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# On conformally related *pp*-waves

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**Abstract.** Brinkmann [1] has shown that conformally related distinct Ricci flat solutions are *pp*-waves. Brinkmann's result has been generalized to include the conformally invariant source terms. It has been shown that [4] if $g_{ik}$ and $\bar{g}_{ik}(= w^{-2}g_{ik}, w$: a scalar function), are distinct metrics having the same Einstein tensor, $G_{ik} = \bar{G}_{ik}$, then both represent (generalized) *pp*-waves and $w_{,i}$ is a null covariantly constant vector of $g_{ik}$. Thus *pp*-waves are the only candidates which yield conformally related nontrivial solutions of $G_{ik} = T_{ik} = \bar{G}_{ik}$, with $T_{ik}$ being conformally invariant source.

In this paper the functional form of the conformal factor for the conformally related *pp*-waves/generalized *pp*-waves has been obtained. It has been shown that the most general *pp*-wave, conformally related to $\mathrm{d}s^2 = -2\mathrm{d}u[\mathrm{d}v - m\mathrm{d}y + H\mathrm{d}u] + P^{-2}[\mathrm{d}y^2 + \mathrm{d}z^2]$, turns out to be $(au + b)^{-2}\mathrm{d}s^2$, where $a, b$ are constants. Only in the special case when $m = 0, H = 1$, and $P = P(y, z)$, the conformal factor is $(au + b)^{-2}$ or $(a(u + v) + b)^{-2}$.

**Keywords.** Conformal transformations; conformal Killing vectors; *pp*-waves.

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## 1. Introduction

The *pp*-wave space-times are plane-fronted gravitational waves with parallel rays. These space-times represent the characteristic property of having parallel rays of plane electromagnetic waves of Minkowaski space. These waves serve as the idealized models of gravitational fields far from isolated radiating bodies. Their unique role in conformal transformations was first pointed out by Brinkmann [1, 2]. He proved that conformally related Einstein spaces are either Ricci flat *pp*-waves or conformally flat spaces with one as the de Sitter space and the other being flat. Thus one can have conformally related Ricci flat solutions only in case of *pp*-waves. This motivates the study of conformal transformations of *pp*-waves as they are the only candidates to give non-trivial conformally related solutions.

A relation between conformally related Einstein spaces and conformal Killing vector has been established [3]. We [3] have shown that any two of the following statements imply the third. (1) $g_{ij}$ is Einstein. (2) $\bar{g}_{ij}(= w^{-2}g_{ij}, w$: a scalar function) is Einstein. (3) $w_{,i}$ is a conformal Killing vector of $g_{ij}$. Using this we have given an alternative proof of Brinkmann's theorem. Hence the problem of finding conformally related *pp*-waves is reduced to that of finding covariantly constant null vectors of *pp*-wave metric [3,4].

Brinkmann's work has further been generalized [4] to include conformally invariant source terms such as massless Klein–Gordon, null radiation fields. It has been proved that [4] if $g_{ik}$ and $\bar{g}_{ik}(= w^{-2}g_{ik})$ are distinct metrics with $G_{ik} = \bar{G}_{ik}$, then both represent (generalized) $pp$-waves and $w_{,i}$ is a null covariantly constant vector.

In the present paper we obtain the functional form of the conformal factor for the conformally related $pp$-wave/generalized $pp$-wave. It has been shown that the generalized $pp$-wave, conformally related to $\mathrm{d}s^2 = -2\mathrm{d}u[\mathrm{d}v - m\mathrm{d}y + H\mathrm{d}u] + P^{-2}[\mathrm{d}y^2 + \mathrm{d}z^2]$, turns out to be $(au+b)^{-2}\mathrm{d}s^2$, where $a, b$ are constants. Only in the special case when $m = 0, H = 1$, and $P = P(y,z)$, the conformal factor is either $(au+b)^{-2}$ or $(a(u+v)+b)^{-2}$.

The present paper has been organized as follows. Section 2 deals with basic definitions and properties of conformal Killing vectors, conformal transformations and $pp$-waves. In §3 we find covariantly constant null vectors of $pp$-wave metric which in turn yields the functional form of the conformal factor of conformally related $pp$-waves.

## 2. Preliminaries

### 2.1 *Conformal Killing vectors*

We present below some basic definitions regarding conformal Killing vectors. A conformal Killing vector (CKV) $V^i$ satisfies

$$L_V g_{ij} = 2\phi(x^i)g_{ij}, \tag{1}$$

where $\phi$ is called as conformal factor. If $\phi = 0$, it is called Killing vector (KV), $\phi$ is a constant, then it is called as homothetic KV. A CKV is called special if $\phi_{,i;j} = 0$ and $\phi_{,i} \neq 0$. A CKV is called gradient CKV (GCKV) if it is a gradient of some scalar. A vector field is called as covariantly constant vector (CCV) if $V_{i;j} = 0$. Note that every CCV is GCKV.

### 2.2 *pp-Waves*

Exact solutions to Einstein's equations having properties similar to those of plane waves in the linearized theory of gravitation, can be obtained by demanding that they are solutions of [5]

$$R_{ij} = 0, R_{ijkl}k^l = 0, k_ik^i = 0. \tag{2}$$

The stronger conditions [5]:

$$R_{ij} = 0, k_{i;j} = 0, k_ik^i = 0, \tag{3}$$

imply that the rays associated with the null vector $k_i$ are parallel. Hence the waves are called as plane fronted waves with parallel rays, ($pp$-waves). Thus a space-time represents $pp$-wave if it admits a null CCV.

It can be shown that [5, 6] if a space-time with zero Ricci scalar (Ricci flat space is special case) admits a null CCV, then there exists a space-time in which the metric takes the following form:

$$ds^2 = -2H(u,y,z)du^2 - 2dudv + dy^2 + dz^2, \tag{4}$$

with $u$ and $v$ as retarded/advanced time coordinates (and $H_{,x,x} + H_{,y,y} = 0$ for Ricci-flat solutions). The space-like 2-surface $u, v =$ constant, are called wave surfaces. Since they are flat, the waves are called plane fronted. Ehlers and Kundt [2] have studied symmetries of *pp*-waves. They have found all Killing symmetries admitted by *pp*-waves in vacuum.

In the case when Ricci scalar is non-zero, the space-time admitting null CCV takes the form:

$$ds^2 = -2du[dv - m dy + H dx] + P^{-2}[dy^2 + dz^2], \tag{5}$$

where $m, H$ and $P$ are functions of $u, y, z$ [7].

### 2.3 *Conformal transformation*

We recollect some results regarding conformal tranformations, which will be used further. Consider the conformal transformation of the metric tensor

$$\bar{g}_{ik} = w^{-2} g_{ik}, \tag{6}$$

which leads to the following relation [4]:

$$w\bar{G}_{ik} = wG_{ik} + \left[2\Box w - \frac{3w_i w^i}{w}\right] g_{ik} - 2w_{,i;k}, \tag{7}$$

where $G_{ik}$ is Einstein tensor of the metric $g_{ik}$, and $\bar{G}_{ik}$ is the Einstein tensor of the metric $\bar{g}_{ik}$, $w_i = w_{,i}$, $w_{ik} = w_{,i;k}$, $\Box w = g^{ik} w_{ik}$. Here a semicolon refers to a covariant derivative for the metric $g_{ik}$.

It can be shown that [4] if $g_{ik}$ and $\bar{g}_{ik}$ are distinct solutions having $\bar{G}_{ik} = G_{ik}$, then both admit null CCV, representing generalized *pp*-waves, and $w_{,i}$ is a null CCV of $g_{ik}$. Conversely if $w_{,i}$ is a null CCV of $g_{ik}$ then $\bar{g}_{ik}$ is a *pp*-wave solution. In other words the problem of finding conformally related *pp*-wave space-time is reduced to finding gradient null CCVs of *pp*-wave metrics.

## 3. Conformal transformation of *pp*-waves

### 3.1 *Null CCVs of pp-wave metrics*

As discussed in the preceding section, the problem of finding conformally related *pp*-wave space-times reduces to that of finding gradient null CCVs of *pp*-wave metrics. Now to begin with consider the case of zero Ricci scalar (which includes vacuum *pp*-waves) in which the *pp*-wave metric has the form [6]: $ds^2 = -2H(u,y,z)du^2 - 2dudv + dy^2 + dz^2$.

In order to obtain the conformally related *pp*-wave solutions corresponding to the *pp*-wave $ds^2 = -2H(u,y,z)du^2 - 2dudv + dy^2 + dz^2$, we [3] have to solve equations $w_{,i;j} = 0$, viz.

$$w_{,u,u} - H_{,u} w_{,v} - H_{,y} w_{,y} - H_{,z} w_{,z} = 0, \tag{8}$$

$$w_{,u,y} - H_{,y} w_{,v} = 0, \tag{9}$$

$$w_{,u,z} - H_{,z} w_{,v} = 0, \tag{10}$$

$$w_{,v,v} = w_{,v,y} = w_{,v,z} = w_{,v,u} = w_{,y,z} = w_{,z,z} = w_{,y,y} = 0. \tag{11}$$

The set of eqs in (11) implies

$$w = z\phi_1(u) + kv + y\phi_2(u) + \phi_3(u), \tag{12}$$

where $k$ is an arbitrary constant and $\phi_1, \phi_2, \phi_3$ are arbitrary functions of $u$. Further, eqs (9) and (10) lead to

$$\phi_{2,u} - kH_{,y} = 0, \tag{13}$$

$$\phi_{1,u} - kH_{,z} = 0. \tag{14}$$

Differentiation of eqs (13) and (14) yields either $H_{,y,y} = H_{,z,z} = 0$, or k is zero.

Thus for $k = 0$, $w = k_1 z + k_2 y + \phi_3(u)$, where $k_1$ and $k_2$ are arbitrary constants, with $w_i = (\phi_{3,u}, 0, k_2, k_1)$. As $w_i$ is null; $w_i w^i = k_1^2 + k_2^2 = 0$. Hence $k_1 = k_2 = 0$ and $w = \phi_3(u)$, which by eq. (8) implies $\phi_{3,u,u} = 0$. In which case $\phi_3 = au + b$, where $a$ and $b$ are constants, i.e. $w = au + b$.

For $k \neq 0$, differentiations of (13) and (14) give $H_{,y,y} = H_{,z,z} = H_{,y,z} = 0$. The Riemann tensor for the $pp$-wave metric is [6]

$$R_{abcd} = 4[\delta^u_{[a}\delta^y_{b]}\delta^u_{[c}\delta^y_{d]}H_{,y,y} + \delta^u_{[a}\delta^z_{b]}\delta^u_{[c}\delta^z_{d]}H_{,z,z} + (\delta^u_{[a}\delta^y_{b]}\delta^u_{[c}\delta^z_{d]} + \delta^u_{[a}\delta^z_{b]}\delta^u_{[c}\delta^y_{d]})H_{,y,z}].$$

Hence if $H_{,y,y} = H_{,z,z} = H_{,y,z} = 0$, then the space-time is flat, this case however, is ruled out thereby leaving the unique possibility $w = au + b$. Thus the conformal factor in this case is $au + b$.

Some remarks are in order.

(i) Under the following coordinate transformations:

$$\tilde{u} = -(au+b)^{-2}, \quad \tilde{v} = v/a + (y^2 + z^2)/[2(au+b)],$$
$$\tilde{y} = y/(au+b), \quad \tilde{z} = z/(au+b),$$

the metric $[1/(au+b)^2]\mathrm{d}s^2$, will take the form:

$$\mathrm{d}\tilde{s}^2 = -2\mathrm{d}\tilde{u}\mathrm{d}\tilde{v} - \tilde{H}(\tilde{u},\tilde{y},\tilde{z})\mathrm{d}\tilde{u}^2 + \mathrm{d}\tilde{y}^2 + \mathrm{d}\tilde{z}^2,$$

with $\tilde{H}(\tilde{u},\tilde{y},\tilde{z}) = [(au+b)^2/a^2]H(u,y,z)$.

(ii) The amplitude $A$ and polarization $\theta$ of a $pp$-wave is defined by [6] $Ae^{i\theta} = (H_{,y,y} - H_{,z,z})/2 + iH_{,y,z}$. Hence $\tilde{A} = (au+b)^2 A$, and $\tilde{\theta} = \theta$. So conformally related $pp$-waves have the same polarization angle.

(iii) Most generally the $pp$-wave metric has the form

$$\mathrm{d}s^2 = -2\mathrm{d}u[\mathrm{d}v - m\mathrm{d}y + H\mathrm{d}u] + P^{-2}[\mathrm{d}y^2 + \mathrm{d}z^2], \tag{15}$$

where $m, H$, and $p$ are functions of $u, y, z$. Clearly $(1,0,0,0)$ is a null CCV of the metric (15). This metric admits one more CCV only in the special case when $m = 0, H = 1$, and $P = P(y,z)$ (see appendix of ref. [7]).

*Thus even for the most general case the conformal factor is* $w = au + b$.

### 3.2 *The special case: $m = 0, H = 1$, and $P = P(y,z)$*

Consider the case $m = 0$, $H = 1$, and $P = P(y,z)$.

$$ds^2 = -2dudv - 2du^2 + P^{-2}[dy^2 + dz^2], \tag{16}$$

where $P = P(y,z)$ and $w$ satisfies the following set of equations:

$$w_{,u,u} = w_{,u,v} = w_{,u,y} = w_{,u,z} = w_{,v,v} = w_{,v,y} = w_{,v,z} = 0, \tag{17}$$

$$w_{,y,y} + (P_{,y}/P)w_{,y} - (P_{,z}/P)w_{,z} = 0, \tag{18}$$

$$w_{,y,z} + (P_{,z}/P)w_{,y} + (P_{,y}/P)w_{,z} = 0, \tag{19}$$

$$w_{,z,z} - (P_{,y}/P)w_{,y} + (P_{,z}/P)w_{,z} = 0. \tag{20}$$

Equations in (17) imply $w = au + cv + \phi(y,z)$, where $a$ and $c$ are constants and $\phi$ is an arbitrary function of its arguments. Adding eqs (18) and (20) we get $w_{,y,y} + w_{,z,z} = 0$. Differentiating eq. (19) with respect to $z$ and eq. (20) with respect to $y$ and subtracting one from the other, we get

$$[(P_{,z,z} + P_{,y,y})/P - (P_{,y}^2 + P_{,z}^2)/P^2]w_{,y} = 0, \tag{21}$$

in view of $w_{,y,y} + w_{,z,z} = 0$. As the Ricci scalar for the metric (10) (viz, $(2/P^2)[(P_{,z,z} + P_{,y,y})/P - (P_y^2 + P_z^2)/P^2]$), is non zero (the $R = 0$ case has been dealt with earlier), eq. (21) yields $w_{,y} = 0$. Hence in view of eq. (19) we get $w_{,z} = 0$. So $w = au + cv + b$, where $a, b, c$ are constants. As $w_{,i}$ is null, we get $(ac - c^2) = 0$ and $c = 0$ or $a = c$, yielding either $w = au + b$ or $w = a(u + v) + b$.

### 3.3 *Illustrative example*

The $pp$-wave $ds_1^2 = -[y^2 + z^2]du^2 - 2dudv + dy^2 + dz^2$ satisfies Einstein–Klein–Gordon equations with massless field $\phi = \sqrt{2}u$. Likewise $ds_2^2 = (au + b)^{-2}ds_1^2$ also satisfies Einstein–Klein–Gordon equations with massless scalar field $\phi = \sqrt{2}u$. Note that $ds_1^2$ and $ds_2^2$ are different $pp$-waves, as in $ds_2^2$, $\tilde{H} = (\tilde{y}^2 + \tilde{z}^2)/a^2\tilde{u}^2$.

## 4. Conclusion

Conformally related solutions having the same source term are possible in case of $pp$-waves/generalized $pp$-waves only. In order to obtain conformally related $pp$-waves/generalized $pp$-waves essentially one has to determine null CCV of $pp$-waves/generalized $pp$-waves. We herewith identify the functional form of the conformal factor.

It turns out that conformally related $pp$-wave corresponding to

$$ds^2 = -2du[dv - mdy + Hdu] + P^{-2}[dy^2 + dz^2]$$

has the form $(au + b)^{-2}ds^2$. Only when $m = 0, H = 1$, and $P = P(y, z)$, the conformal factor turns out to be either $(au + b)^{-2}$ or $(a(u + v) + b)^{-2}$, as a special case.

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