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# Inheriting geodesic flows

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**Abstract.** We investigate the propagation equations for the expansion, vorticity and shear for perfect fluid space-times which are geodesic. It is assumed that space-time admits a conformal Killing vector which is inheriting so that fluid flow lines are mapped conformally. Simple constraints on the electric and magnetic parts of the Weyl tensor are found for conformal symmetry. For homothetic vectors the vorticity and shear are free; they vanish for nonhomothetic vectors. We prove a conjecture for conformal symmetries in the special case of inheriting geodesic flows: there exist no proper conformal Killing vectors ($\psi_{;ab} \neq 0$) for perfect fluids except for Robertson–Walker space-times. For a nonhomothetic vector field the propagation of the quantity $\ln(R_{ab}u^a u^b)$ along the integral curves of the symmetry vector is homogeneous.

**Keywords.** Conformal motions; relativistic fluids; propagation equations.

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## 1. Introduction

Many of the classical exact solutions in general relativity are space-times possessing a high degree of symmetry. In this paper we impose the condition that the space-time manifold admits a conformal Killing vector. The advantage of this symmetry requirement are two-fold: it provides a deeper insight into the space-time geometry, and it facilitates generation of exact solutions to the field equations, some of which may be new, e.g. the conformally invariant models of Herrera *et al* [1], Maartens and Maharaj [2], Castejon-Amenedo and Coley [3] and Maartens and Mellin [4]. These treatments indicate it is possible for other known solutions, such as the relativistic astrophysical models of Vaidya and Tikekar [5], Tikekar [6], Patel *et al* [7], Tikekar and Singh [8] and Mukherjee *et al* [9], to be invariantly characterized with a symmetry. Conformal Killing vectors generate constants of the motion along null geodesics for massless particles; this associates the conformal symmetry with a well-defined, physically meaningful conserved quantity.

The geometric and dynamic features, with perfect fluid and anisotropic matter tensors, for space-times with a conformal symmetry **X** have been extensively studied by Maartens *et al* [10] and Coley and Tupper [11, 12]. An additional requirement imposed, by Coley and Tupper [11, 12], on the fluid 4-velocity **u** is that it must be inheriting so that fluid flow lines are mapped conformally onto fluid flow lines. These analyses indicate that perfect fluid space-times admitting inheriting conformal Killing vectors are rare. This has led to the conjecture that the only perfect fluid space-times, with reasonable physical conditions, admitting inheriting conformal Killing vectors are: space-times conformal to flat space-time, space-times with **X** parallel to **u**, and space-times with **X** orthogonal to **u**, apart from Robertson–Walker space-times and some other simple, symmetric exceptional cases. This suggests that there is room for further investigation of perfect fluids with an inheriting conformal symmetry.

Our specific objective in this study is to investigate the behaviour of the kinematic and dynamic variables of space-times that admit an inheriting conformal Killing vector. We assume that the matter distribution is that of a perfect fluid but we do not specify the space-time geometry. The propagation equations are considered in particular for geodesic flows. In §2 we define an inheriting conformal symmetry. The Lie derivative of the kinematical and dynamical quantities are found. In §3 we take the Lie derivative of the expansion, shear and vorticity propagation equations. A number of results pertaining to the active gravitational mass, the shear and the vorticity are found. In particular we establish a conjecture of Coley and Tupper for geodesic inheriting flows. Finally in §4 we discuss the significance of the results obtained, and briefly consider possibilities for future work. Some of our results may be contained in other treatments; we believe that our approach of utilizing the propagation equations is new and permits an unified analysis of the kinematical and dynamical quantities.

## 2. Inheriting perfect fluids

For the fluid 4-velocity **u** we write

$$u_{a;b} = \sigma_{ab} + \tfrac{1}{3}\Theta h_{ab} + \omega_{ab} - \dot{u}_a u_b,$$

where $\Theta = u^a{}_{;a}$ is the rate of expansion, $h_{ab} = g_{ab} + u_a u_b$ is the symmetric projection tensor $(h_{ab}u^b = 0)$, $\sigma_{ab} = \frac{1}{2}(u_{a;c}h^c{}_b + u_{b;c}h^c{}_a) - \frac{1}{3}\Theta h_{ab}$ is the symmetric shear tensor $(\sigma_{ab}u^a = 0 = \sigma^a{}_a)$, $\omega_{ab} = h^c{}_a h^d{}_b u_{[c;d]}$ is the skew-symmetric vorticity tensor $(\omega_{ab}u^b = 0)$, and $\dot{u}_a = u_{a;b}u^b$ is the acceleration vector $(\dot{u}_a u^a = 0)$. The overhead dot denotes covariant differentiation along a fluid particle worldline. Square brackets denote skew-symmetrization. We can decompose the matter tensor in terms of **u**, and consequently the Einstein field equations take the form

$$R_{ab} - \tfrac{1}{2}Rg_{ab} = (\mu + p)u_a u_b + pg_{ab} \tag{1}$$

for a perfect fluid with energy density $\mu$ and isotropic pressure $p$.

Manifolds with structure may admit groups of transformations which preserve this structure. A conformal motion preserves the metric up to a factor. A conformal Killing vector $\mathbf{X}$ is defined by

$$\mathcal{L}_{\mathbf{X}} g_{ab} = 2\psi g_{ab}, \tag{2}$$

where $\psi = \psi(x^a)$ is the conformal factor. The existence of a conformal Killing vector $\mathbf{X}$ is subject to the integrability condition

$$\mathcal{L}_{\mathbf{X}} C^a{}_{bcd} = 0 \tag{3}$$

which indicates that the Weyl tensor **C** is conformally invariant. Equation (3) is identically satisfied for conformally flat space-times, e.g. Robertson–Walker space-times [13]. A vector **X** is said to be an inheriting conformal Killing vector if, in addition to (2), it satisfies

$$\mathcal{L}_{\mathbf{X}} u_a = \psi u_a. \tag{4}$$

Hence inheriting conformal Killing vectors map fluid flow lines onto fluid flow lines. The physical significance of the assumption (4) has been extensively investigated by Maartens *et al* [10] and Coley and Tupper [11,12]. As a consequence of (2) and (4) we observe that

$$\mathcal{L}_{\mathbf{X}} h_{ab} = 2\psi h_{ab} \tag{5}$$

so that the inheriting vector **X** is a conformal motion of the projection tensor. If **u** is also orthogonal to **X** then the conformal vector is intrinsic to the projected hypersurfaces containing **h** as the metric tensor. The role of the intrinsic symmetries will be considered elsewhere.

If **X** is an inheriting conformal Killing vector then (2) and (4) hold, and we can establish the following relations

$$\mathcal{L}_{\mathbf{X}} \dot{u}_a = \psi_{,a} + u_a \dot{\psi}, \tag{6}$$

$$\mathcal{L}_{\mathbf{X}} \Theta = -\psi\Theta + 3\dot{\psi}, \tag{7}$$

$$\mathcal{L}_{\mathbf{X}} \sigma_{ab} = \psi \sigma_{ab}, \tag{8}$$

$$\mathcal{L}_{\mathbf{X}} \omega_{ab} = \psi \omega_{ab}. \tag{9}$$

The inheriting vector **X** is a conformal motion of the shear and the vorticity, but not of the acceleration and expansion in general. Equations (6)–(9) govern the evolution of the kinematical quantities for an inheriting conformal symmetry **X**.

For the connection coefficients, (2) implies

$$\mathcal{L}_{\mathbf{X}} \Gamma^a{}_{bc} = \psi_{,c}\delta^a_b + \psi_{,b}\delta^a_c - g_{bc}\psi^{,a}. \tag{10}$$

We find the Lie derivative of the Ricci tensor

$$\mathcal{L}_{\mathbf{X}} R_{ab} = -2\psi_{;ab} - g_{ab}\Box\psi, \tag{11}$$

where $\Box\psi = g^{ab}\psi_{;ab}$. Contracting the Lie derivative of the Ricci tensor in (11) gives the Lie derivative of the Ricci scalar

$$\mathcal{L}_{\mathbf{X}} R = -2\psi R - 6\Box\psi. \tag{12}$$

Then we find that the Lie derivative of the Einstein field equations (1) becomes

$$u_a u_b \mathcal{L}_{\mathbf{X}} \mu + h_{ab} \mathcal{L}_{\mathbf{X}} p + 2\psi(\mu u_a u_b + p h_{ab}) = 2\Box\psi g_{ab} - 2\psi_{;ab}$$

on utilizing (11) and (12). Contracting this equation with $u^a u^b$, $h^{ab}$, $u^a h^b{}_c$, $h^{ac}h^{bd} - \frac{1}{3}h^{ab}h^{cd}$, yields the following set of equations:

$$\mathcal{L}_{\mathbf{X}}\mu = -2\psi\mu - 2\Box\psi - 2u^a u^b \psi_{;ab}, \tag{13}$$

$$3\mathcal{L}_{\mathbf{X}}p = 4\Box\psi - 6\psi p - 2u^a u^b \psi_{;ab}, \tag{14}$$

$$0 = 2u^a \psi_{;ac} + 2u^a u^b u_c \psi_{;ab}, \tag{15}$$

$$0 = \psi_{;ab}(h^{ac}h^{bd} - \tfrac{1}{3}h^{ab}h^{cd}). \tag{16}$$

The inheriting vector **X** is not a conformal motion of the energy density and pressure in general. Equations (13)–(16) govern the evolution of the dynamical quantities for a geodesic inheriting symmetry **X**. This system severely restricts the behaviour of the gravitational field $\mathbf{g}$. It may be noted that if **X** is homothetic, i.e. $\psi = \text{constant} \neq 0$, then the acceleration $\dot{u}^a$ is conserved along the integral curves, and the other kinematic and dynamic quantities turn inheriting. However observe that $\dot{u}^a$ is conserved even for $\psi = \psi(t)$.

The Weyl tensor splits covariantly, relative to **u**, into electric and magnetic parts respectively:

$$E_{ab} = C_{acbd}u^c u^d, \qquad H_{ab} = \frac{1}{2}\eta_{acef}C^{ef}{}_{bd}u^c u^d,$$

where $\eta^{abcd} = \frac{1}{\sqrt{-g}}\epsilon^{abcd}$ is the space-time permutation tensor. With the help of (2) and (4) we find that

$$\mathcal{L}_{\mathbf{X}}E_{ab} = 0, \tag{17}$$

$$\mathcal{L}_{\mathbf{X}}H_{ab} = [4\psi - (\ln\sqrt{-g})_{,c}X^c]H_{ab}. \tag{18}$$

Here the electric part $E_{ab}$ is invariant under the action of an inheriting conformal Killing vector. However the magnetic part $H_{ab}$ is not. The quantity $H_{ab}$ is also invariant, as a particular case, if

$$\psi = \frac{1}{4}\left(\ln\sqrt{-g}\right)_{,a}X^a.$$

The results (17), (18) are simple constraints on the behaviour of $E_{ab}$ and $H_{ab}$ but do not seem to have been explicitly given before.

## 3. Geodesic flows

We make the assumption that the acceleration vanishes so that fluid flow is geodesic. When $\dot{u}^a = 0$ we observe from (6) that two cases arise: (a) $\psi = \text{constant} \neq 0$, (b) $\psi_{,a} \neq 0$. When $\psi_{,a} \neq 0$, then

$$u_a = -\frac{\psi_{,a}}{\dot{\psi}} \tag{19}$$

and the 4-velocity **u** is specified completely by the conformal factor. This means that the gradient $\psi_{,a}$ is parallel to $u_a$. We note that (19) is consistent with the unit, time-like requirement $u^a u_a = -1$.

We now analyse the geodesic propagation equations for the expansion $\Theta$, the shear $\sigma_{ab}$, and the vorticity $\omega_{ab}$, in an attempt to obtain general results governing the evolution of the relativistic perfect fluid. The expansion propagation equation is given by

$$\Theta_{,a}u^a = -\tfrac{1}{3}\Theta^2 - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} - R_{ab}u^a u^b. \tag{20}$$

This propagation equation is also called the Raychaudhuri equation. The shear propagation equation can be written as

$$\begin{aligned}\sigma_{ab;c}u^c = &-\tfrac{2}{3}\Theta\sigma_{ab} - \sigma_{ac}\sigma^c{}_b - \omega_{ac}\omega^c{}_b + C_{cbad}u^c u^d \\ &+\tfrac{1}{3}h_{ab}\left(\sigma_{cd}\sigma^{cd} - \omega_{cd}\omega^{cd}\right) + \tfrac{1}{2}R^{cd}\left(h_{ac}h_{bd} - \tfrac{1}{3}h_{ab}h_{cd}\right).\end{aligned} \tag{21}$$

The vorticity propagation equation is given by

$$\omega_{ab;c}u^c = -\tfrac{2}{3}\Theta\omega_{ab} + \omega_{bc}\sigma^c{}_a - \omega_{ac}\sigma^c{}_b. \tag{22}$$

In (20)–(22) we have followed the motivation and conventions of Ellis [14].

### 3.1 *Expansion propagation*

We first consider the expansion propagation equation. Taking the Lie derivative of (20) along an inheriting conformal Killing vector **X**, and using (4), (7)–(9) we obtain

$$\mathcal{L}_{\mathbf{X}}R_{ab}u^a u^b = -3\ddot{\psi} - \dot{\psi}\Theta + 2\psi\left(\dot{\Theta} + \frac{1}{3}\Theta^2 - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab}\right) + \Box\psi.$$

Then on substituting the Raychaudhuri equation (20) into the above equation we generate the result

$$\mathcal{L}_{\mathbf{X}}R_{ab}u^a u^b = -2\psi R_{ab}u^a u^b - 3\ddot{\psi} - \dot{\psi}\Theta + \Box\psi. \tag{23}$$

The quantity

$$R_{ab}u^a u^b = \frac{1}{2}(\mu + 3p)$$

is related to the active gravitational mass. Consequently (23) provides us with an indication of how the active gravitational mass changes along the integral curves of the inheriting conformal Killing vector **X**.

We observe that

$$\begin{aligned} &\mathcal{L}_{\mathbf{X}}R_{ab}u^a u^b = -2\psi R_{ab}u^a u^b \\ \Leftrightarrow\quad &3\ddot{\psi} + \dot{\psi}\Theta - \Box\psi = 0\end{aligned}$$

which provides the condition for **X** to be a conformal motion of the active gravitational mass $R_{ab}u^a u^b$. For a homothetic vector the condition $3\ddot{\psi} + \dot{\psi}\Theta - \Box\psi = 0$ is identically satisfied. When $\psi_{,a} \neq 0$, vorticity vanishes, as will be shown below, and this condition provides an additional constraint on the expansion. Then the 4-velocity becomes comoving and hypersurface orthogonal leading to $\psi = \psi(t)$, and the conformal factor is explicitly determined by $(\sqrt{^3g}\psi^{,0})_{,0} = 0$, and $u^a = \delta^a_0$. The line element can then be written in the orthogonal synchronous form

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + H_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta,$$

where the $H_{\alpha\beta}$ can depend on the space-time coordinates $x^a$ but $|H| = \det H_{\alpha\beta}$ is a function of $t$ alone. Note that Coley and Tupper [11] have demonstrated that orthogonal synchronous perfect fluid space-times (other than Robertson–Walker) do not admit any proper ($\psi_{;ab} \neq 0$) inheriting conformal symmetry. Our argument shows that the propagation of $\ln(\mu + 3p)$ along the integral curves of **X** is homogeneous.

### 3.2 *Shear propagation*

We now consider the shear propagation equation. Taking the Lie derivative of (21) along an inheriting conformal Killing vector **X**, and using (3)–(5), (7)–(11) we obtain after some simplification

$$\dot{\psi}\sigma_{ab} + \sigma_{db}u_a\psi^{,d} + \sigma_{ad}u_b\psi^{,d} = -\tfrac{1}{2}\left(2\psi^{;cd} + g^{cd}\Box\psi\right)\left[h_{ac}h_{bd} - \tfrac{1}{3}h_{ab}h_{cd}\right].$$

Substituting the dynamical equation (16) into the above we obtain

$$\dot{\psi}\sigma_{ab} = -2\sigma^{d}{}_{(a}u_{b)}\psi_{,d}$$

for perfect fluids. In both cases (a) and (b) for geodesic flows the right hand side vanishes. Hence

$$\dot{\psi}\sigma_{ab} = 0.$$

If **X** is homothetic then we have an identity leaving $\sigma_{ab}$ free. When **X** is nonhomothetic then $\sigma_{ab} = 0$.

### 3.3 *Vorticity propagation*

Finally we consider the vorticity propagation equation. Taking the Lie derivative of (22) along an inheriting conformal Killing vector **X**, and using (4), (7)–(10) we generate the result

$$\dot{\psi}\omega_{ab} = 2\omega^{d}{}_{[a}u_{b]}\psi_{,d}$$

for perfect fluid space-times. In both cases (a) and (b) for geodesic flows the right hand side vanishes. Hence

$$\dot{\psi}\omega_{ab} = 0.$$

If **X** is homothetic then we have an identity leaving $\omega_{ab}$ free. When **X** is nonhomothetic then $\omega_{ab} = 0$. The final result for $\omega_{ab}$ is the same as that for $\sigma_{ab}$. We can summarize our results in terms of the following theorem.

**Theorem I**. *For inheriting geodesic flows the propagation equations imply*:
1. *For homothetic vectors there is no constraint on the* 4*-velocity, shear and vorticity.*
2. *For nonhomothetic vectors*

$$\omega_{ab} = 0,\ \sigma_{ab} = 0,\ u_a = -\frac{\psi_{,a}}{\dot{\psi}}.$$

It is well-known [14] that shear-free, irrotational and geodesic perfect fluids ($\sigma_{ab} = \omega_{ab} = \dot{u}_a = 0$) must be Robertson–Walker. Consequently we regain Robertson–Walker space-times for nonhomothetic vectors from Theorem I. We can relate our result to the Coley and Tupper conjecture which states that proper conformal Killing vectors (i.e. $\psi_{;ab} \neq 0$) are restricted to Robertson–Walker space-times, and some other exceptional cases, with reasonable physical conditions on the energy-momentum tensor. Note that in our case the result does not depend on any assumptions, such as the energy conditions or equation of state, other than the fact that the inheriting fluid must be geodesic.

We express the nonexistence of proper conformal inheriting symmetries in terms of a corollary.

COROLLARY I

*For inheriting geodesic flows there exist no proper conformal Killing vectors* ($\psi_{;ab} \neq 0$) *for perfect fluids except for Robertson–Walker space-times.*

## 4. Discussion

We have studied the propagation equations for geodesic flows in perfect fluid space-times. The restrictions that the inheriting conformal symmetry **X** places on the space-time manifold are given in Theorem I, which essentially distinguishes between homothetic and nonhomothetic vectors. Corollary I establishes the Coley and Tupper conjecture for the special case of inheriting geodesic flows. We have demonstrated that general results may be found, without specifying the space-time geometry, by utilizing the propagation equations. In future work we intend to generalize these results for nongeodesic flows. This is a nontrivial task as the propagation equations are not as easy to cope with when $\dot{u}^a \neq 0$: the gradient $\psi_{,a}$ is not parallel to $u_a$ and (19) is no longer valid. Also, the matter tensor could be generalized to include anisotropic terms. The analyses of Maartens *et al* [10] and Coley and Tupper [11, 12, 15, 16] indicate that a wider range of possibilities are permitted if the condition of a perfect fluid is relaxed.

The conditions (17) and (18) on the electric $E_{ab}$ and magnetic $H_{ab}$ hold for a general conformal symmetry which is inheriting (and may be nongeodesic). These should be taken into consideration when seeking solutions to the field equations. We briefly comment on the condition $E_{ab} = 0$ for pure magnetic fields. Our result for geodesic flows indicates that solutions with $E_{ab} = 0$ are possible in perfect fluid space-times with a homothetic vector. In this context we observe that recently Lozanovski and Aarons [17] found a purely magnetic solution for a perfect fluid which satisfies the weak, strong and dominant energy conditions. In this class of space-times, the expansion and the shear are nonvanishing but the fluid is irrotational and nonaccelerating. In addition there have been various attempts to study the dynamics of purely magnetic space-times in the case of dust [18, 19, 20]. It has been shown that such ‘anti-Newtonian universes’ are subject to severe integrability conditions, and it is conjectured that no physically acceptable solution exists. Mars [21] has comprehensively studied the existence of magnetic solutions in Petrov type I vacuum space-times, and established a uniqueness result.

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