \tag{4}$$

where $'$ denotes the derivative with respect to τ. Notice that the solution for y must satisfy both the above equations simultaneously. This is possible if one of the equations vanishes identically or the coefficients of each type of term in the above equations are equal. The coefficients in (4) cannot be set equal to zero since this eventually leads to $E_0 = 0$. As a second possibility, if the coefficients of the eqs (3) and (4) are equated, the last term becomes inconsistent. The only possibility left, therefore, is to set the coefficients in (3) to zero. If we now choose

$$b = -\frac{1}{3} \tag{5}$$

$$a = \frac{2}{27} \tag{6}$$

then eq. (3) vanishes. We will see later that these conditions, in fact, lead to one soliton solution. As a consequence, the eq. (4) reduces to

$$y_x + \frac{1}{3}y' + y''' + 4E_0^2(y^3)' = 0 \tag{7}$$

It can straightaway be observed that if we choose a solution of y of the form

$$y = \operatorname{sech}(\gamma\tau - \beta\gamma \mathrm{x}) \tag{8}$$

the eq. (7) will be satisfied, provided the following conditions are obeyed

$$\beta = \gamma^2 + \frac{1}{3} \tag{9}$$

$$E_0^2 = \frac{1}{2}\gamma^2 \tag{10}$$

The first condition determines the group velocity of the envelope wave. The second condition, on the other hand, gives an interesting relation between the input power and the width of the optical pulse. This, in fact, gives an estimate of the required input power for a given width of an optical pulse to propagate in a particular fibre. If this relation mismatches for a particular fibre, the soliton will not be formed.

Finally, the solution of (1) will be of the form

$$E(x,\tau) = \frac{\gamma}{\sqrt{2}} \exp\left[i\frac{2}{27}x - i\frac{1}{3}\tau\right] \operatorname{sech}\left[\gamma\tau - \gamma\left(\gamma^2 + \frac{\gamma}{3}\right)\mathrm{x}\right] \tag{11}$$

which is nothing but one soliton solution [8,12].

It is interesting to note that, apart from the above solution (11), the Sasa Satsuma equation (1) inherits a class of stable solitary wave solutions of more complex nature. We will show that these pulses propagate at a higher bit rate than the most stable pulses, namely solitons under similar input conditions. We may write the trial solutions explicitly in the form

$$y = \sum_{j=0}^{\infty} C_j \operatorname{sech}^{\mathrm{j}}(\gamma\tau - \beta\gamma\mathrm{x}) \tag{12}$$

as a power series of one soliton (11), where C_j are c-number coefficients and $j = 0, 1, 2, \cdots$. Substituting (12) into (7), we obtain a recursion relation among the coefficients C_j. It is found that unless the first coefficient C_0 becomes zero, the solution emerges as a trivial solution. If $C_0 = 0$, all even coefficients consequently become zero

$$C_{2j} = 0 \tag{13}$$

The odd coefficients, however, satisfy a nontrivial recursion relation among themselves for arbitrary *nonzero* real values of C_1. The explicit form of the recursion relation may be given as

$$C_{2j+3} = \frac{2j+1}{2(j+2)} C_{2j+1} - \frac{3}{2(j+1)(j+2)(2j+3)} \sum_{m=0}^{2j+1} \sum_{l=0}^{m} l C_l C_{m-l} C_{2j-m+3} \tag{14}$$

for $j = 0, 1, 2, 3, \cdots$ and $C_1 \neq 0$. The existence of a power series solutions for the Sasa Satsuma equation (1) has some interesting consequences. A few remarks about the solutions (12), (13), (14) are in order.

1. The recursion relation (14) has a universal form and does not depend on the parameters of the dynamical equation.

2. It is obvious to observe from the recursion relation (14) that if the arbitrary coefficient C_1 is chosen to be unity all higher order coefficients starting from C_3 become zero and the series is left with only one term, $y = \mathrm{sech}(\gamma\tau - \beta\gamma x)$. Thus for $C_1 = 1$, the solution (12) reduces to the one soliton solution (11) itself.

3. Depending on the values of C_1, the Sasa Satsuma equation admits an infinite number of localised solutions of the form (12). But all the localised solutions may not be stable and eventually become chaotic depending on the values of C_1. The Lyapunov exponent

$$\lambda(C_1) = \lim_{j\to\infty} \frac{1}{j} \ln \left| \frac{\mathrm{d}C_{2j+1}}{\mathrm{d}C_1} \right| \tag{15}$$

for different values of C_1 yields to the onset of chaos for $C_1 > 2.4$. If $C_1 \leq 2.4$, the new localised optical pulses are expected to be almost solitonic in nature and are stable. As expected the maximum stability of the solution is found at $C_1 = 1$.

4. It is observed that if $C_1 \geq 1$, the intensity of the pulse remains constant upto the value $C_1 = 2.4$ and shoots up for $C_1 > 2.4$. But for $C_1 < 1$, the behaviour of the pulses becomes extremely interesting. Both the intensity and the width of the pulses decrease together with the decrease of the value of C_1 from $C_1 = 1$. Thus for a particular fibre with a given input power, the widths of these non-solitonic pulses will be smaller than those of most stable solitons. This ensures the higher bit rate of propagation of non-solitonic pulses over the solitons under similar input conditions.

To conclude, localised solitary wave solutions are obtained for the Sasa Satsuma equation. These pulses are stable in a certain domain of the parameter C_1. For $C_1 > 2.4$, the pulses become chaotic. But for $C_1 < 1$, the behaviour of the waves becomes interesting. With the decrease in the values of C_1, both the width and the intensity of the envelope waves decrease. This behaviour of non-solitonic pulses leads to higher bit rate of propagation over the solitons under similar input conditions. This definitely has an interesting consequence in the communication system through the nonlinear fibre.

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