Gauge transformation between retarded and multipolar gauges

A M STEWART
Department of Applied Mathematics, Research School of Physical Sciences and Engineering, The Australian National University, Canberra ACT 0200, Australia
Email: andrew.stewart@anu.edu.au

MS received 24 July 2000; revised 9 January 2001

Abstract. The gauge function, expressed in terms of the sources, required for a gauge transformation between the retarded electromagnetic gauge and the three-vector version of the multipolar gauge is obtained.

Keywords. Gauge; transformation; retarded; multipolar.

PACS Nos 03.50.De; 11.15.q

The retarded solutions to the inhomogeneous wave equations for the electromagnetic scalar and vector potentials \( \phi(r, t) \) and \( A(r, t) \)

\[ \frac{\partial^2 A}{\partial (ct)^2} - \nabla^2 A + \nabla(\nabla \cdot A - c^{-2} \partial \phi / \partial t) = \mu_0 J \]  

and

\[ \nabla^2 \phi + (\partial / \partial t) \nabla \cdot A = -\rho / \varepsilon_0 \]  

are

\[ \phi_1(r, t) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(r', t') \delta(t' - t + |r - r'| / c)}{|r - r'|} dr'dt' \]  

and

\[ A_1(r, t) = \frac{\mu_0}{4\pi} \int \frac{J(r', t') \delta(t' - t + |r - r'| / c)}{|r - r'|} dr'dt' \]  

where \( \delta \) is the Dirac delta function and \( c \) is the velocity of light. They describe the potentials at position \( r \) and time \( t \) arising from charge and current densities \( \rho \) and \( J \) at position \( r' \) and time \( t' \) [1, 2] and satisfy the Lorentz gauge condition \( \nabla \cdot A_1 + c^{-2} \partial \phi_1 / \partial t = 0 \). The electromagnetic fields \( E(r, t) \) and \( B(r, t) \) are obtained from the potentials by the relations

\[ B = \nabla \times A \quad \text{and} \quad E = -\nabla \phi - \partial A / \partial t, \]  

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where $\nabla$ is the spatial gradient operator with respect to $r$. In consequence, if the potentials are transformed to

$$\mathbf{A} \rightarrow \mathbf{A'} = \mathbf{A} + \nabla \chi \quad \text{and} \quad \phi \rightarrow \phi' = \phi - \partial \chi / \partial t,$$

the electromagnetic fields are unchanged. The gauge function $\chi(r, t)$ is required to satisfy the condition \{($\partial / \partial i)(\partial / \partial j) - (\partial / \partial j)(\partial / \partial i)$\}$\chi = 0$, where $i$ and $j$ are any pair of the coordinates $x, y, z$ and $t$. The principle of gauge invariance requires all observable quantities, such as the fields $\mathbf{E}$ and $\mathbf{B}$, to be independent of the gauge function [3,4].

In the retarded gauge above, the potentials, denoted by the subscripts 1 are described in terms of the charge and current source densities $\rho$ and $\mathbf{J}$ at the retarded time $t' = t - \|r - r'/c$. Another gauge that is of interest in the theory of magnetism [5,6] and in semiclassical electrodynamics which describes the interaction of atoms with radiation [6] is the three-vector version of the multipolar gauge [5,7–9]. In this gauge the potentials, denoted by the subscript 2, are described in terms of the instantaneous but non-local values of the fields $\mathbf{E}$ and $\mathbf{B}$

$$\phi_2(r, t) = -\mathbf{r} \cdot \int_0^1 \mathbf{E}(\mathbf{r}, \mathbf{r}, t) \, du \quad \text{and} \quad A_2(r, t) = -\mathbf{r} \times \int_0^1 \mathbf{B}(\mathbf{r}, \mathbf{r}, t) \, du.$$

The multipolar gauge satisfies the condition $\mathbf{r} \cdot A_2(r, t) = 0$ and is obtained from a gauge function that is essentially given by $-\int_0^1 \mathbf{r} \cdot A_2(\mathbf{r}, \mathbf{r}, t) \, du$ [9].

It is the purpose of this note to obtain the gauge function $\chi(r, t)$ that effects a gauge transformation between these two important gauges by means of the relations $A_2 = A_1 + \nabla \chi$ and $\phi_2 = \phi_1 - \partial \chi / \partial t$. The procedure that is used is to get $\mathbf{B}_1(r, t)$ and $\mathbf{E}_1(r, t)$ from eqs (3)-(5) and substitute them in eq. (7) to get $A_2(r, t)$ and $\phi_2(r, t)$. A gauge function $\chi(r, t)$ is then found that relates the two sets of potentials.

First we get $\mathbf{B}_1$ by taking the curl of $\mathbf{A}_1$ with respect to $\mathbf{r}$. Noting that $\mathbf{J}$ is a function of $\mathbf{r}'$ but not of $\mathbf{r}$ this gives

$$B_{1z}(\mathbf{r}, t) = -\frac{\mu_0}{4\pi} \int \, d\mathbf{r}'d\mathbf{t}' \mathbf{J}(\mathbf{r}', t') \times \nabla_{\mathbf{r}'} h(\mathbf{r}),$$

where $h(\mathbf{r}) = \delta(\mathbf{t}' - t + |\mathbf{r} - \mathbf{r}'|/c)/|\mathbf{r} - \mathbf{r}'|$ and the gradient is taken with respect to the parameter $\mathbf{r}$. Hence

$$A_2(r, t) = \frac{\mu_0}{4\pi} \int \, d\mathbf{r}'d\mathbf{t}' \int_0^1 \, du \, \mathbf{r} \times \{\mathbf{J}(\mathbf{r}', t') \times \nabla_{\mathbf{r}'} h(\mathbf{r})\}.$$

The triple vector product may be expanded as

$$\mathbf{r} \times \{\mathbf{J}(\mathbf{r}', t') \times \nabla_{\mathbf{r}'} h(\mathbf{u})\} = \mathbf{J}(\mathbf{r}', t')\{\mathbf{r} \cdot \nabla_{\mathbf{r}'} h(\mathbf{r})\} - \{\mathbf{J}(\mathbf{r}', t') \cdot \mathbf{r}\} \nabla_{\mathbf{r}'} h(\mathbf{u})$$

and this gives rise to two terms in (9). When the relations between derivatives $u \nabla_{\mathbf{r}'} h(\mathbf{u}) = \nabla h(\mathbf{u})$ and $(\mathbf{r} \cdot \nabla) h(\mathbf{u}) = u(\partial h(\mathbf{u}) / \partial u)$ derived in the appendix are used, where $h$ is any function of $\mathbf{u}$, the first term in the integrand becomes $u(\partial h(\mathbf{u}) / \partial u) = (\partial / \partial u)(uh) - \frac{h}{u}$ and so (9) is

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\[ A_2(r, t) = \frac{\mu_0}{4\pi} \int \, dr' dt' J(r', t') \times \int_0^1 \, du \left[ \partial / \partial u \{ uh(\text{ur}) \} - h(\text{ur}) \right] \cdot r \nabla_r h(\text{ur}), \]

where in the last term the vector dot product is between \( J \) and \( r \). The integral of \( u \) over the perfect differential can be carried out to give

\[ A_2(r, t) = \frac{\mu_0}{4\pi} \int \, dr' dt' J(r', t') \left[ h(r) - \int_0^1 \, du \{ h(\text{ur}) + r \nabla_r h(\text{ur}) \} \right]. \]

The first term can be recognized to be \( A_1(r, t) \) of eq. (4) so,

\[ A_2(r, t) - A_1(r, t) = -\frac{\mu_0}{4\pi} \int \, dr' dt' J(r', t') \int_0^1 \, du \{ h(\text{ur}) + r \nabla_r h(\text{ur}) \}. \]

Next we calculate \( \phi_2(r, t) \). From eq. (5), \( E \) is the sum of two parts, denoted by the superscripts \( a \) and \( b \), which, from eq. (7), give rise to two terms in the potential. The first, involving the gradient, is

\[ \phi_2^a(r, t) = \int_0^1 \, du (r \cdot \nabla_{ur}) \phi_1(\text{ur}, t), \]

so using the results in the appendix, \( r \cdot \nabla_{ur} h(q) = \partial h/\partial u \) we obtain \( \phi_2^a(r, t) - \phi_1(r, t) = -\phi_1(0, t) \). The other term is \( \phi_2^b(r, t) = r \cdot \int_0^1 (\partial / \partial t) A_1(\text{ur}, t) \, du \) and leads to

\[ \phi_2(r, t) - \phi_1(r, t) = -\phi_1(0, t) - \frac{\mu_0}{4\pi} \int \, dr' dt' \{ r \cdot J(r', t') \} \times \int_0^1 \, du \frac{\partial}{\partial u} \{ t' - t + |\text{ur} - r'|/c \} / |\text{ur} - r'|, \]

where \( \partial' \) is the derivative of the delta function with respect to its argument.

Consider now the scalar function \( \chi(r, t) = f(r, t) + g(t) \) where

\[ f(r, t) = -\frac{\mu_0}{4\pi} \int \, dr' dt' \{ r \cdot J(r', t') \} \int_0^1 \, du h(\text{ur}) \]

and

\[ g(t) = -\frac{1}{4\pi \varepsilon_0} \int \, dr' dt' \rho(r', t') \theta \{ t' - t + |r'|/c \} / |r'| \]

and \( \theta(x) \) is the function which is 1 for \( x > 0 \) and zero otherwise; its derivative is the delta function. The gradient of \( \chi \), which is the gradient of \( f \), is obtained by noting that

\[ \nabla \{ r \cdot J(r', t') h(\text{ur}) \} = h(\text{ur}) \nabla \{ r \cdot J(r', t') \} + \{ r \cdot J(r', t') \} \nabla h(\text{ur}) = h(\text{ur}) J(r', t') + \{ r \cdot J(r', t') \} \nabla h(\text{ur}) \] since \( \nabla \{ J(r', t') \cdot r \} = J(r', t') \). Hence
The time derivative $\partial \chi / \partial t$ has two terms coming from $f$ and one from $g$.

$$\frac{\partial f}{\partial t} = \frac{\mu_0}{4\pi} \int dr'dt' \rho(r', t') \int_0^1 du \delta(t' - t + |ur - r'|/c) \frac{|ur - r'|}{|ur - r'|}$$

and

$$\frac{\partial g}{\partial t} = \frac{1}{4\pi} \int dr'dt' \rho(r', t') \delta(t' - t + |r'|/c)/|r'| = \phi_1(0, t).$$

By comparing eqs (18) and (20) with (13) and (15) it can be seen that the gauge function $X^\alpha = f + g$ is indeed able to transform the retarded gauge into the three-vector version of the multipolar gauge.

The integral over $u$ in eq. (16) can be simplified. Using the standard relation

$$\delta[f(u)] = \sum_i \delta[u - u^i]/\partial f/\partial u|_{u^i},$$

where $f[u^i] = 0$, in this case with $f[u] = t' - t + |ur - r'|/c$, we obtain the roots $u^i$ from

$$(u^i - r')^2 - c^2(t - t')^2 = 0$$

to be

$$u^{i\pm} = \frac{r'}{r} \left[ \cos \varphi \pm \sqrt{c^2(t - t')^2/r^2 - \sin^2 \varphi} \right],$$

where $r$ and $r'$ are the lengths of the vectors $r$ and $r'$ and $\varphi$ is the angle between them. From the relation $c^2(t - t')^2 = u^2r^2 + r'^2 - 2urr' \cos \varphi$ it follows that $c^2(t - t')^2 - r'^2 \sin^2 \varphi = (ur - r' \cos \varphi)^2 \geq 0$ so the square root is always real. Next, it is straightforward to show that

$$\frac{\partial f}{\partial u} = \frac{r \cdot (ur - r')}{|ur - r'|}$$

and that

$$r \cdot (ur - r')|_{u^\pm} = \pm \sqrt{c^2(t - t')^2 - r'^2 \sin^2 \varphi},$$

so

$$\int_0^1 du h(ur) = \frac{c}{r \sqrt{c^2(t - t')^2 - r'^2 \sin^2 \varphi}} \int_0^1 du \delta(u - u^+ + \delta(u - u^-)), $$

as the $|ur - r'|$ terms in the numerator and denominator cancel and where $u^+$ and $u^-$ are given by eq. (22) so

$$f(r, t) = -\frac{\mu_0}{4\pi} \int \frac{dr'dt'c(r \cdot J(r', t'))}{r \sqrt{c^2(t - t')^2 - r'^2 \sin^2 \varphi}} \int_0^1 du \delta(u - u^+ + \delta(u - u^-)).$$

In carrying out the integrations over $r'$ and $t'$ in (26) the integral over $u$ gives plus one when $u^+$ and $u^-$ calculated from eq. (22) lie between zero and unity and zero otherwise.
Appendix

We show that \( r \cdot \nabla h(q) = u(\partial h / \partial u) \) where \( \nabla \) is the gradient operator with respect to \( r \), \( q = ur \) and \( h \) is any function of the vector \( q \). Noting that \( \partial q^i / \partial x^j = u_{ij} \) and \( \partial q^j / \partial u = x^j \) we find that \( \partial h / \partial x^i = u(\partial h / \partial q^i) \) so \( \nabla h(u) = u \nabla_u h(u) \). Also \( \partial h / \partial u = \sum_i u x^i (\partial h / \partial q^i) \) so \( u \partial h(q) / \partial u = \sum_i u x^i \partial h / \partial q^i \). Next, \( r \cdot \nabla h(q) = \sum_i u x^i \partial h / \partial q^i \) so it follows that \( r \cdot \nabla h(q) = u(\partial h / \partial u) \) and \( r \cdot \nabla_u h(q) = \partial h / \partial u \).

References