

Periodic wavetrains for systems of coupled nonlinear Schrödinger equations

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Abstract. Exact, periodic wavetrains for systems of coupled nonlinear Schrödinger equations are obtained by the Hirota bilinear method and theta functions identities. Both the bright and dark soliton regimes are treated, and the solutions involve products of elliptic functions. The validity of these solutions is verified independently by a computer algebra software. The long wave limit is studied. Physical implications will be assessed.

Keywords. Coupled nonlinear Schrödinger equations; periodic solutions.

PACS Nos 42.65.Tg; 42.81.Dp; 0.2.30Jr

1. Introduction

Systems of coupled nonlinear Schrödinger equations (cNLS) have received tremendous attention recently because of their large potential in applications, e.g., hydrodynamics and optics. The main objective of this paper is to show that a combination of the Hirota bilinear method and theta functions provides a powerful, convenient, and effective way to obtain periodic solutions for such cNLS systems. To be precise, we shall consider cNLS systems of N components in the unstable (or anomalous dispersion) regime, or cNLS+,

$$i \frac{\partial \phi_n}{\partial t} + \frac{\partial^2 \phi_n}{\partial x^2} + \left(\sum_{m=1}^N \phi_m \phi_m^* \right) \phi_n = 0, \quad n = 1, 2, \dots, N, \quad (1.1)$$

and the stable (or normal dispersion) regime, or cNLS–,

$$i \frac{\partial \psi_n}{\partial t} + \frac{\partial^2 \psi_n}{\partial x^2} - \left(\sum_{m=1}^N \psi_m \psi_m^* \right) \psi_n = 0, \quad n = 1, 2, \dots, N. \quad (1.2)$$

We shall use x (space) and t (time) following the conventional mathematical treatment of such equations. However, one must realize that the roles of x and t should be reversed in optical solitons applications. The focus will first be on a review of selected works in the literature. Specific reference to the optics situation will be made in the Conclusions section. Periodic solutions of (1.1) and (1.2), $N = 2$, in terms of one or product of two Jacobi elliptic

functions were obtained earlier [1–4]. The existence of some of these product elliptic functions solutions is especially striking. They arise purely due to the coupling, and would otherwise be absent for the single mode, uncoupled case ($N = 1$).

The objective now is to study (1.1) and (1.2) for a higher degree of freedom (larger N). Periodic solutions in terms of product of up to two elliptic functions are given for a system of three cNLS ($N = 3$) [5]. Periodic solutions in terms of product of up to three elliptic functions are also derived for the $N = 3$ case using the classical Legendre polynomials and Lamé equation [6].

The mathematical details of this combination of the Hirota method and theta functions will be described briefly (§2). For illustration purposes, a system of four cNLS ($N = 4$ in (1.1)) will be chosen as an example in this paper. Periodic solutions in terms of product of up to two elliptic functions are readily deduced. There are two degrees of freedom in the choice of the amplitude parameters, and by a suitable choice these new expressions generalize the known solutions in the literature [1–4]. To illustrate the power of the present technique, a periodic solution in terms of product of up to four elliptic functions is derived for a system of four cNLS. In optical applications this corresponds to a more general index of refraction or sum of intensities, and generalizes our earlier result. In addition similar periodic waves for (1.2) are derived. There are now no free parameters and the amplitudes must generally be determined numerically from a set of constraints. A total of three sets of new results are presented (§§3–5).

The long wave limit is studied, and the result is a stationary sequence of solitary waves. The validity of all these new solutions is confirmed by direct differentiation and substitution in (1.1) or (1.2) with a computer algebra software. Finally, conclusions are drawn (§6).

2. Method

The first step is to seek special solutions of the cNLS systems in bilinear forms. Since the treatment for (1.2) is similar, we shall just consider system (1.1) :

$$\phi_n(x, t) = \frac{g_n(x) \exp(-i\Omega_n t)}{f(x)}, \quad f \text{ real}, \quad (2.1)$$

$$f[D_x^2 g_n \cdot f + \Omega_n g_n f] + g_n \left[-D_x^2 f \cdot f + \sum_{m=1}^N g_m g_m^* \right] = 0, \quad (2.2)$$

where D is the Hirota operator. The crucial difference between the present situation and the case of solitary waves is that the bilinear form (2.2) must now be used as a single equation, and not as two decoupled equations. For simplicity we shall assume g_n to be real as well.

Secondly we choose

$$f = (\theta_4(\alpha x))^p \quad (2.3)$$

where p is a positive integer (2 or 4 in this paper). The precise forms of the theta functions are found in standard references [7–9] and in the Appendix.

Finally g_n must be chosen such that (2.2) is satisfied. The Hirota derivatives of theta functions are handled by theta identities (the Appendix). The crucial step in deducing the correct form of g_n is that sufficient powers of $\theta_3(x)$ must be cancelled in (2.2) for the matching to be performed.

Guided by the forms given earlier in the literature, a generalized solution of (1.1) is

$$\begin{aligned}\phi_1 &= \frac{A_1 \theta_1(\alpha x) \theta_2(\alpha x) \exp(-i\Omega_1 t)}{\theta_4^2(\alpha x)}, \\ \phi_2 &= \frac{A_2 \theta_1(\alpha x) \theta_3(\alpha x) \exp(-i\Omega_2 t)}{\theta_4^2(\alpha x)}, \\ \phi_3 &= \frac{A_3 \theta_2(\alpha x) \theta_3(\alpha x) \exp(-i\Omega_3 t)}{\theta_4^2(\alpha x)},\end{aligned}\tag{2.4}$$

$$\begin{aligned}\phi_4 &= \frac{A_4 (c \theta_4^2(\alpha x) - \theta_3^2(\alpha x)) \exp(-i\Omega_4 t)}{\theta_4^2(\alpha x)}, \\ A_2^2 &= \frac{6\alpha^2 \theta_3^4(0) \theta_4^2(0)}{\theta_2^2(0)} + \frac{A_4^2 (2c \theta_3^2(0) - \theta_4^2(0))}{\theta_2^2(0)} - \frac{A_1^2 \theta_3^6(0)}{\theta_2^6(0)}, \\ A_3^2 &= \frac{6\alpha^2 \theta_3^2(0) \theta_4^4(0)}{\theta_2^2(0)} + \frac{A_4^2 (2c \theta_4^2(0) - \theta_3^2(0))}{\theta_2^2(0)} - \frac{A_1^2 \theta_4^6(0)}{\theta_2^6(0)}.\end{aligned}\tag{2.5}$$

There are two degrees of freedom in the amplitude parameters (A_1, A_4 arbitrary). The angular frequencies Ω_n are given in terms of theta constants and $A_n \cdot c$ satisfies

$$3c^2 + 1 - 2 \left(\frac{\theta_3^2(0)}{\theta_4^2(0)} + \frac{\theta_4^2(0)}{\theta_3^2(0)} \right) c = 0.$$

Alternatively the solution (2.4) and (2.5) can also be expressed in elliptic functions format (k is the modulus of the elliptic functions)

$$\begin{aligned}\phi_1 &= \frac{A_1 k \operatorname{sn}(rx) \operatorname{cn}(rx) \exp(-i\Omega_1 t)}{(1-k^2)^{1/4}}, \\ \phi_2 &= \frac{A_2 \sqrt{k} \operatorname{sn}(rx) \operatorname{dn}(rx) \exp(-i\Omega_2 t)}{(1-k^2)^{1/4}}, \\ \phi_3 &= \frac{A_3 \sqrt{k} \operatorname{cn}(rx) \operatorname{dn}(rx) \exp(-i\Omega_3 t)}{(1-k^2)^{1/2}},\end{aligned}\tag{2.6}$$

$$\begin{aligned}\phi_4 &= A_4 \left[c - \frac{(\operatorname{dn}(rx))^2}{(1-k^2)^{1/2}} \right] \exp(-i\Omega_4 t), \\ A_2^2 &= \frac{6r^2 (1-k^2)^{1/2}}{k} - \frac{A_1^2}{k^3} + \frac{A_4^2 (2c - (1-k^2)^{1/2})}{k}, \\ A_3^2 &= \frac{6r^2 (1-k^2)}{k} - \frac{A_1^2 (1-k^2)^{3/2}}{k^3} + \frac{A_4^2 (2c(1-k^2)^{1/2} - 1)}{k}.\end{aligned}\tag{2.7}$$

Similarly, solutions involving three elliptic or theta functions can be deduced. However, a system of four cNLS will be considered in this work to illustrate the applicability of the

present procedure, as most works in the literature treated (1.1), (1.2) for up to $N = 3$ only. A little experimentation shows that the arrangements

$$g_n = A[c\theta_4^2(\alpha x) - \theta_3^2(\alpha x)]\theta_1(\alpha x)\theta_2(\alpha x)\exp(-i\Omega t), \quad (2.8)$$

$$g_n = A[c\theta_4^2(\alpha x) - \theta_3^2(\alpha x)]\theta_1(\alpha x)\theta_3(\alpha x)\exp(-i\Omega t), \quad (2.9)$$

$$g_n = A[c\theta_4^2(\alpha x) - \theta_3^2(\alpha x)]\theta_2(\alpha x)\theta_3(\alpha x)\exp(-i\Omega t), \quad (2.10)$$

are possible candidates for solving (2.2). The combination (2.8), (2.9) is rejected since the highest power of $\theta_3(\alpha x)$ does not cancel in the $\Sigma g_m g_m^*$ term of (2.2). Expressions (2.9), (2.10) form an acceptable solution [10]. Four linearly independent solutions are obtained by suitably choosing A and c . Similar calculations with further details will be presented in §§3, 4 and 5. The sum of intensity, which is related to the ‘refractive index’ [11], is

$$\sum_{m=1}^4 \psi_m \psi_m^* = 20r^2 \text{dn}^2(rx). \quad (2.11)$$

The present work starts by considering the pair (2.8), (2.10). The counterpart of (2.11) is then ($h_0 = \text{constant}$)

$$\sum_{m=1}^4 \psi_m \psi_m^* = h_0 + 20r^2 \text{dn}^2(rx), \quad h_0 \neq 0,$$

and this thus generalizes the previous result [10].

In the subsequent sections all the results will be expressed in terms of the more compact elliptic functions notations. We emphasize, however, that all the intermediate calculations are performed entirely by theta functions.

3. Periodic solutions of cNLS+

A periodic solution of cNLS+, (1.1), $N = 4$ is ($k = \text{modulus of the elliptic functions}$)

$$\phi_1 = A_1 \left[c_1 - \frac{\text{dn}^2(rx)}{(1-k^2)^{1/2}} \right] \frac{k \text{sn}(rx) \text{cn}(rx) \exp(-i\Omega_1 t)}{(1-k^2)^{1/4}}, \quad (3.1)$$

$$\phi_2 = A_2 \left[c_2 - \frac{\text{dn}^2(rx)}{(1-k^2)^{1/2}} \right] \frac{k \text{sn}(rx) \text{cn}(rx) \exp(-i\Omega_2 t)}{(1-k^2)^{1/4}}, \quad (3.2)$$

$$\phi_3 = A_3 \left[c_3 - \frac{\text{dn}^2(rx)}{(1-k^2)^{1/2}} \right] \frac{\sqrt{k} \text{cn}(rx) \text{dn}(rx) \exp(-i\Omega_3 t)}{(1-k^2)^{1/2}}, \quad (3.3)$$

$$\phi_4 = A_4 \left[c_4 - \frac{\text{dn}^2(rx)}{(1-k^2)^{1/2}} \right] \frac{\sqrt{k} \text{cn}(rx) \text{dn}(rx) \exp(-i\Omega_4 t)}{(1-k^2)^{1/2}}, \quad (3.4)$$

A_1, A_2, A_3, A_4 satisfy

$$\begin{aligned} & -\sqrt{1-k^2}A_1^2 - \sqrt{1-k^2}A_2^2 + kA_3^2 + kA_4^2 = 0, \\ & -\left(2c_1\sqrt{1-k^2} + 2-k^2\right)A_1^2 - \left(2c_2\sqrt{1-k^2} + 2-k^2\right)A_2^2 \end{aligned} \quad (3.5)$$

$$+k \left(2c_3 + \sqrt{1-k^2} \right) A_3^2 + k \left(2c_4 + \sqrt{1-k^2} \right) A_4^2 = 0, \quad (3.6)$$

$$\begin{aligned} & - \left[2c_1(2-k^2) + (c_1^2 + 1)\sqrt{1-k^2} \right] A_1^2 \\ & - \left[2c_2(2-k^2) + (c_2^2 + 1)\sqrt{1-k^2} \right] A_2^2 \\ & + kc_3 \left(c_3 + 2\sqrt{1-k^2} \right) A_3^2 + kc_4 \left(c_4 + 2\sqrt{1-k^2} \right) A_4^2 = 0, \end{aligned} \quad (3.7)$$

$$\begin{aligned} & c_1 \left[2\sqrt{1-k^2} + c_1(2-k^2) \right] A_1^2 + c_2 \left[2\sqrt{1-k^2} + c_2(2-k^2) \right] A_2^2 \\ & - c_3^2 k \sqrt{1-k^2} A_3^2 - c_4^2 k \sqrt{1-k^2} A_4^2 = 20r^2 k^2 \sqrt{1-k^2}. \end{aligned} \quad (3.8)$$

c_1, c_2 are roots of

$$7z^2 - \left(\frac{4}{\sqrt{1-k^2}} + 4\sqrt{1-k^2} \right) z + 1 = 0, \quad (3.9)$$

while c_3, c_4 are roots of

$$7z^2 - \left(\frac{6}{\sqrt{1-k^2}} + 4\sqrt{1-k^2} \right) z + 3 = 0. \quad (3.10)$$

The angular frequencies Ω_n are

$$\begin{aligned} \Omega_n &= h_0 + 14r^2 \sqrt{1-k^2} c_n - 9r^2(2-k^2), \quad n = 1, 2 \\ \Omega_n &= h_0 + 14r^2 \sqrt{1-k^2} c_n - r^2(25-9k^2), \quad n = 3, 4 \end{aligned}$$

where

$$h_0 = - \frac{\sqrt{1-k^2}(c_1^2 A_1^2 + c_2^2 A_2^2)}{k^2}. \quad (3.11)$$

The sum of intensities, which is related to the refractive index, is (h_0 given by (3.11))

$$\sum_{m=1}^4 \phi_m \phi_m^* = h_0 + 20r^2 \text{dn}^2(rx). \quad (3.12)$$

We also verified by direct differentiation with the computer algebra software MATHEMATICA that (3.1)–(3.4) satisfy (1.1).

The long wave limit ($k \rightarrow 1$) of (3.1)–(3.4) can be studied by suitably redefining the variables c_n , and A_n . Omitting the intermediate algebraic calculations the end result is

$$\phi_1 = \frac{3\sqrt{35}r \text{sech}^3(rx) \tanh(rx) \exp(9ir^2t)}{2}, \quad (3.13)$$

$$\phi_2 = \frac{7\sqrt{5}r \text{sech}(rx) \tanh(rx)}{2} \left(\frac{4}{7} - \text{sech}^2(rx) \right) \exp(ir^2t), \quad (3.14)$$

$$\phi_3 = \frac{\sqrt{70}r \text{sech}^4(rx) \exp(16ir^2t)}{2}, \quad (3.15)$$

$$\phi_4 = \frac{7\sqrt{10}r \text{sech}^2(rx)}{2} \left(\frac{6}{7} - \text{sech}^2(rx) \right) \exp(4ir^2t). \quad (3.16)$$

Physically (3.11)–(3.14) correspond to a pattern of stationary (or steadily propagating) solitons.

4. Periodic solution of cNLS-

By examining (2.5) and (2.8)–(2.10) one way for constructing new solutions is to consider polynomials of even powers in $\theta_3(x)$ and $\theta_4(x)$, or $\text{dn}(x)$. Working along this line of reasoning produces a new solution for cNLS-, or (1.2), in the following forms (k = modulus of the elliptic functions):

$$\psi_1 = A_1 \left[c_1 - \frac{\delta_1 \text{dn}^2(rx)}{(1-k^2)^{1/2}} - \frac{\text{dn}^4(rx)}{1-k^2} \right] \exp(-i\Omega_1 t), \quad (4.1)$$

$$\psi_2 = A_2 \left[c_2 - \frac{\delta_2 \text{dn}^2(rx)}{(1-k^2)^{1/2}} - \frac{\text{dn}^4(rx)}{1-k^2} \right] \exp(-i\Omega_2 t), \quad (4.2)$$

$$\psi_3 = A_3 \left[c_3 - \frac{\text{dn}^2(rx)}{(1-k^2)^{1/2}} \right] \frac{\sqrt{k} \text{sn}(rx) \text{dn}(rx) \exp(-i\Omega_3 t)}{(1-k^2)^{1/4}}, \quad (4.3)$$

$$\psi_4 = A_4 \left[c_4 - \frac{\text{dn}^2(rx)}{(1-k^2)^{1/2}} \right] \frac{\sqrt{k} \text{sn}(rx) \text{dn}(rx) \exp(-i\Omega_4 t)}{(1-k^2)^{1/4}}. \quad (4.4)$$

A_1, A_2, A_3, A_4 satisfy

$$-A_1^2 - A_2^2 + \frac{\sqrt{1-k^2}}{k} (A_3^2 + A_4^2) = 0, \quad (4.5)$$

$$2\delta_1 A_1^2 + 2\delta_2 A_2^2 + \frac{(1+2c_3\sqrt{1-k^2})A_3^2}{k} + \frac{(1+2c_4\sqrt{1-k^2})A_4^2}{k} = 0, \quad (4.6)$$

$$-(\delta_1^2 - 2c_1)A_1^2 - (\delta_2^2 - 2c_2)A_2^2 + \frac{c_3(2+c_3\sqrt{1-k^2})A_3^2}{k} + \frac{c_4(2+c_4\sqrt{1-k^2})A_4^2}{k} = 0, \quad (4.7)$$

$$-2c_1\delta_1 A_1^2 - 2c_2\delta_2 A_2^2 + \frac{c_3^2 A_3^2}{k} + \frac{c_4^2 A_4^2}{k} = -20r^2 \sqrt{1-k^2}. \quad (4.8)$$

$\delta_n, n = 1, 2$ are any two roots of

$$49\delta^3 + \frac{98(2-k^2)\delta^2}{\sqrt{1-k^2}} + \left[52 + \frac{48(2-k^2)^2}{1-k^2} \right] \delta + \frac{48(2-k^2)}{\sqrt{1-k^2}} = 0. \quad (4.9)$$

Numerically three real, negative roots are usually found for eq. (4.9) for $0 < k < 1$.

$c_n, n = 1, 2$ are related to δ_n by

$$c_n = \frac{\delta_n}{7\delta_n + 8 \left[\sqrt{1-k^2} + 1/\sqrt{1-k^2} \right]}. \quad (4.10)$$

c_3, c_4 are roots of

$$7z^2 - \left(\frac{4}{\sqrt{1-k^2}} + 6\sqrt{1-k^2} \right) z + 3 = 0. \quad (4.11)$$

The angular frequencies are

$$\begin{aligned}\Omega_n &= h_0 - 14r^2\sqrt{1-k^2}\delta_n - 16r^2(2-k^2), \quad n = 1, 2, \\ \Omega_n &= h_0 + 14r^2\sqrt{1-k^2}c_n - r^2(25-16k^2), \quad n = 3, 4, \\ h_0 &= c_1^2A_1^2 + c_2^2A_2^2.\end{aligned}\tag{4.12}$$

The sum of intensities is

$$\sum_{m=1}^4 \psi_m \psi_m^* = h_0 - 20r^2 \operatorname{dn}^2(rx),$$

where h_0 is given by (4.12). Computer algebra is used to verify that (4.1)–(4.4) satisfy (1.2).

The long wave limit of (4.1)–(4.4) can be studied by taking the limit $k \rightarrow 1$. $V = \delta\sqrt{1-k^2}$ satisfies

$$49V^3 + 98V^2 + 48V = 0$$

to leading order as $k \rightarrow 1$. The root $V = -8/7$ must be among one of the two roots for δ_n in (4.1), (4.2), as this choice will render $c_n \rightarrow \infty$, or $[c_n(1-k^2)]$ finite in the long wave limit of (4.10). Algebraic manipulations now lead to the desired result:

$$\psi_1 = \frac{35\sqrt{2}r}{2} \left[-\frac{8}{35} + \frac{8\operatorname{sech}^2(rx)}{7} - \operatorname{sech}^4(rx) \right] \exp(-32ir^2t),\tag{4.13}$$

$$\psi_2 = \frac{7\sqrt{30}r\operatorname{sech}^2(rx)}{2} \left[\frac{6}{7} - \operatorname{sech}^2(rx) \right] \exp(-28ir^2t),\tag{4.14}$$

$$\psi_3 = \frac{7\sqrt{5}r\operatorname{sech}^3(rx) \tanh(rx) \exp(-23ir^2t)}{2},\tag{4.15}$$

$$\psi_4 = \frac{35\sqrt{3}r\operatorname{sech}(rx) \tanh(rx)}{2} \left[\frac{4}{7} - \operatorname{sech}^2(rx) \right] \exp(-31ir^2t).\tag{4.16}$$

Physically (4.13) corresponds to a sequence of dark solitons, while (4.14)–(4.16) represent three bright solitons. Hence, the cross phase modulation terms $\psi_m \psi_m^* \psi_n$, $m \neq n$, of (1.2) are important, as the three localized bright solitons probably cannot exist in this dark soliton regime.

5. Another solution for cNLS–

The results of the previous section correspond to the choice of (2.9) in conjunction with a polynomial of degree four in (4.1), (4.2). It is possible to arrive at another solution by combining (2.8) and the polynomial of degree four ($k =$ modulus of the elliptic functions):

$$\psi_1 = A_1 \left[c_1 - \frac{\delta_1 \operatorname{dn}^2(rx)}{(1-k^2)^{1/2}} - \frac{\operatorname{dn}^4(rx)}{1-k^2} \right] \exp(-i\Omega_1 t),\tag{5.1}$$

$$\psi_2 = A_2 \left[c_2 - \frac{\delta_2 \operatorname{dn}^2(rx)}{(1-k^2)^{1/2}} - \frac{\operatorname{dn}^4(rx)}{1-k^2} \right] \exp(-i\Omega_2 t), \quad (5.2)$$

$$\psi_3 = A_3 \left[c_3 - \frac{\operatorname{dn}^2(rx)}{(1-k^2)^{1/2}} \right] \frac{k \operatorname{sn}(rx) \operatorname{cn}(rx) \exp(-i\Omega_3 t)}{(1-k^2)^{1/4}}, \quad (5.3)$$

$$\psi_4 = A_4 \left[c_4 - \frac{\operatorname{dn}^2(rx)}{(1-k^2)^{1/2}} \right] \frac{k \operatorname{sn}(rx) \operatorname{cn}(rx) \exp(-i\Omega_4 t)}{(1-k^2)^{1/4}}. \quad (5.4)$$

The equations governing the amplitudes are

$$A_1^2 + A_2^2 - \frac{\sqrt{1-k^2} A_3^2}{k^2} - \frac{\sqrt{1-k^2} A_4^2}{k^2} = 0, \quad (5.5)$$

$$2\delta_1 A_1^2 + 2\delta_2 A_2^2 + \frac{(2-k^2 + 2c_3 \sqrt{1-k^2}) A_3^2}{k^2} + \frac{(2-k^2 + 2c_4 \sqrt{1-k^2}) A_4^2}{k^2} = 0, \quad (5.6)$$

$$(\delta_1^2 - 2c_1) A_1^2 + (\delta_2^2 - 2c_2) A_2^2 - \frac{[2c_3(2-k^2) + (c_3^2 + 1)\sqrt{1-k^2}] A_3^2}{k^2} - \frac{[2c_4(2-k^2) + (c_4^2 + 1)\sqrt{1-k^2}] A_4^2}{k^2} = 0, \quad (5.7)$$

$$-2\delta_1 c_1 A_1^2 - 2\delta_2 c_2 A_2^2 + \frac{c_3 [c_3(2-k^2) + 2\sqrt{1-k^2}] A_3^2}{k^2} + \frac{c_4 [c_4(2-k^2) + 2\sqrt{1-k^2}] A_4^2}{k^2} = -20r^2 \sqrt{1-k^2}, \quad (5.8)$$

where $c_n, \delta_n, n = 1, 2$, are still given by (4.9), (4.10). $c_n, n = 3, 4$ are roots of (3.9). The angular frequencies Ω_n follow the same pattern as before, but now with a different h_0 :

$$\begin{aligned} \Omega_n &= h_0 - 14r^2 \sqrt{1-k^2} \delta_n - 16r^2(2-k^2), \quad n = 1, 2, \\ \Omega_n &= h_0 + 14r^2 \sqrt{1-k^2} c_n - 9r^2(2-k^2), \quad n = 3, 4, \\ h_0 &= c_1^2 A_1^2 + c_2^2 A_2^2 - \frac{c_3^2 \sqrt{1-k^2} A_3^2}{k^2} - \frac{c_4^2 \sqrt{1-k^2} A_4^2}{k^2} \end{aligned} \quad (5.9)$$

The sum of intensities is still given in the form of

$$\sum_{m=1}^4 \psi_m \psi_m^* = h_0 - 20r^2 \operatorname{dn}^2(rx), \quad (5.10)$$

but now h_0 is given by (5.9). Computer algebra is used to verify that (5.1)–(5.4) satisfy (1.2). Figures 1–4 show the intensity of each component, ψ_n , (5.1)–(5.4). They are clearly linearly independent. The long wave limit can also be investigated, but the resulting solution is again (4.13)–(4.16). This is not surprising as both $\operatorname{cn}(rx)$ and $\operatorname{dn}(rx)$ tend to $\operatorname{sech}(rx)$ as $k \rightarrow 1$. The long wave limits, (4.13)–(4.16), are illustrated in figures 5–8.

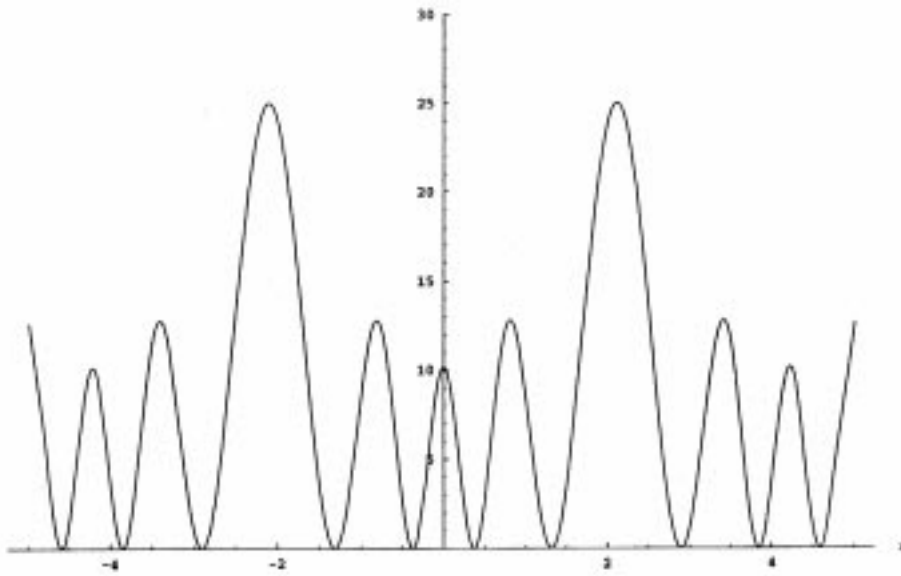


Figure 1. Plot of the intensity $|\psi_1|^2$ versus x , eq. (5.1), $r = 1$, $k = 0.85$, $\delta_1 = -2.506$, $c_1 = -1.351$.

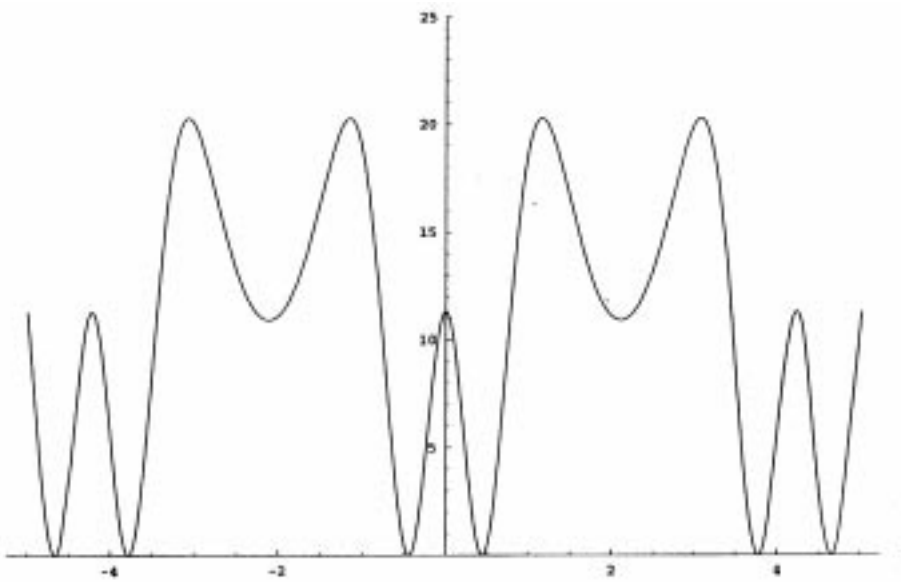


Figure 2. Plot of the intensity $|\psi_2|^2$ versus x , eq. (5.2), $r = 1$, $k = 0.85$, $\delta_2 = -1.824$, $c_2 = -0.275$.

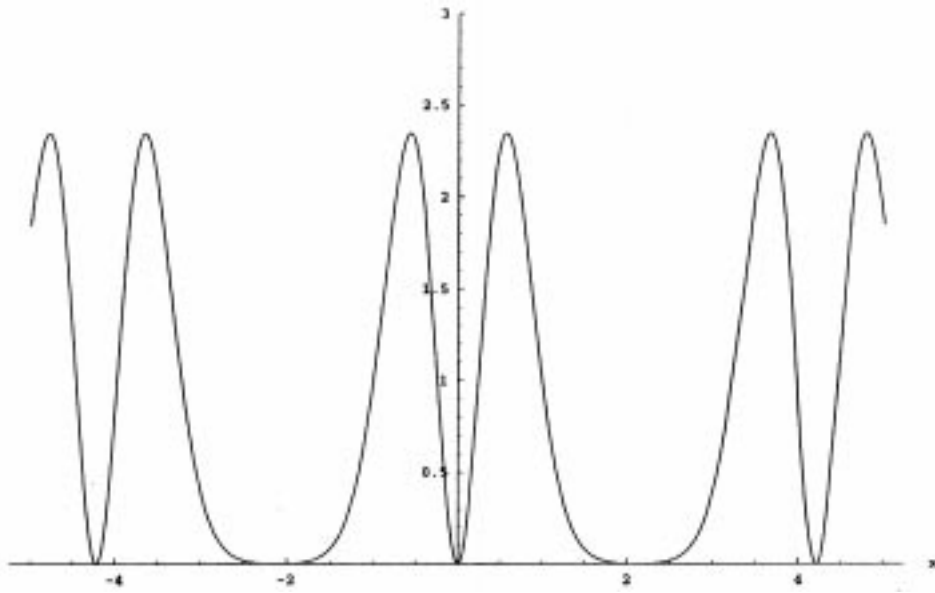


Figure 3. Plot of the intensity $|\psi_3|^2$ versus x , eq. (5.3), $r = 1$, $k = 0.85$, $c_3 = 0.366$.

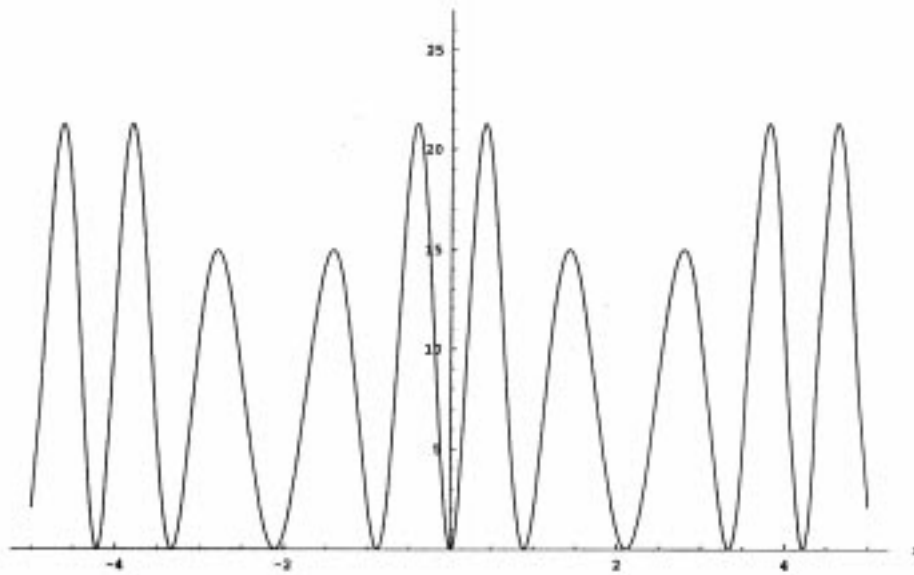


Figure 4. Plot of the intensity $|\psi_4|^2$ versus x , eq. (5.4), $r = 1$, $k = 0.85$, $c_4 = 1.170$.

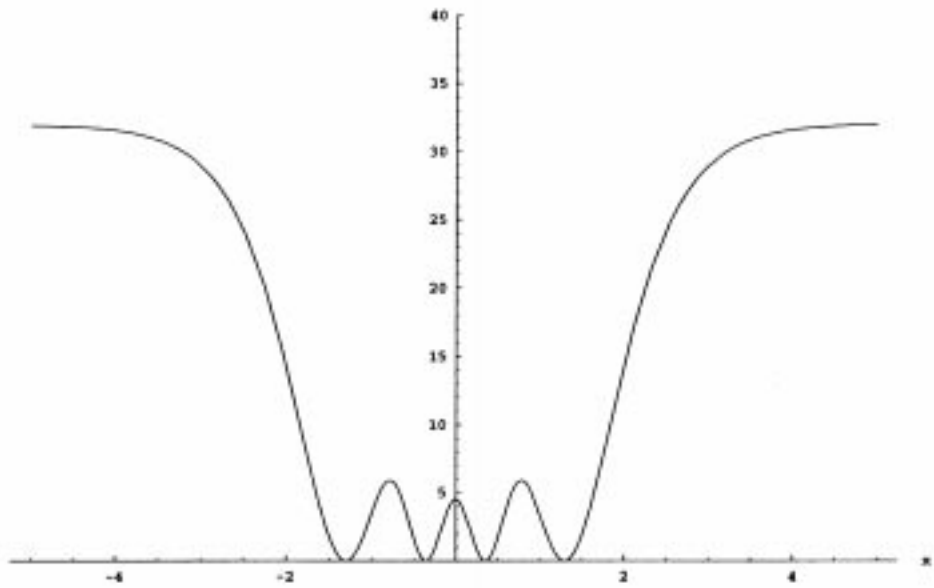


Figure 5. Plot of the intensity $|\psi_1|^2$ in the long wave limit versus x , eq. (4.13), $r = 1$.

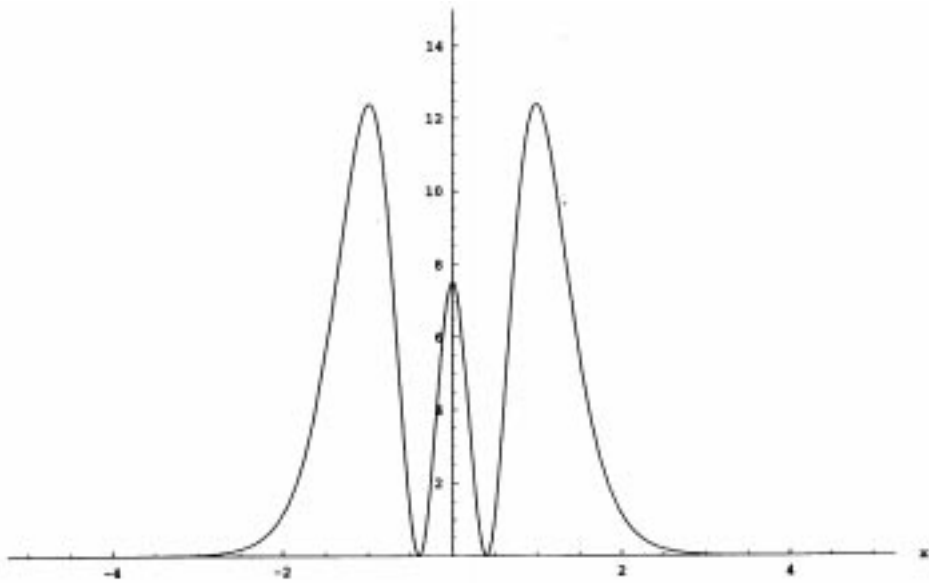


Figure 6. Plot of the intensity $|\psi_2|^2$ in the long wave limit versus x , eq. (4.14), $r = 1$.

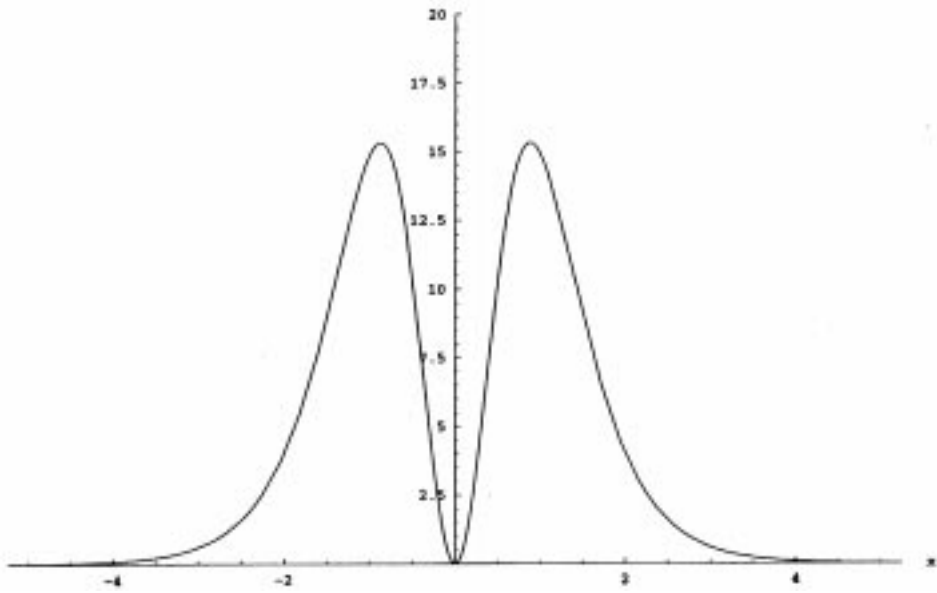


Figure 7. Plot of the intensity $|\psi_3|^2$ in the long wave limit versus x , eq. (4.15), $r = 1$.

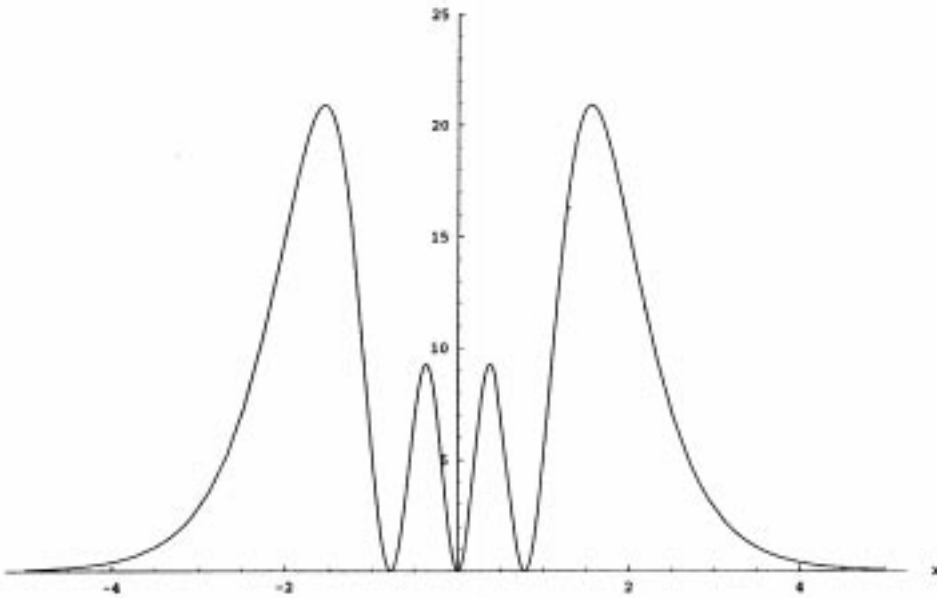


Figure 8. Plot of the intensity $|\psi_4|^2$ in the long wave limit versus x , eq. (4.16), $r = 1$.

6. Conclusions

A class of periodic solutions for coupled systems of nonlinear Schrödinger equations is derived by a combination of the Hirota bilinear method and theta functions identities. In this section we now focus on the potential physical applications of the new solutions developed here, especially in optics.

The propagation of optical solitons along fibers has played an important role in the recent development of long distance communications. Ever since the first theoretical prediction and experimental observation, research in optical solitons has attracted tremendous attention. The physical origin of these solitary waves is that the nonlinear effects of self phase modulation will balance the group velocity dispersion. This leads to the single mode case ($N = 1$ in (1.1, 1.2)). In birefringent fibers the components of polarization in the picosecond regime will typically satisfy a system of coupled nonlinear Schrödinger equations. The $N = 2$ case of (1.1, 1.2), or the Manakov system, constitutes the governing equations for the special cases of random or elliptically birefringent fibers [12].

To increase the information carrying capacity, it will be desirable and perhaps necessary to handle more channels and to transmit ultrashort soliton pulses at a high bit rate. Third order dispersion, self-steepening, and Raman effects are then known to play a critical role [13]. Wavelength division multiplexing will provide a possible solution to this problem. Higher order cNLS systems are then derived [13]. For transmission of three, four or in general N fields, (1.1) and (1.2) will provide the leading order approximation for such coupled, higher cNLS systems. One of the immediate future goals of the present work is to attempt to generalize the class of solutions found here to such coupled, higher order cNLS [14]. In fact in a related development the concept of ‘multi-soliton complex’ (MSC) has been developed recently [15]. MSC is a localized state which is a nonlinear superposition of fundamental solitons. Besides beams and pulses in optics, MSC often arises in many other physical disciplines, e.g., solid state physics. MSC can be governed by a system of cNLS. Results of this paper as periodic wavetrains of cNLS are thus relevant, and solitons are just the long wave limits of the present work.

Another important and recent discovery where the present work is applicable is the observation of incoherent spatial solitons in noninstantaneous nonlinear media like biased photorefractive crystals [11]. Theoretical approaches developed include the coherent density method and the self-consistent multimode method. In the latter approach, multimode soliton solutions with a total intensity equal to the superposition of all the modes propagating along the nonlinear waveguide are sought. Analytical simplifications are possible for such spatial solitons in noninstantaneous Kerr-like media with a special intensity profile. More precisely, when this intensity profile is the square of the hyperbolic secant exact solutions are obtained.

The achievement of the present paper with reference to this application is that exact, periodic solutions are also obtained when this total intensity is described by the square of a Jacobi elliptic function. The hyperbolic secant square is then just the long wave limit of the present result. Some physical assumptions regarding the formulation must also be made [11]. The refractive index of this Kerr-like material should vary linearly with the total optical intensity. Furthermore, the nonlinearity should respond much slower than the phase fluctuation time across the beam.

Finally, we should remark that the stability of these new periodic waves is still an open question. Stability of plane continuous waves, i.e., waves with constant intensity, has been

studied recently [16]. However, the present class of waves has non-constant amplitude, and their stability must also be studied by another set of full scale numerical simulations.

Appendix

The theta functions $\theta_n(x)$ [7, 8], $n = 1, 2, 3, 4$ and the parameters q (the nome), τ (pure imaginary) are defined by

$$\begin{aligned} \theta_1(x) &= \sum_{n=0}^{\infty} (-1)^n q^{(n+1/2)^2} \sin(2n+1)x \\ &= - \sum_{m=-\infty}^{\infty} \exp\left(\pi i \tau \left(m + \frac{1}{2}\right)^2 + 2i \left(m + \frac{1}{2}\right) \left(x + \frac{\pi}{2}\right)\right), \end{aligned} \quad (\text{A1})$$

$$\begin{aligned} \theta_2(x) &= 2 \sum_{n=0}^{\infty} q^{(n+1/2)^2} \cos(2n+1)x \\ &= \sum_{m=-\infty}^{\infty} \exp\left(\pi i \tau \left(m + \frac{1}{2}\right)^2 + 2i \left(m + \frac{1}{2}\right) x\right), \end{aligned} \quad (\text{A2})$$

$$\theta_3(x) = 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \cos 2nx = \sum_{m=-\infty}^{\infty} \exp(\pi i \tau m^2 + 2imx), \quad (\text{A3})$$

$$\begin{aligned} \theta_4(x) &= 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos 2nx \\ &= \sum_{m=-\infty}^{\infty} \exp\left(\pi i \tau m^2 + 2im \left(x + \frac{\pi}{2}\right)\right), \end{aligned} \quad (\text{A4})$$

$$0 < q < 1, \quad q = \exp(\pi i \tau) = \exp\left(-\frac{\pi K'}{K}\right).$$

K and K' are the complete elliptic integrals. Relationships between the theta and elliptic functions are

$$\text{sn}(u) = \frac{\theta_3(0)\theta_1(z)}{\theta_2(0)\theta_4(z)}, \quad \text{cn}(u) = \frac{\theta_4(0)\theta_2(z)}{\theta_2(0)\theta_4(z)}, \quad \text{dn}(u) = \frac{\theta_4(0)\theta_3(z)}{\theta_3(0)\theta_4(z)}, \quad (\text{A5})$$

$$z = \frac{u}{\theta_3^2(0)}, \quad k = \frac{\theta_2^2(0)}{\theta_3^2(0)}, \quad k' = \frac{\theta_4^2(0)}{\theta_3^2(0)}, \quad k^2 + (k')^2 = 1. \quad (\text{A6})$$

Theta functions possess a huge variety of product identities, e.g.

$$\theta_3(x+y)\theta_3(x-y)\theta_2^2(0) = \theta_4^2(x)\theta_1^2(y) + \theta_3^2(x)\theta_2^2(y). \quad (\text{A7})$$

$$\theta_4(x+y)\theta_4(x-y)\theta_2^2(0) = \theta_4^2(x)\theta_2^2(y) + \theta_3^2(x)\theta_1^2(y), \quad (\text{A8})$$

Differentiating (A7), (A8) with respect to y and setting $y = 0$ yield

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$$D_x^2 \theta_3(x) \cdot \theta_3(x) = \frac{2\theta_2''(0)\theta_3^2(x)}{\theta_2(0)} + 2\theta_3^2(0)\theta_4^2(0)\theta_4^2(x),$$

$$D_x^2 \theta_4(x) \cdot \theta_4(x) = 2\theta_3^2(0)\theta_4^2(0)\theta_3^2(x) + \frac{2\theta_2''(0)\theta_4^2(x)}{\theta_2(0)}.$$

Hence formulas for $D_x \theta_m \cdot \theta_n, D_x^2 \theta_m \cdot \theta_n$ can be developed for m, n integers [9]. Derivatives for products of theta functions can be obtained by repeated use of identities like

$$D_x ab \cdot cd = bdD_x a \cdot c + acD_x b \cdot d,$$

$$D_x^2 ab \cdot cd = bdD_x^2 a \cdot c + acD_x^2 b \cdot d + 2(D_x a \cdot c)(D_x b \cdot d).$$

A typical result is

$$D_x^2 \theta_2(\alpha x) \theta_3(\alpha x) \theta_4^2(\alpha x) \cdot \theta_4^4(\alpha x) = \alpha^2 \theta_2(\alpha x) \theta_3(\alpha x) \theta_4^4(\alpha x)$$

$$\times \left[2\theta_3^2(0)\theta_4^2(0)\theta_3^2(\alpha x) + \left(\frac{5\theta_2''(0)}{\theta_2(0)} + \frac{\theta_3''(0)}{\theta_3(0)} \right. \right.$$

$$\left. \left. + \frac{2\theta_4''(0)}{\theta_4(0)} + 2\theta_3^4(0) \right) \theta_4^2(\alpha x) \right].$$

Similar results are obtained for the Hirota derivatives of other theta functions, but details will be omitted here for brevity.

Acknowledgement

Partial financial support has been provided by the Hong Kong Research Grants Council contracts HKU 7067/98E and HKU 7066/00E.

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