PRAMANA — journal of physics
© Indian Academy of Sciences
Vol. 56, No. 4
April 2001
pp. 513–518

Bianchi type IX string cosmological model in general relativity

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MS received 20 January 2000; revised 16 October 2000

Abstract. We have investigated Bianchi type IX string cosmological models in general relativity. To get a determinate solution, we have assumed a condition $\rho = \lambda$ i.e. rest energy density for a cloud of strings is equal to the string tension density. The various physical and geometrical aspects of the models are also discussed.

Keywords. String; cosmological model.

PACS Nos 04.20.-q; 04.20.Jb; 04.20.Cv; 04.20.Ex

1. Introduction

Many relativist have taken keen interest in studying Bianchi type IX universe because familiar solutions like Robertson-Walker universe, the de-Sitter universe, the Taub-NUT solutions etc. are of Bianchi type IX space-times. Bianchi type IX cosmological models include closed FRW models. These models allow not only expansion but also rotation and shear and in general are anisotropic.

The string theory was developed to describe an event at the early stages of evolution of the universe. Cosmic strings arise during phase transitions after the big-bang explosion as the temperature goes down below some critical temperature [1–3]. These strings have stress energy and couple in a simple way to the gravitational field. The general relativistic formalism of cosmic strings is due to Letelier [4,5]. Stachel [6] has considered massless strings. Banerjee *et al* [7] have investigated some cosmological solutions of massive strings for Bianchi type I space-time. They have also obtained a class of solution corresponding to string cosmology with and without magnetic field. In the absence of magnetic field, a string solution obeying Takabayashi equation of state $\rho = (1+w)\lambda$, where $w > 0$ is constant for massive string, is obtained. Krori *et al* [8], and Chakraborty and Nandy [9] have studied Letelier string's model for Bianchi types II, VIII and IX space-times. Chakraborty [10] has investigated a class of cosmological solutions of massive strings in Bianchi type IX space-time using a supplementary condition $a = \alpha b^n$ between metric potentials, α and n are constants. Some solutions obeying Takabayashi equation of state $\rho = (1+w)\lambda, w > 0$, a constant are also obtained.

Tikekar and Patel [11] have obtained some exact solutions of string cosmology in Bianchi type III space-time of massive strings in the presence of magnetic field following the technique used by Letelier and Stachel. Some string solutions in which magnetic field is absent, are also discussed. Roy and Banerji [12] have obtained some LRS Bianchi type II string cosmological models for massive and geometrical string. The condition $\sigma/\theta =$ constant is used for massive string where σ is shear and θ the scalar expansion in the model.

In this paper, we have investigated Bianchi type IX string cosmological models. For the complete determination of the model, we assume that rest energy density is equal to the string tension density. The various physical and geometrical aspects of the models are also discussed.

We consider the homogeneous anisotropic Bianchi type IX metric in the form

$$\begin{aligned} \mathrm{d}s^2 = &-\mathrm{d}t^2 + a^2(t)\mathrm{d}x^2 + b^2(t)\mathrm{d}y^2 + (b^2\sin^2 y + a^2\cos^2 y)\mathrm{d}z^2 \\ &-2a^2\cos y\,\mathrm{d}x\mathrm{d}z, \end{aligned} \tag{1.1}$$

where a and b are functions of t-alone.

The energy momentum tensor for a cloud of massive strings, takes the form

$$T_i^j = \rho v_i v^j - \lambda x_i x^j \tag{1.2}$$

as given by Letelier [4,5], Stachel [6] and Banerjee [7].

Here ρ is the rest energy density for a cloud of strings with particle attached to them. Thus we can write

$$\rho = \rho_p + \lambda, \tag{1.3}$$

ρ_p being the particle energy density, λ the string tension density, v^i the four velocity for the cloud of particles and x^i the four vector which represents the strings direction which is essentially the direction of anisotropy. Thus we have

$$v_i v^i = -1 = -x_i x^i \tag{1.4}$$

and

$$v_i x^i = 0. \tag{1.5}$$

The Einstein field equation

$$R_i^j - \frac{1}{2}R g_i^j = -T_i^j \quad (\text{assuming } 8\pi G = C = 1), \tag{1.6}$$

for the line-element (1.1) leads to

$$\frac{2\dot{a}\dot{b}}{ab} + \frac{\dot{b}^2}{b^2} + \frac{1}{b^2} - \frac{1}{4}\frac{a^2}{b^4} = \rho, \tag{1.7}$$

$$2\frac{\ddot{b}}{b} + \frac{\dot{b}^2}{b^2} + \frac{1}{b^2} - \frac{3}{4}\frac{a^2}{b^4} = \lambda, \tag{1.8}$$

$$\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\dot{a}\dot{b}}{ab} + \frac{1}{4}\frac{a^2}{b^4} = 0. \tag{1.9}$$

For the complete determination of the set, we assume that

$$\rho = \lambda \tag{1.10}$$

which leads to

$$\frac{2\dot{a}\dot{b}}{ab} - 2\frac{\ddot{b}}{b} + \frac{1}{2}\frac{a^2}{b^4} = 0. \tag{1.11}$$

From eqs (1.9) and (1.11), we have

$$\frac{\ddot{a}}{a} = -\frac{2\ddot{b}}{b}. \tag{1.12}$$

Taking $a = \mathrm{e}^{\alpha t}$, eq. (1.12) leads to

$$\ddot{b} + K^2 b = 0, \tag{1.13}$$

where

$$K^2 = \alpha^2/2. \tag{1.14}$$

Equation (1.13) leads to

$$b = N\sin(Kt + \ell). \tag{1.15}$$

After suitable transformation of coordiates, the metric (1.1) leads to

$$\begin{aligned} \mathrm{d}s^2 = &-\frac{\mathrm{d}T^2}{K^2} + \mathrm{e}^{2\sqrt{2}(T-\ell)}\mathrm{d}X^2 \\ &+N^2\sin^2 T\mathrm{d}Y^2 + \left[N^2\sin^2 T\sin^2 Y + \mathrm{e}^{2\sqrt{2}(T-\ell)}\cos^2 Y\right]\mathrm{d}z^2 \\ &-2\mathrm{e}^{2\sqrt{2}(T-\ell)}\cos Y\,\mathrm{d}X\,\mathrm{d}Z. \end{aligned} \tag{1.16}$$

2. Some physical and geometrical features

The rest energy density and string tension density are given by

$$\begin{aligned} \rho = \lambda = &\, 2\sqrt{2}K^2\cot T + K^2\cot^2 T + \frac{1}{N^2}\mathrm{cosec}^2 T \\ &-\frac{1}{4N^4}\mathrm{e}^{2\sqrt{2}(T-\ell)}\mathrm{cosec}^4 T. \end{aligned} \tag{2.1}$$

The scalar of expansion (θ) and shear (σ) are given by

$$\theta = 2K\cot T + \sqrt{2}K \tag{2.2}$$

and

$$\sigma^2 = \frac{2}{3}(K^2 \cot^2 T + 2K^2 - 2\sqrt{2}K^2 \cot T). \tag{2.3}$$

The model starts expanding with a big-bang at $T = 0$ and the expansion in the model at $T = \pi/2$ is given as $\theta = \sqrt{2}K$. Thus expansion decreases slowly and it stops when $K = 0$. At $T = \pi/2, \sigma$ has finite value. Also when $T = \pi/2$ then $(\sigma/\theta) \neq 0$. When $T \to 0$ then $\rho \to \infty$ which shows that there is a massive mass at $T = 0$.

At $T = \pi/2$,

$$\rho = \frac{1}{N^2} - \frac{1}{4N^4}\mathrm{e}^{2\sqrt{2}(\frac{\pi}{2}-\ell)}.$$

The string tension density is maximum at $T = 0$ and it has finite value at $T = \pi/2$.

Thus at an initial epoch $T = 0$ i.e. $t = -(\ell/K)$, ρ, λ, θ and σ^2 diverge while in Chakraborty's model [10], $\rho, \lambda, \theta, \sigma^2$ diverge at the initial epoch $(t - t_0)^2 = (\mu^2/4C)$ where μ and C are constants. In Bianchi type I model, Banerjee *et al* [7] have shown that there is an upper limit for the expansion of the model and one can not obtain physical string model in the absence of magnetic field, while we have obtained Bianchi type IX string cosmological model which is physically valid.

3. Special case

If we assume

$$a = f(\alpha + \beta t)$$

and

$$b = \phi(\alpha - \beta t),$$

then from (1.12), we have

$$\frac{2}{\phi^2}\dot{\phi} + \frac{2\beta}{\phi^2} = \frac{1}{f^2}\dot{f} - \frac{\beta}{f^2} = M \tag{3.1}$$

which leads to

$$\phi = -\sqrt{\frac{2\beta}{M}} \cot h \left\{ \sqrt{\frac{M\beta}{2}} T \right\} \tag{3.2}$$

and

$$f = \sqrt{\frac{\beta}{M}} \tan(\sqrt{\beta M}\, T), \tag{3.3}$$

where

$$t + n = T. \tag{3.4}$$

After suitable transformation of coordinates, the metric (1.1) reduces to

$$ds^2 = -dT^2 + \frac{\beta}{M}\tan^2(\sqrt{\beta M}T)dX^2 + \frac{2\beta}{M}\cot h^2\left(\sqrt{\frac{\beta M}{2}}T\right)dY^2$$
$$+\left\{\frac{2\beta}{M}\coth^2\left(\sqrt{\frac{\beta M}{2}}T\right)\sin^2 Y + \frac{\beta}{M}\tan^2 X(\sqrt{\beta M}\,T\cos^2 Y)\right\}dZ^2$$
$$-\frac{2\beta}{M}\tan^2(\sqrt{\beta M}\,T)\cos Y\,dX\,dZ. \qquad (3.5)$$

The string tension density and rest energy density for the model (3.5) are given by

$$\lambda = \frac{M\beta\,\mathrm{cosec}\,h^2\left[\sqrt{\frac{M\beta}{2}}T\right]}{\cot h^2\left[\sqrt{\frac{M\beta}{2}}T\right]} + \frac{3\beta M}{2\cot h^2\left[\sqrt{\frac{M\beta}{2}}T\right]} + \frac{M}{2\beta\cot h^2\left[\sqrt{\frac{M\beta}{2}}T\right]}$$
$$-\frac{3M\tan^2[\sqrt{\beta M}\,T]}{16\beta\cot h^4\left[\sqrt{\frac{M\beta}{2}}T\right]},$$

$$\rho = \frac{\sqrt{2}\,M\beta}{\cot h\left[\sqrt{\frac{M\beta}{2}}T\right]\tan[\sqrt{\beta M}\,T]} + \frac{\beta M}{2\cot h^2\left[\sqrt{\frac{M\beta}{2}}T\right]}$$
$$+\frac{M}{2\beta\cot h^2\left[\sqrt{\frac{M\beta}{2}}T\right]} - \frac{M\tan^2[\sqrt{\beta M}\,T]}{16\beta\coth^4\left[\sqrt{\frac{M\beta}{2}}T\right]}.$$

The scalar expansion (θ) and shear (σ) for the model (3.5) are given by

$$\theta = \sqrt{\beta M}\left[\frac{\sqrt{2}}{\cot h\left\{\sqrt{\frac{M\beta}{2}}T\right\}} + \frac{1}{\tan\left\{\sqrt{\beta M}\,T\right\}}\right],$$

$$\sigma^2 = -\frac{2}{3}M\left[\frac{1}{2\cot h^2\left\{\sqrt{\frac{M\beta}{2}}T\right\}} + \frac{1}{\tan^2\left\{\sqrt{\beta M}\,T\right\}}\right.$$
$$\left.-\frac{\sqrt{2}}{\cot h\left\{\sqrt{\frac{M\beta}{2}}T\right\}}\,\frac{1}{\tan\left\{\sqrt{\beta M}\,T\right\}}\right].$$

The model (3.5) starts expanding with a big-bang at $T = 0$ and the expansion in the model decreases as T increases. The energy density, $\rho \to \infty$ when $T \to 0$. At $T = 0$, the string tension density $\lambda = M\beta$.

Acknowledgement

The authors are thankful to the referee for his valuable comments for the improvement of the paper.

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