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Introduction to solitons

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Abstract. As an introduction to the special issue on nonlinear waves, solitons and their significance in physics are reviewed. The soliton is the first universal concept in nonlinear science. Universality and ubiquity of the soliton concept are emphasized.

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1. Discovery of soliton

Soliton is a nonlinear wave which has the following two properties:

(1) A localized wave propagates without change of its properties (shape, velocity etc.),
(2) Localized waves are stable against mutual collisions and retain their identities.

The first is a solitary wave condition known in hydrodynamics since the 19th century. The second means that the wave has the property of a particle. In modern physics, a suffix-on is used to indicate the particle property, for example, phonon and photon. Zabusky and Kruskal [1] named a solitary wave with the particle property a 'soliton'.

The history leading to the discovery of soliton is interesting and impressive. The first documented observation of the solitary wave was made in 1834 by the Scottish scientist and engineer, John Scott-Russell (1844):

I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped—not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished and after a chase of one or two miles I lost it in the windings of the channel. Such in the month of August 1834 was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation....

The word 'solitary wave' was coined by Scott-Russell himself. This phenomenon attracted some attention of scientists including Stokes, Boussinesq and Rayleigh, but a theoretical confirmation had to wait until 1898 when two Dutch physicists, Korteweg and de Vries, presented their now famous equation [2] (KdV equation for short),

$$u_t + \alpha u u_x + \beta u_{xxx} = 0, \quad \alpha, \beta \text{ constants.} \tag{1.1}$$

Here, $u(x,t)$ represents the height from the average water surface and x is the coordinate moving with the velocity of a linearized wave. The KdV equation (1.1) has a solitary wave (1-soliton) solution,

$$u(x,t) = \frac{3v}{\alpha}\mathrm{sech}^2\left[\frac{1}{2}\sqrt{\frac{v}{\beta}}(x - vt)\right]. \tag{1.2}$$

In 1955, Fermi, Pasta and Ulam investigated how the equilibrium state is approached in a one-dimensional nonlinear lattice [3]. It was expected that the nonlinear interactions among the normal modes of the linear system would lead to the energy of the system being evenly distributed throughout all the modes, that is, the system would be ergodic. The results of numerical analysis contradicted this idea. The energy is not distributed equally into all the modes, but the system returns to the initial state after some period (the recurrence phenomena). In 1965, Zabusky and Kruskal solved the KdV equation numerically as a model for nonlinear lattice and observed the recurrence phenomena. Further, they found an unexpected property of the KdV equation. From a smooth initial waveform, waves with sharp peaks emerge. Those pulse-waves move almost independently with constant speeds and pass through each other after collisions. A detailed analysis confirmed that each pulse is a solitary wave of sech^2-type as in eq. (1.2) and the solitary waves behave like stable particles. Thus, the soliton was discovered.

2. Establishment of soliton concept

How do we confirm analytically the properties of solitons? Why are solitons stable like particles? Is soliton a specific phenomenon of the KdV equation? Solutions to these questions played an important role in establishing the soliton concept.

After suitable scalings of the independent and dependent variables the KdV equation has a form,

$$u_t - 6uu_x + u_{xxx} = 0. \tag{2.1}$$

The second and third terms represent the nonlinear and dispersion effects, respectively. The nonlinear effect causes the steepening of waveform, while the dispersion effect makes the waveform spread. Due to the competition of these two effects, a stationary waveform (solitary wave) exists. The reason why each solitary wave is stable inspite of mutual interactions is that the KdV equation has an infinite number of conserved quantities. Dynamical properties of the system are severely restricted by the existence of an infinite number of conservation laws. The conserved quantities guarantee the time-independence of parameters which characterize the solitons, and therefore the solitons are stable. Corresponding to an infinite number of conserved quantities (recall the field variable has infinite degrees of freedom), arbitrary number of solitons may coexist.

Fundamental properties of solitons are investigated by the inverse scattering method. In 1967, Gardner, Greene, Kruskal and Miura [4] introduced a linear problem (eigenvalue problem) where the potential $u(x,t)$ is the solution of the KdV equation (2.1) ,

$$-\psi_{xx} + u(x,t)\psi = \lambda\psi. \tag{2.2}$$

It can be shown that when u evolves obeying (2.1), the eigenvalue λ does not depend on time. Equation (2.2) is nothing but the Schrödinger equation in quantum mechanics. A problem that for a given u one calculates the transmission coefficient $1/a(k)$, the reflection coefficient $b(k)/a(k)$, discrete eigenvalues $\lambda_n = -\eta_n^2$ etc. is called a scattering problem. Conversely, a problem that for a given scattering data $a(k), b(k), \lambda_n$ etc. one determines the potential is called an inverse scattering problem. The latter problem for eq. (2.2) was solved by Gelfand-Levitan and Marchenko. It can be shown further that the time-development of the eigenfunction $\psi(x,t)$ is

$$\psi_t = -4\psi_{xxx} + 3u_x\psi + 6u\psi_x. \tag{2.3}$$

From the boundary condition $u \to 0(|x| \to \infty)$, the time-dependence of the scattering data is determined as

$$\begin{aligned} a(k,t) &= a(k,0), \\ b(k,t) &= b(k,0)\mathrm{e}^{-8ik^3 t}, \\ c_n(t) &= c_n(0)\mathrm{e}^{-4\eta_n^3 t}, \end{aligned} \tag{2.4}$$

where c_n is a normalization constant of the eigenfunction ψ_n. To summarize, the solution of the KdV equation is given by

$$u(x,t) = 2\frac{\partial}{\partial x}K(x,x;t), \tag{2.5}$$

where

$$K(x,y;t) + F(x+y;t) + \int_{-\infty}^{x} K(x,z;t)F(z+y;t)\mathrm{d}z = 0, \tag{2.6}$$

$$F(x;t) = \sum_{n=1}^{N} c_n^2(t)\mathrm{e}^{\eta_n x} + \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{b(k,t)}{a(k,0)}\mathrm{e}^{-ikx}\mathrm{d}k. \tag{2.7}$$

The integral equation (2.6) is called Gelfand-Levitan equation. It should be emphasized that all the calculations in the above are linear problems. In this way, the initial-value problem of the KdV equation is solved. Each bound state with the discrete eigenvalue corresponds to a soliton. In particular, when the reflection coefficient $r(k,0) = b(k,0)/a(k,0)$ is identically zero (the reflectionless potential), we can readily solve the Gelfand-Levitan equation and obtain the N-soliton solution corresponding to N bound states. From the exact expression of the N-soliton solution, we can prove that the soliton is stable against mutual collisions and that the collisions are pair-wise and induce only the position shifts of solitons [5] . The above method of solution is called the inverse scattering method.

The initial-value problem of the KdV equation could be solved. At that time, however, it seemed a fluke. Five years later, by extending the inverse scattering method, Zakharov and Shabat [6] solved the nonlinear Schrödinger (NLS) equation,

$$i\psi_t + \psi_{xx} + 2|\psi|^2\psi = 0, \tag{2.8}$$

and subsequently Wadati solved the modified KdV (mKdV) equation [7] ,

$$u_t + 6u^2u_x + u_{xxx} = 0. \tag{2.9}$$

Further, the Sine-Gordon equation has been solved [8] and we have now more than 100 soliton equations.

Through the inverse scattering method, we are in a position to define the soliton in a rigorous manner. A transformation from the field variables to the scattering data is a canonical transformation, and the action-angle variables are defined in the scattering data space. Thus, the soliton equation is a completely integrable system and the soliton is a fundamental mode of the system [9].

The discovery of the inverse scattering method is the most important development in the theory of solitons. When the scattering data space is regarded as the extension of the momentum space, the inverse scattering method is considered as the extension of the Fourier transformation into nonlinear problems. The Fourier transformation was introduced in 1811 to solve the diffusion equation. About 150 years later, it was developed into a unified method to solve nonlinear evolution equations.

3. Soliton physics

We saw that the studies of solitary waves propagating in shallow waters such as a canal are related to the ergode problem in nonlinear lattices and lead to the discovery of solitons. It was the KdV equation that encompassed the two different fields, hydrodynamics and lattice dynamics. This is merely one example of the universality and ubiquity of soliton equations.

We consider some salient aspects of soliton physics taking the Sine-Gordon (SG) equation as an example,

$$\phi_{tt} - \phi_{xx} + \sin\phi = 0. \tag{3.1}$$

The SG equation has various interesting solutions: as x varies from $-\infty$ to ∞, a solution which changes from 0 to 2π is a soliton (kink). A solution which changes from 2π to 0 is an anti-soliton (anti-kink) and the bound state of kink and anti-kink is a breather. In 1939, Frenkel and Kontrova [10] introduced the SG equation as a model for the dislocations in crystals. The displacement $\phi(x,t)$ of atoms connected by linear springs may propagate as a kink in the periodic crystal field. Around 1960, Perring and Skyrme [11] considered the SG equation as a model for elementary particles (more rigorously, baryons). They examined collisions of kink-kink and kink-antikink and confirmed the particle-like stability of kinks (Historically, Seeger, Donth and Kochendörfer [12] found the kink-kink solution and the kink-antikink solution in the study of the SG equation as a dislocation model). Further, Perring and Skyrme defined a topological charge,

$$Q = \frac{1}{2\pi}\{\phi(\infty,t) - \phi(-\infty,t)\}. \tag{3.2}$$

The topological charge is a conserved quantity under the boundary condition imposed on the field variable. In general, a discrete invariant quantity due to the topology of the field variables is called topological quantum number. In 1967, McCall and Hahn [13] discovered an interesting phenomenon in the field of nonlinear optics. Coherent light propagating in the system of 2-level atoms obeys the SG equation when the spectral widths are neglected (perfect resonance). The observed soliton-like behavior is called self-induced transparency (SIT). The 2π-pulse is the soliton and 0π-pulse is the breather. In the other limit, that is, the interaction between the medium and the light wave is not resonant, the envelop of the electric field is described by the NLS equation [14]. As an application of the SG equation related to new technology, a propagation of magnetic fluxes in the Josephson junction is important. The Josephson junction consists of two superconductors (1 and 2) and the insulator. We denote by ϕ_1 and ϕ_2 the phases of Cooper-pair wave functions in the superconducting plates. Due to the Josephson current $J = J_0 \sin\phi$ caused by the phase difference $\phi = \phi_1 - \phi_2$, the motion of ϕ is described by the SG equations. There are many applications of the SG equation including one-dimensional organic conductors, one-dimensional ferromagnets, He^3 and liquid crystals.

A wide applicability of the soliton equation implies soliton phenomena which are common in various fields of physics. This is the essence of soliton physics. Solitons appear in almost all branches of physics, such as hydrodynamics, plasma physics, nonlinear optics, condensed matter physics, low temperature physics, particle physics, nuclear physics, biophysics and astrophysics.

4. Further developments and open problems

The study of solitons is the first systematic research on nonlinear phenomena with a consistent leading principle. While the soliton concept makes a new viewpoint on nature, there are many problems to be studied.

1) *Soliton mathematics*

The soliton concept has developed with mathematical methods. There are many new approaches such as the inverse scattering method, Hirota method, the Bäcklund transformation, the Darboux transformation and Painlevé analysis. The most fundamental problem continues to be a criterion and a classification of the completely integrable systems.

2) *Multi-dimensional solitons*

Most of the soliton systems are $(1+1)-$ dimensional. In the cases of the Ernst equation and the cylindrical KdV equation, the number of independent variables is reduced because of the symmetry. Well known examples of $(2+1)-$ dimensional soliton systems are

$$(u_t + 6uu_x + u_{xxx})_x + \alpha u_{yy} = 0, \alpha = \pm 1. \tag{4.1}$$

and

$$iu_t - u_{xx} - \sigma^2 u_{yy} + |u|^2 u - 2uv = 0,$$
$$-\sigma^2 v_{xx} + v_{yy} - (|u|^2)_{yy}, \sigma^2 = \pm 1. \tag{4.2}$$

We call (4.1) the 2-dimensional KdV equation or Kadomtsev-Petviashvili equation, and (4.2) the Davey-Stewartson equation. A characterization of multi-dimensional solitons requires further investigations. The geometrical approach is one of the promising ones [15].

3) *Coupled soliton systems*

There has been renewed interest in multi-component soliton equations. For example, the coupled NLS equations (the $m = 2$ case is called Manakov equation [16]),

$$iq_{j,t} + q_{j,xx} + 2\sum_{k=1}^{m} |q_k|^2 \cdot q_j = 0, \tag{4.3}$$

and the coupled mKdV equation [17]

$$v_{j,t} - 6\left(\sum_{k=1}^{m} \varepsilon_k v_k^2\right) v_{j,x} + v_{j,xxx} = 0,$$
$$\varepsilon_j = \pm 1, \tag{4.4}$$

are known to be integrable. Dynamical properties of multi-component solitons have not been studied in detail.

4) *Solitons under external forces and dissipations*

To control a nonlinear dynamical system in practice, we need to investigate the effects due to external forces, noises, impurities and dissipations. For soliton systems with such effects, there are many reported numerical works and perturbative analyses. Among many, an interesting scenario is the competition of spatial order (solitons) and temporal disorder (chaos) [18]. On the other hand, the effect of the gaussian noise on the KdV solitons has been analytically studied.

Related to recent applications in nonlinear optics and condensed matter physics, nonlinear wave propagations under the periodic potential are interesting. This would be a standard problem which is important both theoretically and experimentally.

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