

Quantum chaos, thermalization and dissipation in nuclear systems

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Abstract. Nuclei have complex energy-level sequence with statistical properties in agreement with canonical random matrix theory. This agreement appears when the one-particle one-hole states are mixed completely with two-particle two-hole states. In the transition, there is a new universality which we present here, bringing about a relation between dynamics and statistics. We summarize also the role of chaos in thermalization and dissipation in isolated systems like nuclei. The methods used to bring forth this understanding emerge from random matrix theory, semiclassical physics, and the theory of dynamical systems.

Keywords. Chaos; dissipation; quantum transport; random matrix theory.

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1. Introduction

Various studies on complex energy-level sequences and eigenfunctions have shown that nuclei are examples of chaotic quantum systems. These systems display quantum chaos and also stand as paradigms of random matrix systems [1]. Recent studies indicate a new universal regime as a function of the strength of residual interaction [2]. This new universality is explained in terms of the short-range Dyson model (SRDM) [3]. New results on level correlations are given here.

Nuclear processes occur at such short scales that nuclei are best thought of as isolated systems [4]. The complexity of eigenfunctions leads to an understanding that these systems thermalize due to the presence of chaos in nuclear dynamics [5]. Also, it has been shown that dissipation is related to presence of classical chaos. It was shown by Gaspard and Jain [6] that the time-correlations decay at a rate dependent on the Liapunov exponents for finite Fermi systems. The energy distribution evolves in an irreversible manner with time, in terms of a quantum diffusion equation obtained by Jain [7]. This leads to a better understanding of one-body dissipation, going beyond the chaos-weighted-wall-formula (CWWF) due to Pal and Mukhopadhyay [8]. With the development of a systematic semiclassical theory [5] based on Gutzwiller trace formula, the dissipation rates are expressible in terms of periodic orbits governed by the shape of nuclei.

The semiclassical methods are also fruitfully employed in understanding large-amplitude collective motion [9,10]. This leads to a semiclassical theory which can treat

anomalous diffusion in collective coordinates, leading to expressions for lifetimes of collective states.

We conclude with a list of open problems which will be quite worthy of future considerations.

2. Canonical and non-canonical random matrices, and order-chaos transition

2.1 Canonical RMT universality

Nearly 60 years ago, it was realized by Wigner and Landau that Hamiltonian matrix of nuclei may be well-represented as a random matrix in the absence of an expression for the realistic interaction. The status of our knowledge has not changed much in terms of the end-result, although it is now expected that quantum chromodynamics will perhaps, one day, present to us a form of the nuclear interaction – on outstanding problem today. The random matrix theory (RMT) laid down by Wigner, Dyson, Mehta and others is based on invariances and a classification of matrix algebra over the real field. By virtue of an old theorem of Frobenius, there emerge three ensembles – orthogonal, unitary, and symplectic. There is no preferred basis here. The consequence of RMT is in the universality displayed by the fluctuation properties of energy levels. The energy level sequences are quite complex and the nearest-neighbour level spacing distribution (NNLS) agrees very well with the Wigner surmise, $P(s) = (\pi s/2) \exp(-\pi s^2/4)$ with linear level repulsion at an excitation energy of about 4 MeV [1]. Since random matrix theory is much more general, it is also applicable to chaotic quantum systems [11], acoustic resonators, microwave cavities, conductance fluctuations in disordered conductors, and so on. In the case of nuclei, collecting 1407 resonance energies from 30 sequences of 27 nuclei, it was shown by Haq, Pandey, and Bohigas [12] that there is complete agreement with the statistical properties of eigenvalues of the Gaussian orthogonal ensemble (GOE) of random matrices. This ensemble is most frequently used in realistic applications. The matrix elements are drawn from a Gaussian distribution, the orthogonal ensemble corresponds to time-reversal invariant systems and the unitary ensemble (GUE) corresponds to time-reversal non-invariant systems. For the GUE, the NNLS displays quadratic level repulsion, $P(s) \sim s^2$. For systems with regular dynamics, i.e. where there are f constants of the motion for systems with f degrees of freedom, $P(s) \sim \exp(-s)$. Since the statistical properties of chaotic dynamical systems and nuclei (beyond excitation energy of 4 MeV) follow random matrix universality, it is commonly believed that nuclear dynamics is chaotic. Since the Hamiltonian of the nucleus is not known, it is not clear if the dynamics will be chaotic or integrable or mixed. We turn to this point in the next section. Before we close, it must be noted that only the local features agree with random matrix theory, not the global features like the average level density. For a nucleus the average level density grows exponentially, in total disagreement with the semicircle form for the Gaussian ensembles, $\sim \sqrt{a^2 - E^2}$. It is indeed a rare circumstance when both global and local features are successfully modelled, like in the work on disordered conductors by Ahmed and Jain [14].

2.2 New universality

Studies on the NNLS for 899 0^+0 ($J^\pi T$) states of 12 valence particles in the sd -shell model with Wildenthal–Brown interaction of variable strength, λ , has shown that around $\lambda \sim 0.2$, Wigner surmise appears [2]. This corresponds to the situation when the $1p - 1h$ states are mixed with the states of the next exciton level $2p - 2h$ [15]. $P(s)$ interpolates between integrable and chaotic dynamics. Empirical fits suggested that the level repulsion is neither s nor s^2 , but s^β with $0 \leq \beta \neq 1$.

Due to selection rules, the Hamiltonian matrices are not full of non-zero entries, but they can be banded. Generally, it happens that the numerical magnitude of matrix elements becomes smaller as one goes away from the principal diagonal. For such random matrices, the joint probability distribution of eigenvalues is given by

$$P(x_1, x_2, \dots, x_N) \sim \prod_{i=1}^{N-1} |x_i - x_{i+1}|^\beta \exp \left[-\sum_{i=1}^N x_i^2 \right], \quad (1)$$

where β is a real parameter, depending on interaction. This distribution can be re-written as a partition function for equilibrium positions of charged particles on a line with a neutralizing background and interacting by a screened Coulomb interaction of the form: $\log |x_i - x_{i+1}|^{-1}$. This is the short-range Dyson model (SRDM) introduced by Grèmaud and Jain [3]. The NNLS was found by Pandey [13]:

$$P^\beta(s) = \frac{(\beta + 1)^{\beta+1}}{\Gamma(\beta + 1)} s^\beta e^{-(\beta+1)s}. \quad (2)$$

This gives us the level repulsion as s^β , as desired from the numerical investigations mentioned in the beginning of the subsection. Other important quantities like spectral rigidity, number variance, stem from the two-point correlation function found in [16,17].

A word on the title of the subsection. It is very interesting that a class of pseudointegrable billiards where the phase space has singular vector fields [18], or, the phase space surface is topologically equivalent to a sphere with many handles [19], also show the same behaviour for the spectral statistics. Also, the fluctuation properties of the eigenvalues of transfer matrices of Anderson model in three dimensions at the metal-insulator transition point [20].

3. Chaos and thermalization

The connection between chaos and equilibration or thermalization of many-body systems has been always believed in. The tenets of equilibrium statistical mechanics rest on the assumption of ‘molecular chaos’ – an idea that goes back to Boltzmann. With the advancement of our understanding of the theory of dynamical systems and its relation with statistical mechanics, it has become important to extend this understanding to quantal many-body systems. Only recently, such a connection has been established. Since this connection and the present approach is an underlying theme of this paper, we briefly present the status of our understanding for want of a better perspective.

Let us mention the numerical results on the eigenfunctions of systems whose classical counterparts are chaotic.

(N1) The amplitude distribution of eigenfunctions are found to agree with a Gaussian distribution – we call these states as the generic ones [21]. Of course, we are not referring to eigenfunctions corresponding to the ground state which is special. Nevertheless, this observation is more general as the Gaussian distribution remains good even for pseudointegrable (non-integrable and non-chaotic) systems like a $\pi/3$ -rhombus billiard [19] where the Kolmogorov–Sinai entropy is zero.

(N2) The spatial correlation function

$$C_\alpha(\vec{s}) = \frac{1}{V} \int d\mathbf{r} \psi_\alpha^* \left(\mathbf{r} - \frac{\mathbf{s}}{2} \right) \psi_\alpha \left(\mathbf{r} + \frac{\mathbf{s}}{2} \right), \quad (3)$$

corresponding to an eigenfunction $\psi_\alpha(\mathbf{r})$ is found to agree with a (cylindrical) Bessel function [22], where V is volume. For two-dimensional billiards, it has been found to agree with $J_0(k\mathbf{s})$, $k = \sqrt{2mE/\hbar^2}$ [21].

(N3) The nodal lines are quite complicated, for pseudointegrable as well as chaotic billiards [21,23].

As far as eigenvalues or the energy levels are concerned, the measures quantifying spectral statistics agree with the results known from random matrix theory [11]. Even for pseudointegrable billiards, which are non-integrable, the nearest-neighbour level spacing distribution can be explained in terms of new ensembles [3].

The numerical results (N1), (N2) can be explained if we assume that the complicated eigenfunctions are written as a random superposition of plane waves. This has been advocated by Berry [22] and written explicitly by Srednicki [24]. For an N -particle system, the canonical pair of coordinates are (\mathbf{X}, \mathbf{P}) where $\mathbf{X} = (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N)$ and $\mathbf{P} = (\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_N)$. The energy eigenfunction $\psi_\alpha(\mathbf{X})$ is thus written as

$$\psi_\alpha(\mathbf{X}) = N_\alpha \int d^{dN} \mathbf{P} A_\alpha(\mathbf{P}) \delta(P^2 - 2mE_\alpha) e^{i\mathbf{X} \cdot \mathbf{P}} \quad (4)$$

for a d -dimensional system, where N_α is the normalization constant and the amplitudes satisfy the two-point correlation function

$$\langle A_\alpha^*(\mathbf{P}) A_\gamma(\mathbf{P}') \rangle_{\text{ME}} = \delta_{\alpha\gamma} \frac{\delta^{dN}(\mathbf{P} - \mathbf{P}')}{\delta(P^2 - P'^2)}. \quad (5)$$

This correlator has been shown to be consistent with random matrix theory, hence the appearance of a statistical average on the left side of the equation (ME stands for a matrix-ensemble average) [25]. Beginning from this ansatz for the eigenfunction, the momentum distributions for ideal gases have been shown to concur with the well-known results (like Maxwell–Boltzmann, Fermi–Dirac, Bose–Einstein, or fractional statistics) by Srednicki [24], and, Jain and Alonso [26]. Even the first few virial coefficients have been calculated. The Wigner distribution corresponding to an eigenfunction is microcanonical. Thus, generically, corresponding to an eigenfunction, there is entire energy surface in phase space. Interestingly enough, this was anticipated by van Hove in 1956.

An important question in this theory, what is temperature? Temperature quantifies heat, and heat is defined as a mode of motion. Even if there is no reservoir, one can still consistently use the kinetic theory, and write for a system with f freedoms at energy E_α , a

temperature, T_α , with $E_\alpha = \frac{1}{2}fkT_\alpha$, where k is the Boltzmann constant. As the average over the entire system involves an averaging over the level density, the average temperature, T will automatically get defined. This average temperature is the one we measure as we obtain the correct momentum distributions with this temperature playing exactly the same role as the ‘usual’ temperature.

For the case of a many-body system like a nucleus where a nucleon trajectory is chaotic, a typical eigenfunction will be written as above. Thus, as the system evolves and shape of the nucleus changes, the eigenfunction will acquire a geometric phase. It is this phase which has been shown to be related to the absorptive part of generalized susceptibility [27].

4. Irreversibility of energy dissipation

In this section, following Ott [28], Jarzynski [29] and Jain [7] we show irreversibility of energy dissipation occurs when the single particle dynamics is chaotic [4]. Quantum mechanically, the difficulty is in showing that the energy distribution, $\eta(E)$, defined below follows an equation which implies irreversibility. Most often, in literature, this equation is taken as the Smoluchowski equation. Within the assumptions of classical linear response, this is the correct equation, as shown by Jarzynski [29].

Recently, we have derived evolution equations for the energy distributions and shown that these are, indeed, irreversible on an observational time scale. We have assumed that the single-particle dynamics is non-integrable. It has been known for chaotic and pseudo-integrable quantum systems that there is level repulsion [11,3] in their energy spectra. That is, the probability of clustering of levels is zero. In the quantum case, the equation contains an additional term to the Smoluchowski equation. However, this equation reduces to the classical equation as $\hbar \rightarrow 0$ [5].

In the first subsection let us see how it might be possible to place the assumption of having well-separated time-scales in a proper perspective.

4.1 Theoretical basis of well-separated time-scales

Following Balian [30], if we study the foundations of non-equilibrium quantum statistical mechanics in Liouville representation, we have the von Neumann equation:

$$\frac{\partial \rho_k}{\partial t} = \mathsf{L} \rho_k, \tag{6}$$

where ρ_k are the coordinates of a statistical state. The density operator $\hat{\rho} = \sum_k \hat{\Omega}_k \rho_k$ and $\{\hat{\Omega}_k\}$ is a dual basis. L is the superoperator given by

$$\mathsf{L}_{\alpha\beta,\gamma\delta} = \frac{1}{i\hbar} (H_{\alpha\delta} \delta_{\beta\gamma} - H_{\gamma\beta} \delta_{\alpha\delta}) \tag{7}$$

with the Hamiltonian operator, \hat{H} . The expectation value of an operator \hat{A} is given by

$$\mathcal{A} = \sum_k A_k \rho_k := (\hat{A}; \hat{\rho}). \tag{8}$$

Writing the total density operator as a sum of density operators for relevant and irrelevant observables, respectively $\hat{\rho}_R$ and $\hat{\rho}_I$, and defining a projection superoperator,

$$P = \hat{\rho}_R \otimes \hat{I} + \sum_j \frac{\partial \hat{\rho}_R}{\partial \mathcal{A}_j} \otimes (\hat{A}_j - \mathcal{A}_j \hat{I}), \quad (9)$$

the evolution equation of \mathcal{A}_j is

$$\frac{d\mathcal{A}_j}{dt} = (\hat{A}_j; L\hat{\rho}_R) + (\hat{A}_j; \int_{t_0}^t dt' LW(t, t')L\hat{\rho}_R(t')), \quad (10)$$

with

$$W(t, t') = T \exp \left[\int_{t'}^t dt'' (1 - P(t''))L(1 - P(t'')) \right]. \quad (11)$$

From this exact description, in going to a short-memory approximation, the relevant set should be chosen as the set of slowest variables, replacing $\hat{\rho}_R(t')$ by $\hat{\rho}_R(t)$. The rapid variables are the random ones and we shall assume that their dynamics is dictated by an ergodic Hamiltonian. Indeed, quoting Balian, ‘...the reconciliation of microscopic reversibility and macroscopic irreversibility thus requires the existence of two well-separated time-scales’.

It is important to note that a similar classical analysis exists for a long time, due to Ernst and Dorfman [31].

With this background, let us now present some results on evolution of energy distribution.

4.2 From von Neumann equation to quantum diffusion equation

Let us consider the Hamiltonian,

$$\hat{H}(t) = \hat{H}_0 + F(t)\hat{V}, \quad (12)$$

where \hat{H}_0, \hat{V} are linear operators, and $F(t)$ is a function of time. We assume that the evolution in time is adiabatic which corresponds to the existence of a small parameter, ϵ .

At any instant, the system admits an eigenvalue spectrum given by the eigenvalue problem for the ‘frozen’ Hamiltonian, $\hat{H}(\epsilon t)$,

$$\hat{H}(\epsilon t)|n(\epsilon t)\rangle = E_n(\epsilon t)|n(\epsilon t)\rangle. \quad (13)$$

From time $t = 0$, the levels evolve in time, the resulting density operator, $\hat{\rho}$ satisfies

$$i\hbar \frac{\partial \hat{\rho}}{\partial t} = [\hat{H}, \hat{\rho}]. \quad (14)$$

Our objective is to derive an equation for the energy distribution,

$$\eta(E) = \int^{E'} dE' \text{tr}\{\delta(E' - \hat{H})\hat{\rho}\}. \quad (15)$$

Since there are well-separated time-scales in the problem, we employ the multiple time-scale method for treating the partial differential equation (14). Accordingly, denoting the set of instantaneous states by $\{|n(\epsilon t)\}$, we can write an expansion for the density operator,

$$\hat{\rho}(\{|n\rangle\}, t) = \hat{\rho}_0(\{|n\rangle\}, \epsilon t) + \epsilon \hat{\rho}_1(\{|n\rangle\}, t, \epsilon t) + \dots \quad (16)$$

with initial conditions,

$$\hat{\rho}_0 = \hat{\rho}_{00}(\hat{H}(\epsilon t)), \hat{\rho}_1 = \hat{\rho}_2 = \dots 0. \quad (17)$$

Substituting (16) in (14), we get a system of equations separated by different orders of ϵ :

$$\begin{aligned} [\hat{\rho}_0, \hat{H}] &= 0, \\ i\hbar \frac{\partial \hat{\rho}_j}{\partial t} + [\hat{\rho}_j, \hat{H}] &= -i\hbar \frac{\partial \hat{\rho}_{j-1}}{\partial t}, j = 1, 2, \dots \end{aligned} \quad (18)$$

It is useful to convert the traces into energy-integrals. For that, we define an energy average over the ‘frozen’ Hamiltonian by

$$\frac{1}{\Sigma} \sum_n \langle n | \delta(E - \hat{H}) \dots | n \rangle = \langle \dots \rangle_{E, \epsilon t}. \quad (19)$$

With this definition, it is easy to verify that

$$\sum_n \langle n | \dots | n \rangle = \int dE \Sigma \langle \dots \rangle_{E, \epsilon t}. \quad (20)$$

Let us define the average two-time autocorrelation function of $\partial_{\epsilon t} \hat{H}$. The correlated quantities are separated in time by s , which is reflected in the instantaneous frozen bases, $\{|n\rangle\}$ and $\{|N\rangle\}$, respectively at times t and $t-s$. Thus, the correlation function is the trace or the energy-average, as defined by (19), of the product of the operators at two different times:

$$C_{\epsilon t}(s, E) = \langle \{ \partial_{\epsilon t} \hat{H}(\{|n\rangle\}) - u \} \{ \partial_{\epsilon t} \hat{H}(\{|N\rangle\}) - u \} \rangle_{E, \epsilon t}. \quad (21)$$

A coarse-grain over the energy spectrum leads to replacing $f(E_n)$ by $\bar{f}(E)$. This can be effected in many ways. For instance, using procedure similar to the Strutinsky procedure, or, over a probability distribution of the distances between states which, in turn, could be related to entropy changes.

Also, for times of $\mathcal{O}(\epsilon^{-1})$,

$$\int_{-t}^0 ds C(s) = \frac{1}{2} \int_{-\infty}^{\infty} ds C(s) = \frac{1}{2} G_2. \quad (22)$$

We have used the following definition:

$$\langle \partial_{\epsilon t} \hat{H} \rangle_{E, \epsilon t} = u(E, \epsilon t). \quad (23)$$

The energy distribution follows the equation,

$$\frac{\partial \eta}{\partial t} = -\epsilon \frac{\partial}{\partial E} (u\eta) + \epsilon^2 \frac{\partial}{\partial E} \left[G_2 \Sigma \frac{\partial}{\partial E} \left(\frac{\eta}{\Sigma} \right) \right] + \frac{\epsilon^2}{2} \Sigma \frac{\partial G_2}{\partial E} \frac{\partial}{\partial E} \left(\frac{\eta}{\Sigma} \right), \quad (24)$$

the quantum diffusion equation. It should be noted that this equation corresponds to the solution of the von Neumann equation up to $\mathcal{O}(\epsilon)$. Thus, the diffusion in quantum systems could be qualitatively and quantitatively different from the one we have in the Smoluchowski equation.

About the quantal nature of the third term in (24), a comment is in order. To understand this rather subtle point, let us recall that whereas the classical derivation uses states in phase space to find energy-averages, the quantal derivation employs a sum over states in the Hilbert space. When one derives the equation, the sums over more than one index appear and the summand consists of non-commuting operators. It is, indeed, due to this basic difference between the algebra of classical and quantum operators that the third term appears.

There are different important time-scales in the system. First of all, the time-scale associated with the decay of correlation function, t_c . If the quantum system considered is modelled as a random matrix of dimension N [32] with large N , we know that correlation function will decay rapidly. Thus, t_c can be very small if the quantum systems possess the following properties: (a) the number of eigenvalues is very large, and the energy spectrum is complex, and, (b) the corresponding classical system is chaotic. Chaos in the underlying classical system plays a fundamental role in the decay of correlation functions. It was shown by Gaspard and Jain [6] that the quantum time-dependent correlation functions in a fermionic system are dominated by the classical correlation function. The decay of the correlation function is shown in this work to be governed by the eigenvalues of the Liouvillian operator. Thus, t_c is related to the Liapunov exponents and other detailed features of chaos. In classical ergodic adiabatic systems, the time t (fast scale) is much larger than t_c , thus the third term of (24) is zero. However, in quantal systems, we have the quantum mechanical scale, $t_q \hbar / \bar{\sigma}$ ($\bar{\sigma}$ being the mean level spacing) which is why the third term is explicitly present. If $t_q \ll t_c \ll t$, the quantum effects will dominate, and all the terms of (24) will be important. If $t_c \ll t_q \ll t$, then the system will behave classically initially and eventually, quantum phenomena will become important; so initial evolution will be Smoluchowski-like and then the non-Smoluchowski regime sets in. If, however, $t_c \ll t \ll t_q$, then the evolution will be according to the classical equation. Notice, as \hbar becomes small and the system is classical, t_q will become vanishingly small. In the limit, the correlations will become largely independent of energy and the third term of (24) will drop.

We now turn to a situation relevant to large-amplitude collective motion, and see how we can treat this problem semiclassically.

5. Semiclassical study for coupled collective surfaces

We have seen above that order-chaos transition in nucleonic motion takes place at some energy. Chaos in quantum many-body systems leads to equilibration and dissipation of collective motion. Response of many-body systems can be semiclassically related to dynamics of single particle, as shown by Jain [5]. We now see how large-amplitude collective motion can be treated semiclassically. The method described below can be effectively used in many situations where spin-orbit interaction is important.

Although there are experimental signatures of large-amplitude collective motion, it is well-known that for a strongly interacting many-body system like a nucleus, there is no *a priori* obvious separation of collective degrees of freedom. However, from the interacting fermionic theory developed by Klein, Do Dang, and Walet [9], it has been shown that a many-body Hamiltonian separates into a collective part, $H_{\text{coll}}(Q, P)$ and a non-collective part [10]. Thus, there are several mean-field configurations and corresponding collective surfaces which are coupled. Collective diffusion, resulting from two such surfaces, was considered by Kusnezov [34], and we treat the same tractable Hamiltonian here semiclassically. It is interesting to note that modern semiclassical methods have been employed recently to obtain expressions for time correlations and response functions in terms of periodic orbits. Here, we present an example where at the super-adiabatic level, complex periodic orbits play a role. The Hamiltonian considered here includes residual interactions schematically, coupled to collective motion :

$$H = \frac{P^2}{2} + V, \quad (25)$$

with

$$V = \begin{pmatrix} \sin Q & \Delta \\ \Delta & -\sin Q \end{pmatrix}. \quad (26)$$

Assume that H is known in the vicinity of several level-crossings, (Q, P) are collective variables, and 2Δ is the energy gap at the avoided crossing. This matrix Hamiltonian can be transformed to a set of Hamiltonians, following the method developed by Littlejohn and Flynn [35]. We present the simplest situation where the expansion is carried out to order $\mathcal{O}(\hbar^2)$.

First of all, we diagonalize V by an orthogonal transformation and get

$$v = W^\dagger V W = \pm \sqrt{\Delta^2 + \sin^2 Q} \quad (27)$$

with

$$W = \begin{pmatrix} \cos \theta/2 & -\sin \theta/2 \\ \sin \theta/2 & \cos \theta/2 \end{pmatrix}, \quad (28)$$

with $\tan \theta = \Delta / \sin Q$. Obviously, the matrix W consists of the local eigenvectors of the potential matrix, $(\tau^{(+)}, \tau^{(-)})$. Local completeness and orthonormality relations exist for τ 's and one also has $W^\dagger W = 1 = W W^\dagger$. The orthogonal transformation will be applied to the full Hamiltonian now. Since the kinetic energy does not commute with Q , new terms in powers of \hbar arise.

We have

$$\begin{aligned} H^{(1)} &= W^\dagger H W \\ &= \frac{1}{2}(W^\dagger P W)^2 + v. \end{aligned} \quad (29)$$

Since the transformation depends on Q ,

$$W^\dagger P W = P - i\hbar W^\dagger \frac{\partial W}{\partial Q}, \quad (30)$$

note that the second term is Hermitian. Let us define a matrix of vector potentials,

$$A = iW^\dagger \frac{\partial W}{\partial Q} = \begin{pmatrix} 0 & -\frac{i}{2} \frac{d\theta}{dQ} \\ \frac{i}{2} \frac{d\theta}{dQ} & 0 \end{pmatrix}, \quad (31)$$

with

$$\frac{d\theta}{dQ} = -\frac{\Delta \cos Q}{\sqrt{\Delta^2 + \sin^2 Q}}. \quad (32)$$

Thus the diagonalization leads to a gauge structure into the Hamiltonian. We can now write the new Hamiltonian,

$$H^{(1)} = H_0^{(1)} + H_1^{(1)} + H_2^{(1)}, \quad (33)$$

with

$$\begin{aligned} H_0^{(1)} &= \frac{P^2}{2} + v \rightarrow \text{diagonal}, \\ H_1^{(1)} &= -\frac{\hbar}{2}(A.P + P.A) \rightarrow \text{non-diagonal}, \\ H_2^{(1)} &= \frac{\hbar^2}{2}A^2 \rightarrow \text{non-diagonal}. \end{aligned} \quad (34)$$

For the non-diagonal parts here, we can continue the transformations as above repeatedly and keep getting new terms. The important, interesting, and useful point is that the new terms that appear are at the higher powers of \hbar . Thus, we get a systematic semiclassical expansion. For the present problem, we get two Hamiltonians at the order \hbar^2 :

$$\begin{aligned} h^{(\pm)} &= -\frac{\hbar^2}{2} \left[1 \pm \frac{\hbar^2}{4} \frac{\Delta^2 \cos^2 Q}{(\sqrt{\Delta^2 + \sin^2 Q})^5} \right] \frac{d^2}{dQ^2} \\ &\pm \sqrt{\Delta^2 + \sin^2 Q} \left[1 \mp \frac{\hbar^2}{8} \frac{\Delta^2 \cos^2 Q}{(\Delta^2 + \sin^2 Q)^2} \right] + \mathcal{O}(\hbar^3). \end{aligned} \quad (35)$$

Note that $h^{(+)}$ supports bound states, whereas $h^{(-)}$ is of scattering type.

Interestingly, at the second order, the coupling between surfaces leads to renormalization of the mass into effective masses associated with each surface.

To find the energy levels of the upper surface, $v^{(+)}$, at the leading order, we have

$$E_n = E_n^{(0)}(\hbar) + \hbar^2 E_n^{(1)}(\hbar) + \dots \quad (36)$$

with the condition on the energy at the lowest order:

$$S^{(+)}(E_n^{(0)}) = 2\pi\hbar \left(n + \frac{1}{2} \right). \quad (37)$$

The reduced action is

$$S^{(+)}(E) = 2 \int_{Q_{<}^{(+)}}^{Q_{>}^{(+)}} \sqrt{2(E - \sqrt{\Delta^2 + \sin^2 Q})} dQ, \quad (38)$$

where the integration limits are turning points. For large E , the turning points are $-\sqrt{E}$ and \sqrt{E} . The reduced action is

$$S^{(+)}(E) = 4\sqrt{2} \left\{ \sqrt{E-1} \mathbf{E} \left[\frac{\pi}{4} + \frac{\sin^{-1} E}{2}, \frac{2}{1-E} \right] - \sqrt{E-1} \mathbf{E} \left[\frac{\pi}{4} - \frac{\sin^{-1} E}{2}, \frac{2}{1-E} \right] \right\}, \quad (39)$$

where \mathbf{E} is the incomplete elliptic integral of the second kind,

$$\mathbf{E}(x, m) = \int_0^x \frac{dt \sqrt{1 - mt^2}}{\sqrt{1 - t^2}}. \quad (40)$$

From (37) and (39), we get discrete and real energy levels.

In reality, these are metastable states and their lifetime is exponentially small in \hbar . The lifetimes can be obtained by considering complex periodic orbits. The system jumps from upper to lower surface at the points of intersection, if they exist. In this example, the surfaces meet at two points, $\pm i \sinh^{-1} \Delta$. Thus, there will be a complex periodic orbit making excursions at the points of intersection to imaginary Q axis. This is a non-adiabatic process. The probability of transition between upper and lower surface is $1 - e^{-|\Im S_c|/\hbar}$. And the half-width of the quantum state is

$$\begin{aligned} \Gamma_n &= \frac{\hbar}{\tau_n} \\ &= \frac{2\hbar}{T^{(+)}(E_n)} e^{-|\Im S_c|/\hbar}, \end{aligned} \quad (41)$$

where $T^{(+)}(E_n) = \partial_E S^{(+)}(E_n)$. The quantity S_c is the imaginary part of the complex action

$$\begin{aligned} \Im S_c &= \int_0^{\sinh^{-1} \Delta} \sqrt{2(E + \sqrt{\Delta^2 + \sin^2 Q})} dQ \\ &\quad - \int_0^{\sinh^{-1} \Delta} \sqrt{2(E - \sqrt{\Delta^2 + \sin^2 Q})} dQ. \end{aligned} \quad (42)$$

For high energies, this quantity can be calculated, we get

$$\begin{aligned} \Im S_c &= 2\sqrt{2(E+1)} \left[\mathbf{E} \left(\frac{\pi}{4}, \frac{2}{E+1} \right) - \mathbf{E} \left(\frac{\pi}{4} - \frac{\sinh^{-1} \Delta}{2}, \frac{2}{E+1} \right) \right] \\ &\quad - 2\sqrt{2(E-1)} \left[\mathbf{E} \left(\frac{\pi}{4}, \frac{2}{-E+1} \right) \right. \\ &\quad \left. - \mathbf{E} \left(\frac{\pi}{4} - \frac{\sinh^{-1} \Delta}{2}, \frac{2}{-E+1} \right) \right]. \end{aligned} \quad (43)$$

The imaginary part of the action decreases with energy. Thus, for higher energies, the probability of transition between the two collective surfaces decreases. Thus, from the above equations one can deduce the transition rates, lifetimes of all the states in terms of quantities coming from complex periodic orbits which the system supports.

6. Open problems

Here, we present a (subjective) list of open problems which can really be brought to fruition in near future.

- In addition to single particle chaos, there is many-body chaos as Landau's picture of weakly interacting elementary excitations breaks down at higher excitation energies. The residual interactions become strong enough to mix quasiparticle configurations. The connection of one-body chaos and many-body chaos exists but has not been treated systematically. Thus, any transition of dynamical behaviour or transition of shapes of certain multipole resonances (particularly giant dipole resonances (GDR) [2]) will be understood only with an understanding of this connection. The semi-classical methods developed in [5] can be generalized to realize this connection.
- Evolution of hot GDR as a function of temperature and time has been studied by the double time Green function approach by Dang, Tanabe, and Arima [36]. They study the time correlation functions and also find a co-existence of order and chaos. To understand deeply, the correlation functions are very important objects. A variety of directions are possible in the spirit of the exhaustive study on multiple-time correlations due to Jain and Gaspard [32].
- In the connection between quantum chaos and thermalization, it has been shown by Lacroix [33] by taking an example of ^{16}O that the effective quasiparticle energies fit the Fermi-Dirac distribution as the system evolves from an initially compressed state. There are fundamental questions relating different temperatures, and there is the very impressive work of Zelevinsky [2].
- The irreversibility of energy distribution shown by us is possible due to coarse graining. And coarse-graining can be effected in many ways – is there a way guided by some basic principle? The answer to this question is in the affirmative. The coarse-graining can be done by employing the probability density of distances in the state-space [7]. Since the metric has been shown to be related to entropy [37], the coarse-graining procedure will be guided by entropy.
- Order-to-chaos transition in spectral distributions and the mixing of states can be understood in a random matrix framework as a perturbation of a Poisson ensemble by a Gaussian ensemble [38].
- The evaporation of a nucleus can be treated as a kicked system, the kicks corresponding to the recoils [39]. Thus the evaporation of a nucleus can be considered as a kicked chaotic systems. Immediately, the connection between localization theory and transitions among spectra [40] becomes related and interesting results are possible.

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