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Cloning and superluminal signaling*

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Abstract. It is shown that the no-signaling constraint generates the symmetric as well as the asymmetric $1 \to 2$ optimal universal quantum cloning machine of single qubits.

Keywords. Cloning; cloning fidelity; superluminal signaling; state discrimination.

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1. Introduction

One of the fundamental discoveries of quantum physics is the 'no-cloning theorem', which states that one can not exactly copy an arbitrarily given quantum mechanical state [1]. This fact is very much counter-intutive in contrast with the classical physical ideas, where one can always, in principle, make one (or more than one) copy of an arbitrarily given object. We have at our hand the (classical) xerox machine, which can produce exact copy (or copies) of an arbitrarily given data written on a paper, with the help of blank paper(s). At this position one may raise two *different* points: (1) What is the basic principle that restrains us in producing exact copy (or copies) of an arbitrarily given quantum mechanical state? (2) How much better one can make inexact copy (or copies) of an arbitrarily given quantum mechanical state?

Bužek and Hillery [2] (and then Bruß *et al* [3]) provided the answer to the 2nd question in the case of $1 \to 2$ universal symmetric (as the two copies produced are identical) cloning of qubits. This has also been generalized to produce optimal universal $N \to M$ (where N, M are positive integers and $N < M$) symmetric cloning machine in the case of qubits (i.e., for quantum mechanical states described in d-dimensional Hilbert space) [4]. Recently Cerf [5] has provided a concept of asymmetric cloning when the two output states of the cloner are not identical, but at the same time, these two output states are specifically related to the input state. The cloning operation presented in [5] is universal for qubits, i.e., the

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fidelity of cloning does not depend on the input qubit state. A universal $1 \rightarrow 2$ cloning network, for asymmetric cloning, has been provided by Bužek *et al* [6] using local unitary operations and controlled NOT operations. And the symmetric optimal universal cloning machine of Bužek and Hillery [2] has been reproduced.

Gisin [7] has given the answer to the first question. It can be shown that if exact cloning is possible, we can then have (usable) superluminal signaling [8]. The basic principle behind no exact cloning is the principle of no-signaling (i.e., no superluminal signaling). Allowing no-signaling, Gisin [7] has reproduced the $1 \rightarrow 2$ optimal universal symmetric cloning machine of Bužek and Hillery [2] for qubits. We shall reproduce here the result of Bužek *et al* [6] using this no-signaling condition. And our derivation will show that the universal $1 \rightarrow 2$ asymmetric cloning machine of Bužek *et al* [6] is optimal.

If one is asked to clone an unknown state taken from a *a priori* given set of states, the question of state-dependent cloning arises. If this set consists of only mutually orthogonal states, one can then exactly clone an arbitrarily given state of this set. This is possible because we can always do some projective measurement which enables us to discriminate all of these states with certainty, and once we are able to discriminate all of these states with certainty, we can always clone each of these states exactly (as we then know exactly what are these states). But due to the unitarity of evolutions of quantum mechanical states, one can never clone exactly an arbitrarily taken state from a *a priori* given set of two non-orthogonal states [9]. Further we see that if we can clone an arbitrarily taken state of this set of two non-orthogonal states exactly, we can discriminate these two non-orthogonal states with certainty [10] – an impossible event in quantum mechanics. But one may then ask what is the basic principle that resists us from discriminating with certainty one unknown state from another unknown but non-orthogonal state from a *a priori* given set of two non-orthogonal states? Is it the principle of no-signaling? It is known from the work of Bruß *et al* [3] that the fidelity of optimal state-dependent cloning machine of any set of two non-orthogonal qubit states is always higher than that of the optimal universal cloning machine. So no-signaling constraint *does not* produce the former machine, because of the above-mentioned result of Gisin [7]. We shall show here that if we can discriminate all the states, with certainty, from a given set $S = \{|\psi\rangle, |\phi\rangle, |\overline{\psi}\rangle, |\overline{\phi}\rangle\}$ (where $\langle\psi|\overline{\psi}\rangle = \langle\phi|\overline{\phi}\rangle = 0$ and $\langle\psi|\phi\rangle \neq 0$) of four qubit states, we then have the possibility of superluminal signaling. So optimal cloning of the set S may be compatible with no-signaling condition (as exact cloning of all the states of any given set of states implies discrimination of all these states with certainty, and vice-versa). In other words, it is the mathematical problem of finding the *minimal* set of states, the optimal cloning machine corresponding to which matches with the optimal universal cloning machine.

In §2, we shall show that universal exact cloning *implies* superluminal signaling. We shall also show in this section that if we can discriminate all the states, with certainty, from a given set $S = \{|\psi\rangle, |\phi\rangle, |\overline{\psi}\rangle, |\overline{\phi}\rangle\}$ (where $\langle\psi|\overline{\psi}\rangle = \langle\phi|\overline{\phi}\rangle = 0$ and $\langle\psi|\phi\rangle \neq 0$) of four qubit states, we can then have superluminal signaling, which helps to send some physical information instantaneously. In §3, we shall rederive the universal asymmetric cloning machine (of qubits) of Bužek *et al* [6], and show that their machine is optimal, using no-signaling condition. Section 4 will deal with conclusion and future directions.

2. Exact cloning and superluminal signaling

The possibility of superluminal signaling in quantum mechanics stems from the concept of entanglement between two (or more than two) space-like separated parties, which interacted in the past, but they do not have any interaction at present, or in future. Keeping this in mind, we take here two space-like separated parties A and B, each of which being described in a two dimensional Hilbert space, and their combined state is the entangled state

$$|\Psi\rangle = (1/\sqrt{2})\,(|+\rangle_A \otimes |-\rangle_B - |-\rangle_A \otimes |+\rangle_B)\,, \tag{1}$$

the singlet state, where $|\pm\rangle_j$ ($j = A, B$) is the eigenstate of σ_z corresponding to the eigenvalue ± 1 respectively. Now the party A is allowed to operate σ_z [11]. If the outcome is $|+\rangle_A$, the state of the party B will *instantaneously* become $|-\rangle_B$ (as the joint state of A and B is the singlet, given in eq. (1)). And if the outcome is $|-\rangle_A$, the state of the party B will *instantaneously* become $|+\rangle_B$, due to the same reason. Whatever be the outcome, if we assume that universal exact cloning is possible, we can then clone exactly (and, in principle, infinitely many times) each of the states $|\pm\rangle_B$. And this will enable us to specify with certainty, what the state of the party B will be, if σ_z is operated on the state of the party A. Hence we will have superluminal signaling, which will help us to send the information about the outcome of the measurement on the state of the party A to a space-like separated party B (correlated with A), instantaneously.

What about the converse? Does superluminal signaling imply universal exact cloning? As we have no means to use superluminal signaling, we are unable to answer this. On the other hand, we do not have any contradiction of quantum mechanics with no-signaling. So the better way is to assume no-signaling condition, and see how close to exact universal cloning machine is possible. We shall discuss this in the next section. Before we end this section, we shall discuss some relation between state discrimination with certainty and superluminal signaling.

Let entangled state of the two far apart parties A and B be $|\Psi\rangle$, as given by the eq. (1). Let $|\psi\rangle$, $|\overline{\psi}\rangle$ be the eigenstates of the spin observable σ_ψ corresponding to the eigenvalues $+1$ and -1 respectively. Similarly let $|\phi\rangle$, $|\overline{\phi}\rangle$ be the eigenstates of the spin observable σ_ϕ corresponding to the eigenvalues $+1$ and -1 respectively. Here we assume that the states $|\psi\rangle$ and $|\phi\rangle$ are non-orthogonal. Now the party A is allowed to operate σ_ψ or σ_ϕ on its state. Then the outcomes of these operations will be either ($|\psi\rangle_A$ or $|\overline{\psi}\rangle_A$) or ($|\phi\rangle_A$ or $|\overline{\phi}\rangle_A$) respectively; and depending on these outcomes, the state of the party B will instantaneously become either ($|\overline{\psi}\rangle_B$ or $|\psi\rangle_B$) or ($|\overline{\phi}\rangle_B$ or $|\phi\rangle_B$) respectively. So if the party B can discriminate, with certainty, the states from the set

$$S = \{|\psi\rangle_B, |\overline{\psi}\rangle_B, |\phi\rangle_B, |\overline{\phi}\rangle_B\}, \tag{2}$$

it will then be able to tell (with certainty and instantaneously) what is the measurement that has been done by the party A on its own state. Hence we do have then superluminal signaling. Therefore exact cloning (and hence, discrimination with certainty) of all the states of the set S *implies* superluminal signaling.

3. Optimal asymmetric cloning and no-signaling

Let a_0 correspond to the original single qubit, a_1 correspond to the blank copy (which is also in a single qubit state), and b_1 correspond to the machine of the cloning process. Let $\rho^{\rm in}_{a_0}(\mathbf{m}) = \frac{1}{2}(I + \mathbf{m}.\sigma)$ be the density matrix of the input single qubit state (which is unknown, as the Bloch vector $\mathbf{m} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ is unknown) entering into the asymmetric quantum cloning machine (AQCM). We want to clone (asymmetrically) this qubit universally, i.e., input-state independently (i.e., independent of the Bloch vector $\mathbf{m}$), in such a way that the density matrices of the two clones $\rho^{\rm out}_{a_j}(\mathbf{m})$ ($j = 0, 1$) at the output of the AQCM are of the forms

$$\rho^{\rm out}_{a_j}(\mathbf{m}) = s_j \rho^{\rm in}_{a_j}(\mathbf{m}) + \frac{1 - s_j}{2} I, \tag{3}$$

(for $j = 0, 1$) where I is the 2×2 identity matrix. Equation (3) is referred as the isotropy condition. Obviously here $0 \le s_0, s_1 \le 1$. For symmetric QCM, $s_0 = s_1$. Let $\rho^{\rm out}_{a_0 a_1}(\mathbf{m})$ be the two-qubit output density matrix of the AQCM, obtained after employing the trace operation on the machine states in the output state $\rho^{\rm out}_{a_0 a_1 b_1}(\mathbf{m})$, obtained by applying the asymmetric cloning operation on $\rho^{\rm in}_{a_0}(\mathbf{m})$. In full generality, $\rho^{\rm out}_{a_0 a_1}(\mathbf{m})$ can be written as

$$\begin{aligned} \rho^{\rm out}_{a_0 a_1}(\mathbf{m}) = {} & \frac{1}{4} \left(I \times I + s_0 \mathbf{m}.\sigma \otimes I\right) \\ & + \frac{1}{4} \left(s_1 I \otimes \mathbf{m}.\sigma + \sum_{j,k = x,y,z} t_{jk} \sigma_j \otimes \sigma_k \right). \end{aligned} \tag{4}$$

The AQCM will be universal if it acts similarly on all input states, i.e., if

$$\rho^{\rm out}_{a_0 a_1}(\mathbf{R}(\mathbf{m})) = U(\mathbf{R}) \otimes U(\mathbf{R}) \rho^{\rm out}_{a_0 a_1}(\mathbf{m}) U(\mathbf{R})^\dagger \otimes U(\mathbf{R})^\dagger, \tag{5}$$

where $\mathbf{R} \equiv \mathbf{R}(\mathbf{n}, \alpha)$ represents an arbitrary rotation (in SO(3)) about an axis along the unit vector $\mathbf{n}$ through an angle α of the Bloch vector $\mathbf{m}$, and $U(\mathbf{R}) \equiv {\rm e}^{-i\frac{\alpha}{2}\mathbf{n}.\sigma}$ is the corresponding 2×2 unitary operation (it is in SU (2)) acting on the two-dimensional Hilbert spaces corresponding to the two qubits a_0 and a_1 [12]. As a consequence of this property (given by eq. (5)), we see that (see [7]) $\rho^{\rm out}_{a_0 a_1}(\mathbf{m})$ remains invariant under rotations of $\mathbf{m}$, i.e.,

$$\left[{\rm e}^{i\alpha\mathbf{m}.\sigma} \otimes {\rm e}^{i\alpha\mathbf{m}.\sigma},\ \rho^{\rm out}_{a_0 a_1}(\mathbf{m})\right] = 0 \ \text{ for all real } \alpha. \tag{6}$$

Equation (6) imposes the following conditions on the parameters t_{jk} :

$$\left.\begin{array}{lcl} -m_z t_{xy} + m_y t_{xz} - m_z t_{yx} + m_y t_{zx} & = & 0 \\ m_z t_{xx} - m_x t_{xz} - m_z t_{yy} + m_y t_{zy} & = & 0 \\ -m_y t_{xx} + m_x t_{xy} - m_z t_{yz} + m_y t_{zz} & = & 0 \\ m_z t_{xx} - m_z t_{yy} + m_y t_{yz} - m_x t_{zx} & = & 0 \\ m_z t_{xy} + m_z t_{yx} - m_x t_{yz} - m_x t_{zy} & = & 0 \\ m_z t_{xz} - m_y t_{yx} + m_x t_{yy} - m_x t_{zz} & = & 0 \\ -m_y t_{xx} + m_x t_{yx} - m_z t_{zy} + m_y t_{zz} & = & 0 \\ -m_y t_{xy} + m_x t_{yy} + m_z t_{zx} - m_x t_{zz} & = & 0 \\ -m_y t_{xz} + m_x t_{yz} - m_y t_{zx} + m_x t_{zy} & = & 0 \end{array}\right\}. \tag{7}$$

Particularly, for $\mathbf{m} = (0,0,1) \equiv \uparrow$, we have $t^{\uparrow}_{xx} = t^{\uparrow}_{yy}$, $t^{\uparrow}_{xy} = -t^{\uparrow}_{yx}$ and $t^{\uparrow}_{yz} = t^{\uparrow}_{zy} = t^{\uparrow}_{zx} = t^{\uparrow}_{xz} = 0$. For $\mathbf{m} = (0,0,-1) \equiv \downarrow$, we have $t^{\downarrow}_{xx} = t^{\downarrow}_{yy}$, $t^{\downarrow}_{xy} = -t^{\downarrow}_{yx}$ and $t^{\downarrow}_{yz} = t^{\downarrow}_{zy} = t^{\downarrow}_{zx} = t^{\downarrow}_{xz} = 0$. And for $\mathbf{m} = (1,0,0) \equiv \rightarrow$, we have $t^{\rightarrow}_{yy} = t^{\rightarrow}_{zz}$, $t^{\rightarrow}_{yz} = -t^{\rightarrow}_{zy}$ and $t^{\rightarrow}_{zx} = t^{\rightarrow}_{xz} = t^{\rightarrow}_{xy} = t^{\rightarrow}_{yx} = 0$. And for $\mathbf{m} = (-1,0,0) \equiv \leftarrow$, we have $t^{\leftarrow}_{yy} = t^{\leftarrow}_{zz}$, $t^{\leftarrow}_{yz} = -t^{\leftarrow}_{zy}$ and $t^{\leftarrow}_{zx} = t^{\leftarrow}_{xz} = t^{\leftarrow}_{xy} = t^{\leftarrow}_{yx} = 0$. Our motivation is to find out bounds on s_0 and s_1 (or some sort of relation between them), using the conditions imposed on t_{jk}s by eq. (7), the no-signaling condition (to be described below), and the positive semi-definiteness of the density matrices $\rho^{\text{out}}_{a_0a_1}(\mathbf{m})$ for each Bloch vector $\mathbf{m}$.

If it would have been possible to distinguish between different mixtures that can be prepared at a distance (e.g., between $\rho^{\text{out}}_{a_0a_1}(\mathbf{m}) + \rho^{\text{out}}_{a_0a_1}(-\mathbf{m})$ and $\rho^{\text{out}}_{a_0a_1}(\mathbf{m}') + \rho^{\text{out}}_{a_0a_1}(-\mathbf{m}')$), then non-locality in quantum mechanics could be used for signaling (i.e., superluminal signaling) through that distance, and hence we would reach at a contradiction between quantum mechanics and relativity [13]. Thus we have to maintain no-signality, which imposes that the mixtures $\rho^{\text{out}}_{a_0a_1}(\mathbf{m}) + \rho^{\text{out}}_{a_0a_1}(-\mathbf{m})$ and $\rho^{\text{out}}_{a_0a_1}(\mathbf{m}') + \rho^{\text{out}}_{a_0a_1}(-\mathbf{m}')$ (of the output states), corresponding to the indistinguishable mixtures $\frac{1}{2}(I + \mathbf{m}.\sigma) + \frac{1}{2}(I - \mathbf{m}.\sigma)$ and $\frac{1}{2}(I + \mathbf{m}'.\sigma) + \frac{1}{2}(I - \mathbf{m}'.\sigma)$ respectively (of the input states), are themselves indistinguishable [7]. So, without loss of generality, we can write

$$\rho^{\text{out}}_{a_0a_1}(0,0,1) + \rho^{\text{out}}_{a_0a_1}(0,0,-1) = \rho^{\text{out}}_{a_0a_1}(1,0,0) + \rho^{\text{out}}_{a_0a_1}(-1,0,0). \tag{8}$$

Using (7) and (8), we get the following expression for $\rho^{\text{out}}_{a_0a_1}(\uparrow)$, where $\uparrow = (0,0,1)$:

$$\begin{aligned}\rho^{\text{out}}_{a_0a_1}(\uparrow) = \frac{1}{4}&\left[I \otimes I + s_0\sigma_z \otimes I + s_1 I \otimes \sigma_z\right.\\ &\left.+t\left(\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z\right)\right]\\ &+\frac{1}{4}t_{xy}\left[\sigma_x \otimes \sigma_y - \sigma_y \otimes \sigma_x\right],\end{aligned} \tag{9}$$

where $t = t^{\uparrow}_{xx} = t^{\uparrow}_{yy} = t^{\uparrow}_{zz}$ and $t_{xy} = t^{\uparrow}_{xy}$ are both real quantities. The eigenvalues of $\rho^{\text{out}}_{a_0a_1}(\uparrow)$ are:

$$\frac{1}{4}\left\{1 + t \pm (s_0 + s_1)\right\}, \quad \frac{1}{4}\left[1 - t \pm \left\{4t^2 + 4t^2_{xy} + (s_0 - s_1)^2\right\}^{1/2}\right]. \tag{10}$$

All these eigenvalues must be non-negative, and so we must have

$$s_0 + s_1 \leq 1 + t, \tag{11}$$

$$(s_0 - s_1)^2 + 4t^2_{xy} \leq (1+t)(1-3t), \tag{12}$$

$$-1 \leq t \leq \frac{1}{3}. \tag{13}$$

From eq. (11) we see that maximum values of both s_0 and s_1 will occur when

$$s_0 + s_1 = 1 + t. \tag{14}$$

So, using eq. (14), we get from eq. (12) that

$$s_0^2 + s_1^2 + s_0s_1 - s_0 - s_1 + t^2_{xy} \leq 0. \tag{15}$$

The optimal symmetric cloning machine of Bužek and Hillery [2] (where $s_0 = s_1 = 2/3$) will be reproduced here if we take $t_{xy} = 0$, and then condition (15) is exactly eq. (11) of [6]. Also from eq. (14) we see that the relation (15) has to be satisfied by the reduction factors s_0, s_1 of an optimal AQCM, and this implies that the AQCM of Bužek *et al* [6] is optimal.

4. Conclusion

No-signaling constraint was used by Gisin [7] to rederive the universal optimal symmetric cloning machine of Bužek and Hillery [2], while in the present paper, the same constraint has been used to show the optimality of the universal asymmetric cloning machine of Bužek *et al* [6]. And thus we see that if we have any universal asymmetric cloning machine corresponding to the reduction factors s_0, s_1, where $s_0^2 + s_1^2 + s_0 s_1 - s_0 - s_1 > 0$, certainly then we have superluminal signaling.

Universality, unitarity, linearity, and isotropy have been used for deriving the optimal universal cloning machines, both for symmetric [2,3] as well as aymmetric [6] cases. Whereas we have used universality, isotropy, and no-signality in rederiving these optimal universal cloning machines. So one may enquire into the basic characteristics of no-signality to find out some direct or indirect relation of it with unitarity and linearity.

We have seen earlier that exact cloning of all the states of the set $S = \{|\psi\rangle, |\overline{\psi}\rangle, |\phi\rangle, |\overline{\phi}\rangle\}$ implies superluminal signaling (where $\langle\psi|\phi\rangle \neq 0$). So, using the consequences of no-signality, one may find out the optimal cloning (symmetric and/or asymmetric) machine corresponding to the set S, which (machine) can also be derived from unitarity and linearity of quantum mechanics.

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