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On the absence of scalar hair for charged rotating black holes in non-minimally coupled theories

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Abstract. In this work we check the validity of the no scalar hair theorem in charged axisymmetric stationary black holes for a wide class of scalar tensor theories.

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1. Introduction

Recently there has been a considerable resurgence in the no scalar hair theorem for black holes. Investigations regarding no hair theorem, however, had started about thirty years ago [1]. Inspired by Israel's uniqueness theorem for Schwarzschild and Reissner–Nordstrom black holes [2] and Carter [3] and Wald's [4] uniqueness theorem for Kerr black holes, Wheeler anticipated that gravitational collapse leads to black holes endowed with mass, charge and angular momentum and no other free parameters, which he summarized as 'black holes have no hair'. The 'no scalar hair theorem' excludes the availability of any knowledge of a scalar field from the exterior geometry of a black hole even when a scalar field is present in the space-time along with gravity.

The search for such scalar hair were initiated long back. Investigations, involving physical fields like massless scalar [5], massive vector [6], spinor [7] fields go in favour of Wheeler's dictum as any information about these fields from a stationary [6] black hole exterior is excluded. These investigations were mainly limited to the cases where the scalar field is only minimally coupled to gravity. But in the early 90's, solutions for stationary black holes with exterior non-abelian gauge field or Skyrmion field [8–10] have put strong challenge in front of the conjecture. Black hole solutions with new hair like Yang–Mills hair [8], Skyrme hair [9], dilaton hair [11] or others [12] act as counter examples to the conjecture. With a few exceptions [9] many of these black holes are unstable [13]. It is interesting to note that all the hair are not of similar stature [14]. The hair which act as new quantum numbers, i.e, independent of other quantum numbers are primary hair. Skyrme hair [9,14,15] in nonlinear sigma models coupled to gravity are examples of such hair. The hair which grow on other hair, i.e, the new quantum numbers determined by other quantum numbers are examples of secondary hair. Dilaton hair on electrically charged black holes [11], Kaluza Klein black holes [16] fall in this second category.

In spite of the popular name, there is no proof of the no hair theorem, and its status is in fact that of a conjecture. In the absence of a true theorem, one has to consider explicitly various sources of gravity and try to examine the nature of admissible black hole solutions. In this work the validity of the no scalar hair theorem is studied for a class of stationary axisymmetric charged black hole solutions in the context of a wide class of scalar tensor theories. In the rotating space-time there were some investigations [6,17] with minimally coupled scalar fields. It was shown that black hole in its final state cannot be endowed with an exterior scalar field. So the interest in the present work primarily involves the inclusion of a wide class of scalar tensor theories in axially symmetric space-time, where the scalar field is non-minimally coupled to gravity.

In order to check the validity of the no scalar hair theorem we have explicitly studied the space-time metric and scalars both in the cases of minimally and non-minimally coupled scalar fields. But the exact solutions for such fields in various scalar tensor theories are not available in the literature in most cases. So we use an algorithm to generate the exact solutions for charged rotating space-time with a minimally coupled scalar field from the known general relativistic electrovac solution. This solution is then analysed to find the compatibility of a scalar field with a black hole. Then we use a conformal transformation to generate the solutions for a large number of non-minimally coupled scalar tensor theories from Einstein–Maxwell minimally coupled scalar field (EMS) solution and test the no scalar hair theorem against these solutions.

2. A minimally coupled scalar field

2.1 *A technique to generate solutions for minimally coupled scalar field*

We start with a general form of stationary axially symmetric line element

$$\mathrm{d}s^2 = \mathrm{e}^{2\psi}(\mathrm{d}t + \omega\mathrm{d}\phi)^2 - \mathrm{e}^{-2\psi}[\mathrm{e}^{2\gamma}(\mathrm{d}x_1^2 + \mathrm{d}x_2^2) + h^2\mathrm{d}\phi^2], \tag{2.1}$$

where ψ, ω, γ, h are all functions of x_1 and x_2.

The energy momentum tensor for the electromagnetic field is

$$T_{\mu\nu} = g^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}, \tag{2.2}$$

where the Maxwell tensor $F_{\mu\nu}$ is given by

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}, \tag{2.3}$$

A_μ being the vector potential component. A_0 and A_3, (i.e A_t and A_ϕ) are the only existing components of A_μ. They are also functions of x_1 and x_2.

Now if a massless scalar field ϕ is also included, the total energy momentum tensor becomes

$$E_{\mu\nu} = T_{\mu\nu} + S_{\mu\nu}, \tag{2.4}$$

where $T_{\mu\nu}$ = energy momentum tensor for electromagnetic field and $S_{\mu\nu}$ = energy momentum tensor due to massless scalar field $= \phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\phi_{,\alpha}\phi^{,\alpha}$, where ϕ is also function of x_1 and x_2.

The set of equations to be solved are

$$R_{\mu\nu} = -\phi_{,\mu}\phi_{,\nu} - g^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta} + \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}, \tag{2.5}$$

$$\Box\phi = 0, \tag{2.6}$$

and

$$F^{\mu\nu}_{;\nu} = 0. \tag{2.7}$$

For the line element (2.1), the wave equation (2.6) becomes

$$\phi_{11} + \phi_{22} + \frac{h_1}{h}\phi_1 + \frac{h_2}{h}\phi_2 = 0. \tag{2.8}$$

Now according to the generation technique, if γ can be written as

$$\gamma = \gamma^v + \gamma^\phi, \tag{2.9}$$

where γ^v is the solution for γ in the electrovac field for metric (2.1) and γ^ϕ satisfies the equations

$$h_1\gamma_2^\phi + h_2\gamma_1^\phi = h\phi_1\phi_2, \tag{2.10}$$

$$h_1\gamma_1^\phi - h_2\gamma_2^\phi = \frac{h}{2}(\phi_1^2 - \phi_2^2), \tag{2.11}$$

then, the metric coefficients ψ, ω and h, vector potentials A_0 and A_3 of the general relativistic electrovac solutions along with γ as given by (2.9) and ϕ determined by (2.8) form the complete set of solutions for Einstein–Maxwell field minimally coupled with massless scalar field (EMS).

This algorithm is similar to that given by Eris and Gurses [18]. The difference is that our technique holds for a general metric while they [18] have used a different coordinate system, where the metric (2.1) is written in the Weyl–Papapetrou canonical form

$$\mathrm{d}s^2 = \mathrm{e}^{2\psi}(\mathrm{d}t + \omega\mathrm{d}\phi)^2 - \mathrm{e}^{-2\psi}[\mathrm{e}^{2\gamma}(\mathrm{d}\rho^2 + \mathrm{d}z^2) + \rho^2\mathrm{d}\phi^2], \tag{2.12}$$

ρ and z are harmonic functions of x_1 and x_2 and are called canonical cylindrical coordinates. The result of ref. [18] can be recovered from our result for $h = \rho$.

2.2 *Some axisymmetric solutions with minimally coupled scalar field*

The most widely used axially symmetric stationary electrovac solution in general relativity is the Kerr–Newman (KN) metric. We use this solution as a seed for the algorithm described above. The KN solution in the well-known Boyer Lindquist form is given by

$$\begin{aligned} \mathrm{d}s^2 = \mathrm{d}t^2 - \frac{2mr - e^2}{r^2 + a^2\cos^2\theta}(\mathrm{d}t + a\sin^2\theta\mathrm{d}\phi)^2 \\ -(r^2 + a^2\cos^2\theta)\left\{\mathrm{d}\theta^2 + \frac{\mathrm{d}r^2}{r^2 - 2mr + a^2 + e^2}\right\} \\ -(r^2 + a^2)\sin^2\theta\mathrm{d}\phi^2, \end{aligned} \tag{2.13}$$

and the solutions for the vector potentials are

$$A_3 = -\frac{ear\sin^2\theta}{r^2 + a^2\cos^2\theta} \quad \text{and} \quad A_0 = -\frac{er}{r^2 + a^2\cos^2\theta}. \tag{2.14}$$

The constants m, a and e are the mass, angular momentum per unit mass and the electric charge respectively of the axisymmetric distribution.

By a coordinate transformation of the form

$$r = \mathrm{e}^R + m + \frac{m^2 - a^2 - e^2}{4}\mathrm{e}^{-R}. \tag{2.15}$$

Misra *et al* [21] and later Singh *et al* [22] had rewritten the KN metric in the following canonical form

$$\begin{aligned} \mathrm{d}s^2 = \left(\frac{L^2 - 2mL + a^2\cos^2\theta + e^2}{L^2 + a^2\cos^2\theta}\right)\left[\mathrm{d}t - \frac{(2mL - e^2)a\sin^2\theta}{L^2 - 2mL + a^2\cos^2\theta + e^2}\mathrm{d}\phi\right]^2 \\ -\left(\frac{L^2 + a^2\cos^2\theta}{L^2 - 2mL + a^2\cos^2\theta + e^2}\right) \\ \{(L^2 - 2mL + a^2\cos^2\theta + e^2)(\mathrm{d}R^2 + \mathrm{d}\theta^2) \\ +(L^2 - 2mL + a^2 + e^2)\sin^2\theta\mathrm{d}\phi^2\}, \end{aligned} \tag{2.16}$$

where

$$L = \mathrm{e}^R + m + \frac{m^2 - a^2 - e^2}{4}\mathrm{e}^{-R}. \tag{2.17}$$

The vector potentials in the transformed coordinates are

$$A_3 = -\frac{eaL\sin^2\theta}{L^2 + a^2\cos^2\theta} \quad \text{and} \quad A_0 = -\frac{eL}{L^2 + a^2\cos^2\theta}. \tag{2.18}$$

We shall now use this solution to generate the corresponding EMS solution. In terms of the metric (2.16), eq. (2.8) can be written as

$$\phi_{RR} + \phi_{\theta\theta} + \frac{\mathrm{e}^{R} + \frac{M^2}{4}\mathrm{e}^{-R}}{\mathrm{e}^{R} - \frac{M^2}{4}\mathrm{e}^{-R}}\phi_R + \frac{\cos\theta}{\sin\theta}\phi_\theta = 0, \tag{2.19}$$

where

$$M^2 = m^2 - a^2 - e^2. \tag{2.20}$$

Equation (2.19) can be solved in a general way by simply imposing the separability condition. The scalar field ϕ is considered to be separable in functions of R and θ both in product and summed form. The solutions for ϕ in both the ways have been exhibited explicitly in the Appendix. Here we would consider some special cases of the general solutions.

2.2.1 *First set*: We first consider the simplest case of $\phi[\phi = \phi(R)]$, i.e, ϕ is isotropic. The solution for ϕ in such case,

$$\phi = \phi_0 + \frac{\sigma}{M}\ln\left(\frac{\mathrm{e}^{R} - \frac{M}{2}}{\mathrm{e}^{R} + \frac{M}{2}}\right) = \phi_0 + \frac{\sigma}{2M}\ln\frac{L-(m+M)}{L-(m-M)}, \tag{2.21}$$

where σ and ϕ_0 are two constants. We take $\phi_0 = 0$ without any loss of generality.

Once ϕ is specified, we can find γ^ϕ from equations (2.10) and (2.11) as

$$\begin{aligned}\gamma^\phi &= \frac{\sigma^2}{4M^2}\ln\frac{\left(\mathrm{e}^{R} - \frac{M^2}{4}\mathrm{e}^{-R}\right)^2}{\left(\mathrm{e}^{R} - \frac{M^2}{4}\mathrm{e}^{-R}\right)^2 + M^2\sin^2\theta}\\ &= \frac{\sigma^2}{4M^2}\ln\frac{L^2-2mL+a^2+e^2}{(L-m)^2 - M^2\cos^2\theta}.\end{aligned} \tag{2.22}$$

Hence

$$\begin{aligned}\mathrm{e}^{2\gamma} &= \mathrm{e}^{2\gamma^{v}+2\gamma^{\phi}} = (L^2 - 2mL + a^2\cos^2\theta + e^2)\\ &\quad\times\left\{\frac{L^2-2mL+a^2+e^2}{(L-m)^2-M^2\cos^2\theta}\right\}^{\frac{\sigma^2}{2M^2}}.\end{aligned} \tag{2.23}$$

By using the inverse transformation given by (2.15), the line element in EMS field can be written in the well-known Boyer Lindquist form, as

$$\begin{aligned}\mathrm{d}s^2 &= \mathrm{d}t^2 - \frac{2mr - e^2}{r^2 + a^2\cos^2\theta}(\mathrm{d}t + a\sin^2\theta\mathrm{d}\phi)^2\\ &\quad -(r^2 + a^2\cos^2\theta)\left\{\frac{r^2-2mr+a^2+e^2}{(r-m)^2-M^2\cos^2\theta}\right\}^{\frac{\sigma^2}{2M^2}}\\ &\quad \left\{\mathrm{d}\theta^2 + \frac{\mathrm{d}r^2}{r^2-2mr+a^2+e^2}\right\} - (r^2+a^2)\sin^2\theta\mathrm{d}\phi^2,\end{aligned} \tag{2.24}$$

with the vector potentials

$$A_3 = -\frac{ear\sin^2\theta}{r^2 + a^2\cos^2\theta} \quad \text{and} \quad A_0 = -\frac{er}{r^2 + a^2\cos^2\theta}, \tag{2.25}$$

and the scalar field

$$\phi = \frac{\sigma}{2M}\ln\frac{r-(m+M)}{r-(m-M)}. \tag{2.26}$$

The scalar field vanishes for the limit $r \to \infty$ and the metric is asymptotically flat. In this solution, for $\sigma = 0$, the scalar field becomes trivial and the metric goes over to the KN solution. If we put off the electric charge, i.e $e = 0$, and set the angular momentum also to zero, i.e $a = 0$, the metric reduces to one of the solutions given by Penny [23]. Without the electric charge, the solution (2.24) reduces to the Brans–Dicke–Kerr solution given by McIntosh [24] in Dicke's revised units [25].

From the metric (2.24) we see that g_{11} is singular at $r = m \pm M$ surfaces. Simultaneously the scalar field ϕ in (2.26) diverges at these two surfaces if $\sigma \neq 0$. The Ricci scalar is given by

$$\mathcal{R} = -\phi_\alpha\phi^\alpha = \frac{\sigma^2}{[r^2 - 2mr + a^2 + e^2]^{1+\frac{\sigma^2}{2M^2}}} \times\frac{((r-m)^2 - M^2\cos^2\theta)^{\frac{\sigma^2}{2M^2}}}{r^2 + a^2\cos^2\theta}. \tag{2.27}$$

It is evident from this expression that Ricci scalar diverges at the surfaces $r = m \pm M$ for $\sigma \neq 0$ and thus these surfaces fail to act as horizons. However, if $\sigma = 0$, $\mathcal{R}$ also becomes 0 for all values of r and there is no singularity at $r = m \pm M$. For $\sigma = 0$, however, the solution reduces to the KN solution, and the scalar field becomes trivial. So the only black hole solution in this space-time is KN black hole and hence this solution supports the theorem for the non-existence of a scalar hair.

2.2.2 *Second set*: Next we consider the case when ϕ is both function R and θ in the form $\phi = \alpha(R) + \beta(\theta)$. Here we have assumed the integration constants σ and τ to be 0 in (A5). Then from eq. (A5) we get the solution for ϕ to be

$$\phi = \phi_0 + \lambda\ln\left[\left(\mathrm{e}^R - \frac{M^2}{4}\mathrm{e}^{-R}\right)\sin\theta\right], \tag{2.28}$$

where ϕ_0 and λ are two constants. Here also we consider $\phi_0 = 0$ without any loss of generality.

Using the similar technique as before we find the line element for the EMS field in the Boyer–Lindquist coordinate as

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \frac{2mr - e^2}{r^2 + a^2\cos^2\theta}(\mathrm{d}t + a\sin^2\theta\mathrm{d}\phi)^2 -(r^2 + a^2\cos^2\theta)\left\{(r^2 - 2mr + a^2 + e^2)\sin^2\theta\right\}^{\frac{\lambda^2}{2}} \left\{\mathrm{d}\theta^2 + \frac{\mathrm{d}r^2}{r^2 - 2mr + a^2 + e^2}\right\} - (r^2 + a^2)\sin^2\theta\mathrm{d}\phi^2, \tag{2.29}$$

with the same solution for vector potentials A_0 and A_3. The scalar field, when expressed in Boyer–Lindquist coordinates (r,θ), appears as

$$\phi = \frac{\lambda}{2}\ln[(r^2 - 2mr + a^2 + e^2)\sin^2\theta]. \tag{2.30}$$

This solution has metric singularities at $r = m \pm M$ surfaces, as $g_{\theta\theta}$ goes to zero and g_{11} goes to zero or infinity corresponding to $\lambda > \sqrt{2}$ or $< \sqrt{2}$. And like the previous case the scalar field ϕ diverges at these surfaces. To ensure the nature of singularity we find the Ricci scalar

$$\mathcal{R} = -\phi_\alpha\phi^\alpha = \frac{\lambda^2}{(r^2 + a^2\cos^2\theta)}[(r^2 - 2mr + a^2 + e^2)\sin^2\theta]^{\frac{-\lambda^2}{2}} \times \left\{\frac{(r-m)^2 - M^2\cos^2\theta}{r^2 - 2mr + a^2 + e^2}\right\}. \tag{2.31}$$

For $\lambda = 0$, $\mathcal{R}$ becomes 0 for all r and for $\lambda \neq 0$ $\mathcal{R}$ diverges for $r = m \pm M$. So for $\lambda \neq 0$ these surfaces $r = m \pm M$ become singular and hence fail to act as event horizons. As for $\lambda = 0$, $\mathcal{R}$ is finite and consequently the surfaces can act as event horizons. However, the scalar field becomes trivial in that case and the metric reduces to the KN one. So this class of solutions also supports the conjecture.

2.2.3 *Third set*: We choose our third set of solution for ϕ from the general solution (A11) where ϕ is separable in the product form of function of R and θ. The solution, being the product of two infinite series, we take the simplest choice ($n = 1$, i.e., $\lambda = 2$), given by (A16)

$$\phi = \sigma\left(\mathrm{e}^R + \frac{M^2}{4}\mathrm{e}^{-R}\right)\cos\theta, \tag{2.32}$$

where σ is a constant.

Adopting a similar technique we find the line element for EMS field in the Boyer–Lindquist form as

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \frac{2mr - e^2}{r^2 + a^2\cos^2\theta}(\mathrm{d}t + a\sin^2\theta\mathrm{d}\phi)^2 -(r^2 + a^2\cos^2\theta)\left\{\mathrm{e}^{\frac{-\sigma^2}{2}(r^2 - 2mr + a^2 + e^2)\sin^2\theta}\right\} \left\{\mathrm{d}\theta^2 + \frac{\mathrm{d}r^2}{r^2 - 2mr + a^2 + e^2}\right\} - (r^2 + a^2)\sin^2\theta d\phi^2, \tag{2.33}$$

with the same solutions for the vector potentials. In the Boyer–Lindquist coordinate (r,θ) the scalar field looks like

$$\phi = \sigma(r - m)\cos\theta. \tag{2.34}$$

It is quite transparent from the metric (2.33) that there is singularity in g_{11} at $r = m \pm M$ surfaces though the scalar field remains finite at these surfaces. The expression for Ricci scalar is

$$\mathcal{R} = -\phi_\alpha \phi^\alpha = \sigma^2 \frac{(r-m)^2 - M^2 \cos^2\theta}{(r^2 + a^2\cos^2\theta)} \mathrm{e}^{\frac{\sigma^2}{2}(r^2 - 2mr + a^2 + e^2)\sin^2\theta}. \tag{2.35}$$

Unlike the other two cases for $\sigma \neq 0$ (eqs (2.27) and (2.31)), $\mathcal{R}$ remains finite at $r = m \pm M$ surfaces. This implies that even for a non trivial scalar field we have finite $\mathcal{R}$ at $r = m \pm M$ surfaces which, in turn, indicates that these surfaces are no longer singular surfaces. In fact the Kretchman scalar I ($I = R^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta}$) also remains finite everywhere including the surfaces $r = m \pm M$ surfaces for $\sigma \neq 0$. The expression for I is excluded from the text for the economy of space. So these surfaces are not singular and act as event horizons to shield the essential singularity. And as there is a non-trivial contribution of scalar field this solution seems to contradict the 'no scalar hair theorem'. But this solution has two major defects. The solution is not asymptotically flat and the scalar field becomes infinite for $r \to \infty$ limit, while the energy due to scalar field has a finite contribution in that limit. So although this solution has some nontrivial contribution for ϕ on the horizon it could not really be considered as a serious counter example due to its pathological behaviour at $r \to \infty$.

3. A conformal transformation and non-minimally coupled scalar fields

3.1 *Conformal transformation*

The action for a very general scalar tensor theory along with the Maxwell field is given by

$$S[g_{\mu\nu}, \phi, F_{\mu\nu}] = \int [f(\phi)R - h(\phi) g^{\mu\nu} \phi_{,\mu}\phi_{,\nu} - F_{\mu\nu}F^{\mu\nu}]\sqrt{-g}\,\mathrm{d}^4x, \tag{3.1}$$

with $f(\phi) > 0$ and $h(\phi) > 0$ where $g_{\mu\nu}, \phi$ and $F_{\mu\nu}$ are the metric tensor, the scalar field and the Maxwell field respectively. The scalar field is non-minimally coupled to gravity because of the term $f(\phi)$ in the action and the Newtonian constant G thus becomes a function of ϕ instead of being a constant. For different choices of the functions $f(\phi)$ and $h(\phi)$, one obtains various scalar tensor theories of gravitation. With a conformal transformation

$$\bar{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}, \tag{3.2}$$

where $\Omega^2 = f(\phi)$, and by defining a new scalar field $\bar{\phi}$ in terms of ϕ as

$$\bar{\phi}(\phi) = \sqrt{2} \int_{\phi_c}^{\phi} \mathrm{d}\xi \sqrt{\frac{3}{2}\left(\frac{\mathrm{d}}{\mathrm{d}\xi} \ln f(\xi)\right)^2 + \frac{h(\xi)}{f(\xi)}}, \tag{3.3}$$

the action (3.1) can be written in the form

$$\bar{S}[\bar{g}_{\mu\nu}, \bar{\phi}, \bar{F}_{\mu\nu}] = \int \left[\bar{R} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{\phi}_{,\mu}\bar{\phi}_{,\nu} - \bar{F}^2\right]\sqrt{-\bar{g}}\,\mathrm{d}^4x. \tag{3.4}$$

Here ϕ_c is an arbitrary positive constant and the variables with an overhead bar represent those in the transformed version. The action (3.4) clearly resembles that of a minimally coupled scalar field with a Maxwell field. So if the solution for this case is known, one can now easily find out the solutions for the corresponding non-minimally coupled scalar field cases by using the eqs (3.2) and (3.3) with proper choices for $f(\phi)$ and $h(\phi)$. This type of conformal transformation has been used for a long time in the literature [12,19,25]. Reference [26] represents a good set of references on this subject.

It deserves mention at this point [20] that in case of dilaton gravity, obtained from the low energy limit of string theory, there is a coupling between Maxwell field and dilaton field in the action. This coupling makes dilaton gravity different from other scalar tensor theories described by action (3.1). Horne and Horowitz [31] found the black hole solution with dilaton hair for slow rotation in this theory.

3.2 *Some axisymmetric solutions in non-minimally coupled scalar tensor theories*

Amongst the three classes of solutions exhibited in §2 each class of solutions could be used to generate the corresponding new solutions in non-minimally coupled scalar tensor theories and then the no scalar hair theorem will be verified against them. We cite the examples in Brans–Dicke theory, although this method works for other more general theories also.

3.2.1 *First set*: The relevant action in BD theory [27] is

$$S = \int \left[\phi R - \frac{\omega}{\phi} g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - F_{\mu\nu}F^{\mu\nu}\right]\sqrt{-g}\mathrm{d}^4x. \tag{3.5}$$

Hence the conformal transformation for Brans–Dicke field from the minimally coupled scalar field would be of the form

$$g_{\mu\nu} = \frac{1}{\phi}\bar{g}_{\mu\nu}, \tag{3.6}$$

and

$$\bar{\phi}(\phi) = \sqrt{(2\omega+3)}\ln\frac{\phi}{\phi_c}. \tag{3.7}$$

The solution for the Brans–Dicke scalar field corresponding to (2.26) is

$$\phi = \phi_c\left(\frac{r-(m+M)}{r-(m-M)}\right)^{\overline{\frac{\sigma}{2M\sqrt{2\omega+3}}}}, \tag{3.8}$$

and the BDM metric components become

$$g_{\mu\nu} = \frac{1}{\phi_c}\left(\frac{r-(m-M)}{r-(m+M)}\right)^{\overline{\frac{\sigma}{2M\sqrt{2\omega+3}}}}\bar{g}_{\mu\nu}. \tag{3.9}$$

Thus the line element for the BDM metric in the Boyer–Lindquist form is

$$\phi_c \mathrm{d}s^2 = \left(\frac{r-(m-M)}{r-(m+M)}\right)^{\frac{\sigma}{2M\sqrt{2\omega+3}}} \Bigg[\mathrm{d}t^2 - \frac{2mr-e^2}{r^2+a^2\cos^2\theta}(\mathrm{d}t + a\sin^2\theta \mathrm{d}\phi)^2 - (r^2+a^2\cos^2\theta) \left\{\frac{r^2-2mr+a^2+e^2}{(r-m)^2-M^2\cos^2\theta}\right\}^{\frac{\sigma^2}{2M^2}} \left\{\mathrm{d}\theta^2 + \frac{\mathrm{d}r^2}{r^2-2mr+a^2+e^2}\right\} -(r^2+a^2)\sin^2\theta \mathrm{d}\phi^2\Bigg] \tag{3.10}$$

with the solution for the vector potentials being the same as in EMS field. In the limit $r \to \infty$, the metric is flat and the scalar field is constant. For this line element the Ricci scalar takes the from

$$\mathcal{R} = \frac{\omega}{\phi^2} g^{\mu\nu}\phi_\mu\phi_\nu = \frac{\omega\sigma^2}{2\omega+3} \times \frac{1}{r^2+a^2\cos^2\theta} \times \frac{[(r-m)^2 - M^2\cos^2\theta]^{\frac{\sigma^2}{2M^2}}}{[r^2-2mr+a^2+e^2]^{1+\frac{\sigma^2}{2M^2}}}. \tag{3.11}$$

We find that the surfaces $r = m \pm M$ act as physically singular surfaces for $\sigma \neq 0$ as Ricci scalar $\mathcal{R}$ becomes infinitely large at these surfaces. And as there is no other horizon, these singularities are naked. But if $\sigma = 0$, the Ricci scalar remains finite and the scalar field becomes trivial. So these surfaces are only coordinate singularities and act as event horizon to shield the essential singularity. But the metric (3.10) reduces to the KN metric for $\sigma = 0$. So like the minimally coupled counterpart this set of solution for BDM space-time also is in agreement with the no scalar hair theorem as the only black hole solution is the KN solution which indeed has no scalar hair.

3.2.2 *Second set*: The solution for Brans–Dicke scalar field in this case is

$$\phi = \phi_c[(r^2-2mr+a^2+e^2)\sin^2\theta]^{\frac{\lambda}{2\sqrt{2\omega+3}}}, \tag{3.12}$$

and the line element in Brans–Dicke–Maxwell theory corresponding to the metric (2.29) is

$$\phi_c \mathrm{d}s^2 = [(r^2-2mr+a^2+e^2)\sin^2\theta]^{\frac{-\lambda}{2\sqrt{2\omega+3}}} \Bigg[\mathrm{d}t^2 - \frac{2mr-\mathrm{e}^2}{r^2+a^2\cos^2\theta}(\mathrm{d}t + a\sin^2\theta \mathrm{d}\phi)^2 -(r^2+a^2\cos^2\theta)\left\{(r^2-2mr+a^2+e^2)\sin^2\theta\right\}^{\frac{\lambda^2}{2}} \left\{\mathrm{d}\theta^2 + \frac{\mathrm{d}r^2}{r^2-2mr+a^2+e^2}\right\} -(r^2+a^2)\sin^2\theta \mathrm{d}\phi^2\Bigg]. \tag{3.13}$$

The expression for the Ricci scalar is

$$\mathcal{R} = \frac{\omega}{\phi^2} g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}$$
$$= \frac{\lambda^2\omega}{2\omega+3}\frac{[(r^2-2mr+a^2+e^2)\sin^2\theta]^{\frac{-\lambda^2}{2}}}{(r^2+a^2\cos^2\theta)}\left\{\frac{(r-m)^2-M^2\cos^2\theta}{r^2-2mr+a^2+e^2}\right\}. \qquad (3.14)$$

It is quite clear that the surfaces $r = m \pm M$ are physically singular surfaces for non-trivial scalar field i.e, for $\lambda \neq 0$. For $\lambda = 0$ these surfaces become event horizons with finite Ricci scalar and trivial scalar field and the metric becomes KN. So for this class of solution, like the previous case, we find that Brans–Dicke scalar hair is not compatible with the black hole. This solution can be regarded as asymptotically zero curvature solution since the curvature (Ricci scalar) becomes zero at $r \to \infty$.

3.2.3 *Third set*: Now we generate the third class of solutions corresponding to the eqs (2.33)–(2.35) which is fairly interesting in the sense that it goes against the theorem. For Brans–Dicke theory the solution for the Brans–Dicke field in this case is

$$\phi = \phi_c \mathrm{e}^{\frac{\sigma}{\sqrt{2\omega+3}}(r-m)\cos\theta}. \qquad (3.15)$$

The corresponding line element of Brans–Dicke–Maxwell metric is

$$\phi_c \mathrm{d}s^2 = \mathrm{e}^{\frac{-\sigma}{\sqrt{2\omega+3}}(r-m)\cos\theta}\Bigg[\mathrm{d}t^2 - \frac{2mr-e^2}{r^2+a^2\cos^2\theta}(\mathrm{d}t + a\sin^2\theta\mathrm{d}\phi)^2$$
$$-(r^2+a^2\cos^2\theta)\left\{\mathrm{e}^{\frac{-\sigma^2}{2}(r^2-2mr+a^2+e^2)\sin^2\theta}\right\}$$
$$\left\{\mathrm{d}\theta^2 + \frac{\mathrm{d}r^2}{r^2-2mr+a^2+e^2}\right\} - (r^2+a^2)\sin^2\theta d\phi^2\Bigg]. \qquad (3.16)$$

The expression for the Ricci scalar is

$$\mathcal{R} = \frac{\omega}{\phi^2} g^{\mu\nu}\phi_{\mu}\phi^{\nu}$$
$$= \frac{\sigma^2\omega}{2\omega+3}\frac{(r-m)^2-M^2\cos^2\theta}{(r^2+a^2\cos^2\theta)}\mathrm{e}^{\frac{\sigma^2}{2}(r^2-2mr+a^2+e^2)\sin^2\theta + \frac{\sigma}{\sqrt{2\omega+3}}(r-m)\cos\theta}. \qquad (3.17)$$

The expression for Ricci scalar reveals that the surfaces $r = m \pm M$ are not physically singular but rather act as event horizon with a non-trivial scalar field. But this solution, like its minimally coupled counterpart, has the defects of not being asymptotically flat and having a divergent scalar field at infinite r. Thus, although in this we have a horizon with non-trivial scalar field, it cannot be taken seriously for its pathological asymptotic behaviour.

In all these three sets, we have examined some other non-minimally coupled scalar tensor theories [28–30], where the BD parameter ω is a function of the scalar field ϕ. In all these cases the results are the same, i.e., for the first two sets there are no black holes with scalar hair and for the third the scalar hair can exist where the space-time is not asymptotically flat. We do not include the examples in the text for the economy of space.

4. Discussion

Although there are already a lot of results regarding the scalar hair for spherical black holes, the axially symmetric black holes warrants more investigations. The present work studies axisymmetric charged black holes for various scalar tensor theories. All these solutions are essentially the analogue of Kerr–Newman (KN) solutions in general relativity and reduces to KN if the scalar field contribution is put equal to zero.

For the first two sets, it is found that there is no regular horizons if the scalar field exists. If, however, the scalar field is trivial $(\sigma = 0)$, there are event horizons. But in the latter case, $(\sigma = 0)$ the metric is that of Kerr–Newman, and the geometry is endowed with only mass, electric charge and angular momentum.

For the third set of solutions, it appears that the surfaces $r = m \pm M$ will act as event horizon as the scalar curvatures are finite at those surfaces even with a non-trivial scalar field. But these solutions are not asymptotically flat and hence will produce curvature in the space-time even at infinitely large distances from the black hole.

Thus the present investigations are in keeping with Bekenstein's statement [32] that there is no asymptotically flat, stationary and stable black hole solution in general relativity and general scalar tensor theory which is endowed with a scalar field. This also, in a way, is in agreement with the Hawking theorem [33] which states that exterior of a stationary black hole is identical both in general relativity and Brans–Dicke theory and this theorem can be extended to include a wide class of scalar tensor theories represented by action (3.1).

The case of Kerr black hole analogue with the scalar fields can easily be studied from the present work simply by setting the electric charge $e = 0$. It is a trivial matter to see that conclusions regarding the scalar hair will remain exactly the same as in the case when the distribution has a non-zero charge.

It deserves mention that the conformal transformation used in the present work crucially depends on the fact that $f(\phi)$ is positive. This may not be treated as a serious restriction as in the weak field limit $f(\phi)$ gives the inverse of the Newtonian constant G and thus a negative $f(\phi)$ will indicate that G is negative. Anyway, for the sake of completeness the cases with negative $f(\phi)$ should also be investigated. It should be noted that the solutions obtained by the generation techniques do actually solve the relevant field equations.

Appendix A

We obtain the solution for the minimally coupled scalar field ϕ from eq. (2.19) by assuming the separability condition. We first exhibit the solution when ϕ is separable in summation form

$$\phi = \alpha(R) + \beta(\theta), \tag{A1}$$

where α and β are functions of R and θ respectively. Under such an assumption eq. (2.19) attains the form

$$\alpha_{RR} + \frac{\mathrm{e}^R + \frac{M^2}{4}\mathrm{e}^{-R}}{\mathrm{e}^R - \frac{M^2}{4}\mathrm{e}^{-R}}\alpha_R = -\left\{\beta_{\theta\theta} + \frac{\cos\theta}{\sin\theta}\beta_\theta\right\} = \lambda, \tag{A2}$$

where λ is a separation constant.

From (A2) it is quite easy to find the solution for α and β as

$$\alpha = \lambda \ln \left(\mathrm{e}^R - \frac{M^2}{4}\mathrm{e}^{-R} \right) + \frac{\sigma}{M} \ln \left(\frac{\mathrm{e}^R - \frac{M}{2}}{\mathrm{e}^R + \frac{M}{2}} \right) + \text{constant}, \tag{A3}$$

and

$$\beta = \lambda \ln \sin\theta + \tau \ln \tan \frac{\theta}{2} + \text{constant}, \tag{A4}$$

where σ and τ are integration constants. So

$$\phi = \lambda \ln \left[\left(\mathrm{e}^R - \frac{M^2}{4}\mathrm{e}^{-R} \right) \sin\theta \right] + \frac{\sigma}{M} \ln \left(\frac{\mathrm{e}^R - \frac{M}{2}}{\mathrm{e}^R + \frac{M}{2}} \right) + \tau \ln \tan \frac{\theta}{2}. \tag{A5}$$

Next we consider ϕ to be separable as a product of functions of R and θ as

$$\phi = \sigma A(R) B(\theta), \tag{A6}$$

where σ is constant.

With the form like (A6), eq. (2.19) can be split into two equations,

$$A_{RR} + \frac{\mathrm{e}^R + \frac{M^2}{4}\mathrm{e}^{-R}}{\mathrm{e}^R - \frac{M^2}{4}\mathrm{e}^{-R}} A_R - \lambda A = 0 \tag{A7}$$

and

$$B_{\theta\theta} + \frac{\cos\theta}{\sin\theta} B_\theta + \lambda B = 0, \tag{A8}$$

where λ is the separation constant.

Now eq. (A8) can be recast as

$$(1 - X^2)\frac{\mathrm{d}^2 B}{\mathrm{d}X^2} - 2X\frac{\mathrm{d}B}{\mathrm{d}X} + \lambda B = 0, \tag{A9}$$

where $X = \cos\theta$. If λ is taken as $n(n+1)$ where n is an integer, eq. (A9) becomes the Legendre differential equation of second order. Hence the solution of (A9) is given by the Legendre polynomial functions, i.e,

$$B(\cos\theta) = P_n(\cos\theta), \tag{A10}$$

where

$$P_n(X) = \frac{1}{2^n n!} \left(\frac{\mathrm{d}}{\mathrm{d}X} \right)^n (X^2 - 1)^n.$$

Almost in a similar fashion eq. (A7) is recast as

$$(M^2 - Y^2)\frac{\mathrm{d}^2 A}{\mathrm{d}Y^2} - 2Y\frac{\mathrm{d}A}{\mathrm{d}Y} + n(n+1)A = 0, \tag{A11}$$

where $Y = \mathrm{e}^R + \frac{M^2}{4}\mathrm{e}^{-R}$ and $\lambda = n(n+1)$. We find the series solution of this second order differential equation by substituting

$$A = Y^k \sum_{l=0}^{\infty} a_l Y^l \qquad \text{(Frobenius method)} \tag{A12}$$

The series solution for (A11) is

$$A = a_0 \left[1 - \frac{n(n+1)}{2!}\left(\frac{Y}{M}\right)^2 + \frac{n(n+1)(n-2)(n+3)}{4!}(\frac{Y}{M})^4 + \cdots\right] + a_1 \left[Y - \frac{(n-1)(n+2)}{3!}\frac{Y^3}{M^2} + \cdots\right] \tag{A13}$$

Now after normalization $[S_n(M) = 1]$ we find the generating function for the series

$$S_n(Y) = \frac{1}{(2M)^n n!}\left(\frac{\mathrm{d}}{\mathrm{d}Y}\right)^n (Y^2 - M^2)^n. \tag{A14}$$

This function generates similar polynomial as the Legendre ones except an extra factor of M in the denominator.

$$S_0(Y) = 1, \qquad P_0(X) = 1,$$
$$S_1(Y) = \frac{Y}{M}, \qquad P_1(X) = X,$$
$$S_2(Y) = \frac{1}{2M^2}(3Y^2 - M^2), \qquad P_2(X) = \frac{1}{2}(3X^2 - 1),$$
$$S_3(Y) = \frac{1}{2}\left(5\left[\frac{Y}{M}\right]^3 - 3\frac{Y}{M}\right), \qquad P_3(X) = \frac{1}{2}(5X^3 - 3X).$$

So the solution for the scalar field is

$$\phi = \sigma A(R)B(\theta) = \sigma A\left(\mathrm{e}^R + \frac{M^2}{4}\mathrm{e}^{-R}\right)B(\cos\theta) = \sigma A(Y)B(X) = \sigma S_n(Y)P_n(X) = \sigma S_n(r-m)P_n(\cos\theta). \tag{A15}$$

The simplest choice for ϕ from (A15) would be for $n = 1$ (since for $n = 0$, ϕ would be trivial), i.e,

$$\phi = \sigma S_1(Y)P_1(\cos\theta) = \sigma\left(\mathrm{e}^R + \frac{M^2}{4}\mathrm{e}^{-R}\right)\cos\theta = \sigma(r-m)\cos\theta. \tag{A16}$$

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