PRAMANA — journal of physics
© Indian Academy of Sciences
Vol. 56, Nos 2 & 3
Feb. & Mar. 2001
pp. 161–167

Test of Bell's inequality using the one-atom micromaser

A S MAJUMDAR and N NAYAK
S. N. Bose National Centre for Basic Sciences, Block JD, Sector III, Salt Lake, Calcutta 700 091, India
Email: archan@boson.bose.res.in

Abstract. We examine a local realist bound in the case of a one-atom micromaser. It is shown that such a bound is violated using a simplified treatment of the micromaser. We consider the effect of dissipation in a proposed experiment with the real micromaser. It is seen that the magnitude of violation of a Bell-type inequality depends significantly on the cavity parameters.

Keywords. Bell's inequality; micromaser; dissipation.

PACS Nos 03.65.Bz; 42.50.Dv; 42.52.+x

1. Introduction

It was demonstrated by Bell [1] that realistic interpretations of quantum mechanics cannot be consistent with locality. This, and a large amount of subsequent work [2] has had profound implications on our perspective of the underlying nature of quantum mechanics. Furthermore, Bell's inequalities provided a basis for experimental proposals [3] to test the concept of local realism. The advance of technology and the actual feasibility of several of these proposals have led to a renewed upsurge of interest in this field in recent years.

Schemes for demonstrating violations of Bell's inequalities have so far mostly involved spin-$1/2$ particles [3] or photons [4,5]. Other notable proposals include experiments involving mesons [6]. However, the advantages of using two-level Rydberg atoms for testing Bell's inequalities are being increasingly appreciated [7–9]. In such schemes two-level Rydberg atoms are considered in analogy to the spin system in Bell's original reasoning. The role of the polarization axis of the Stern-Gerlach apparatus used for spin-$1/2$ systems is played here by the phase of an auxiliary electromagnetic field, as we shall see later. The velocity of Rydberg atoms is much smaller compared to the velocity of photons used in, for example, the experiment [4]. Hence, the realization of spacelike separation for a pair of detected atoms is less problematic. In addition, the efficiency of detectors used for Rydberg atoms is considerably higher compared to photodetectors.

In this paper we consider the following experimental scenario. A couple of two-level Rydberg atoms initially in their upper excited state $|e\rangle$ pass through a high-Q micromaser cavity one after the other such that there is no spatial overlap of their wavefunctions. The cavity is in a steady state and tuned to mode resonant with the transition $|e\rangle \rightarrow |g\rangle$. The emerging single-atom wavefunction is a superposition of the upper $|e\rangle$ and lower $|g\rangle$ state.

After passing through the cavity, the atoms pass through an electromagnetic field which gives a $\pi/2$ pulse to the atoms, and the phase of which can be varied for different atoms. Finally, each atom is detected in the upper or lower state by a detector placed far away. Note that there is no direct interaction between the two atoms. However, the passage of the first atom through the cavity modifies the cavity (photonic) wavefunction which is subsequently entangled with the wavefunction of the second atom. In this way secondary correlations [8,9] are built up between the two atomic wavefunctions through the atom-photon interactions. This feature is exploited in obtaining Bell-type inequalities, with the phase of the electromagnetic field acting as the variable Bell parameter. To avoid spatial overlap between the two atoms, the rate of pumping from the source is so monitored that the time during which the cavity is devoid of atoms is much longer compared to the atom-cavity interaction time. Dissipation of the cavity photons is an important consideration in actual experiments, and we derive its effects on the Bell sum.

The paper is organised as follows. In §2 we present the derivation of a Bell-type inequality for the case of an idealized micromaser [8] which is initially in a photon number state, and dissipation is neglected. The purpose of this exercise is to show that Bell's inequalities can be set up and proved to be violated using a simple analytical treatment in the context of a one-atom maser. However, the actual experimental situation is more complicated. To get a better handle on this we discuss the theory of a real micromaser in §3. The effects of decoherence on atomic statistics obtained through numerical analysis is presented in §4 where we also make some concluding remarks about the scope of extension of this work.

2. Violation of Bell's inequality in an ideal micromaser

In this section we shall first briefly go through the derivation of a Bell-type inequality which is suitable in context of the micromaser [7,8]. In order to do so, we shall as usual, assume the twin criteria of locality and reality, wherever required.

As stated earlier, the experiment consists of a source of two-level Rydberg atoms which are pumped randomly one at a time. An atom first passes through a high-Q microwave cavity, and then interacts with an electromagnetic field which acts as a $\pi/2$ pulse on the atom and whose phase is labelled by ϕ. Finally, the atom is detected by a detector which assigns value $+1$ whenever the atom is detected in the upper state $|e\rangle$, and -1 for the lower state $|g\rangle$. In any local realistic theory one can define functions

$$f(\phi_1) = \pm 1;\ g(\phi_2) = \pm 1 \tag{1}$$

describing the outcome of measurement on the atoms 1 and 2 when the phase of the e.m. field is set to be ϕ_1 and ϕ_2 for the respective atoms. The ensemble average for double click events is therefore defined as

$$E^{\lambda}(\phi_1, \phi_2) = \int \mathrm{d}\lambda f(\phi_1) g(\phi_2), \tag{2}$$

where λ is a suitable probability measure on the space of all possible sets.

The quantum mechanical expectation value for double click events is calculated from the probabilities of all possible double-click sequences. This is given by

$$E(\phi_1, \phi_2) = P_{ee}(\phi_1, \phi_2) + P_{gg}(\phi_1, \phi_2) - P_{eg}(\phi_1, \phi_2) - P_{ge}(\phi_1, \phi_2), \tag{3}$$

where $P_{eg}(\phi_1, \phi_2)$ stands for the probability that the first atom is found to be in state $|e\rangle$ after traversing the $\pi/2$ pulse with phase ϕ_1, and the second atom is found to be in state $|g\rangle$ with the phase of the $\pi/2$ pulse being ϕ_2 for its case. Defining $E_0 = E^\lambda(\phi_1 = \phi_2)$ and $M_0 = P_{ee}(\phi_1 = \phi_2) + P_{gg}(\phi_1 = \phi_2)$, and assuming perfect detections, i.e.,

$$E(\phi_1, \phi_2) = P_{ee}(\phi_1, \phi_2) + P_{gg}(\phi_1, \phi_2) + P_{eg}(\phi_1, \phi_2) + P_{ge}(\phi_1, \phi_2) = 1 \tag{4}$$

it follows that

$$E_0 = 2M_0 - 1. \tag{5}$$

Returning to the functions f and g, it is easy to see that $f(\phi) = +g(\phi)$ with probability M_0, and $f(\phi) = -g(\phi)$ with probability $(1 - M_0)$. Hence, $E^\lambda(\phi_1, \phi_2)$ can be written as

$$\begin{aligned} E^\lambda(\phi_1, \phi_2) &= M_0 \int \mathrm{d}\rho f(\phi_1) f(\phi_2) - (1 - M_0) \int \mathrm{d}\rho f(\phi_1) f(\phi_2) \\ &= E_0 \int \mathrm{d}\rho f(\phi_1) f(\phi_2), \end{aligned} \tag{6}$$

where the last equality follows from eq. (5). Further, one has

$$\begin{aligned} E^\lambda(\phi_1, \phi_2) - E^\lambda(\phi_1, \phi_3) &= E_0 \int \mathrm{d}\lambda [f(\phi_1) f(\phi_2) - f(\phi_1) f(\phi_3)] \\ &= E_0 \int \mathrm{d}\lambda f(\phi_1) f(\phi_2)[1 - f(\phi_2) f(\phi_3)]. \end{aligned} \tag{7}$$

Taking the modulus of both sides, one obtains the following Bell-type inequality [8]:

$$|E^\lambda(\phi_1, \phi_2) - E^\lambda(\phi_1, \phi_3)| + \text{ sign}(E_0)[E^\lambda(\phi_2, \phi_3) - E_0] \leq 0. \tag{8}$$

We shall now evaluate the quantum mechanical expectation value for double click events using an idealised micromaser, and compare the Bell sum thus obtained with eq. (8). To begin with, we assume that the cavity is initially in a number state and both the atoms enter the cavity in their upper excited state $|e\rangle$. The initial state of the atom-cavity system is thus written as $|ee, n\rangle$. Dissipation is neglected in this simplified treatment. The interaction of the atoms with the cavity photons is governed by the Jaynes-Cummings interaction [10]. After the pair of atoms have passed both the micromaser and the $\pi/2$ pulse (with different phases ϕ_1 and ϕ_2 for the two atoms), the atom-field state is given by [8]

$$\begin{aligned} |\psi\rangle = {} & c_1^2[|ee, n\rangle + \mathrm{e}^{-i\phi_2}|ge, n\rangle] + \mathrm{e}^{-i\phi_1} c_1^2[|eg, n\rangle + \mathrm{e}^{-i\phi_2}|gg, n\rangle] \\ & + i[\mathrm{e}^{i\phi_1} c_2 s_1 + \mathrm{e}^{i\phi_2} c_1 s_1]|ee, n+1\rangle + i[\mathrm{e}^{i(\phi_1 - \phi_2)} c_2 s_1 - c_1 s_1]|ge, n+1\rangle \\ & + i[\mathrm{e}^{-i(\phi_1 - \phi_2)} c_1 s_1 - c_2 s_1]|eg, n+1\rangle - i[\mathrm{e}^{-i\phi_2} c_2 s_1 + \mathrm{e}^{-i\phi_1} c_1 s_1]|gg, n+1\rangle \\ & + \mathrm{e}^{i\phi_1} s_2 s_1[-\mathrm{e}^{-i\phi_2}|ee, n+2\rangle + |ge, n+2\rangle] \\ & - s_2 s_1[-\mathrm{e}^{i\phi_2}|eg, n+2\rangle + |gg, n+2\rangle], \end{aligned} \tag{9}$$

where $c_j = \cos(\eta\sqrt{n+j})$ and $s_j = \sin(\eta\sqrt{n+j})$, and η is the Rabi angle. The Bell sum B, in this case is given by

$$B = |E(\phi_1, \phi_2) - E(\phi_1, \phi_3)| + \text{sign}(E_0)[E(\phi_2, \phi_2) - E_0]. \tag{10}$$

For a particular choice of the phase angles, e.g., $\phi_1 = 0$, $\phi_2 = \pi/3$, and $\phi_3 = 2\pi/3$, it is easy to verify that

$$S = |\mathcal{A}| \geq 0, \tag{11}$$

where

$$\mathcal{A} = \sin^2(\eta\sqrt{n+1})\cos(\eta\sqrt{n+1})\cos(\eta\sqrt{n+2}). \tag{12}$$

Hence, we see that for this particular choice of the phases ϕ_1, ϕ_2 and ϕ_3, the Bell-type inequality (8) is violated for all feasible choices of the Rabi angle η. Of course, the magnitude of violation does depend on the cavity parameter η. However, in order to describe a real experimental situation, one cannot proceed with the above idealised treatment for the micromaser. The corresponding equations are rather complicated for the case of the real micromaser where dissipation of the cavity photons has to be taken into account. One has to take recourse to numerical methods to deal with this case, as we shall see in the following sections.

3. Micromaser dynamics

The micromaser [11], or the one-atom maser, consists of a superconducting microwave high-Q cavity at sub-kelvin temperatures. Rydberg atoms in the upper of two masing levels, the transition frequency of which is in resonance with the single eigenmode of the cavity, are pumped into the cavity at such a rate that at most one atom is present there at any time. The very first atom that enters the cavity interacts with the thermal radiation field having average photon number $\bar{n}_{th}$ and in equilibrium with the cavity temperature. The atom interacts with the field for a duration τ, fixed for every atom. The radiation field, after this interaction, evolves under its own dynamics until the second atom enters the cavity, which interacts with the resultant field in the cavity. The process repeats itself after every $\bar{t}_c$ seconds, where $\bar{t}_c = 1/R$, with R being the number of atoms passing through the cavity per second. Thus one has a repetition time $t_c = \tau + t_{\text{cav}}$ where t_{cav} is the duration for which the cavity is empty of atom. $\bar{t}_c$ is the average of t_c with respect to the distribution in time of the incoming atoms, which for the purpose of the present study, we consider to be Poissonian, such as in the experimental set-up [11]. This dynamics and its steady-state characteristics have been discussed in ref. [12–14]. The above mechanism shows that the cavity radiation field is in interaction with its reservoir all the time whereas the interacting atom is coupled to its reservoir during time τ only. Naturally, the dynamics carries the signature of these two reservoirs to the steady-state situation of the cavity which has been discussed in ref. [14]. We adopt the results of this paper for the present purpose.

The Hamiltonian for the interaction between the single mode of the radiation field and the single atom is

$$H_{\text{int}} = g(s^+ a + s^- a^\dagger), \tag{13}$$

where g is the atom-cavity coupling constant, $a(a^\dagger)$ is the annihilation (creation) operator for the radiation field. s^+ and s^- are the Pauli pseudo-spin operators for the two-level atoms with $[s^+, s^-] = s^z$, where s^z is the population operator. The equations of motion for the composite atom-field density matrix ρ due to the interaction in eq. (13) is

$$\dot{\rho} = -i[H_{\rm int}, \rho]. \tag{14}$$

The interactions of the atoms and the radiation field with their respective reservoirs are considered to be Markovian, and the equation governing these interactions can be seen in ref. [15]. The derivations leading to the steady-state results are given in ref. [14].

The steady-state photon statistics of the cavity field is

$$P_n = P_0 \prod_{m=1}^{n} v_m \tag{15}$$

and P_0 is obtained from the normalisation $\sum_{n=0}^{\infty} P_n = 1$. The v_n is given by the continued fractions

$$v_n = f_3^{(n)}/(f_2^{(n)} + f_1^{(n)} v_{n+1}) \tag{16}$$

with $f_1^{(n)} = (Z_n + C_n)/\kappa$, $f_2^{(n)} = -2N + (Y_n + B_n)/\kappa$ and $f_3^{(n)} = -(X_n + A_n)/\kappa$. κ is the cavity bandwidth and $N = R/2\kappa$ is the number of atoms passing through the cavity in a photon lifetime. $A_n = 2n\kappa\bar{n}_{th}$, $B_n = -2\kappa(n + \bar{n}_{th} + 2n\bar{n}_{th})$ and $C_n = 2(n+1)(\bar{n}_{th}+1)\kappa$. X_n, Y_n and Z_n are given by

$$X_n = R\sin^2(g\sqrt{n}\tau)\exp\{-[\gamma + (2n-1)\kappa]\tau\}, \tag{17}$$

$$\begin{aligned} Y_n = \frac{1}{2}R\Bigg(&\Bigg\{2\cos^2[g\sqrt{n+1}\tau] - \frac{1}{2}(\gamma/\kappa + 2n + 1) \\ &+F_1(n-1)\Bigg\}\exp\{-[\gamma + (2n+1)\kappa]\tau\} \\ &+\left[\frac{1}{2}(\gamma/\kappa + 2n + 1) - F_2(n-1)\right]\exp\{-[\gamma + (2n-1)\kappa]\tau\}\Bigg), \end{aligned}$$

and

$$\begin{aligned} Z_n = \frac{1}{2}R\Bigg(&\left[\frac{1}{2}(\gamma/\kappa + 2n + 3) + F_2(n)\right]\exp\{-[\gamma + (2n+1)\kappa]\tau\} \\ &-\left[\frac{1}{2}(\gamma/\kappa + 2n + 3) + F_1(n)\right]\exp\{-[\gamma + (2n+3)\kappa]\tau\}\Bigg) \end{aligned}$$

where γ represents reservoir induced spontaneous emission from the upper to the lower masing level. The functions F_i are

$$\begin{aligned} F_i(n) = \frac{\kappa/4g}{(\sqrt{n+2} - \sqrt{n+1})^2}\Bigg[&\frac{\gamma}{\kappa}(\sqrt{n+2} - \sqrt{n+1})\sin(2g\sqrt{m}\tau) \\ &-\frac{\gamma}{g}\cos(2g\sqrt{m}\tau) - [2n + 3 + 2\sqrt{(n+1)(n+2)}] \end{aligned}$$

$$\times(\sqrt{n+2}-\sqrt{n+1})\sin(2g\sqrt{m}\tau)\Bigg] + \frac{\kappa/4g}{(\sqrt{n+2}+\sqrt{n+1})^2}$$
$$\times\Bigg[\pm\frac{\gamma}{\kappa}(\sqrt{n+2}+\sqrt{n+1})\sin(2g\sqrt{m}\tau) - \frac{\gamma}{g}\cos(2g\sqrt{m}\tau)$$
$$\mp[2n+3-2\sqrt{(n+1)(n+2)}](\sqrt{n+2}+\sqrt{n+1})\sin(2g\sqrt{m}\tau)\Bigg], \qquad (18)$$

where $m = n+2$ and $n+1$ for $i = 1$ and 2, respectively, with the upper sign for $i = 1$.

4. Decoherence effects on the Bell sum and directions of further study

We evaluate the Bell sum given in eq. (10) numerically. We start with the steady-state photon statistics of the cavity given by eq. (15). This is the field that the first of the two experimental atoms encounter. The second atom interacts with the field modified due to

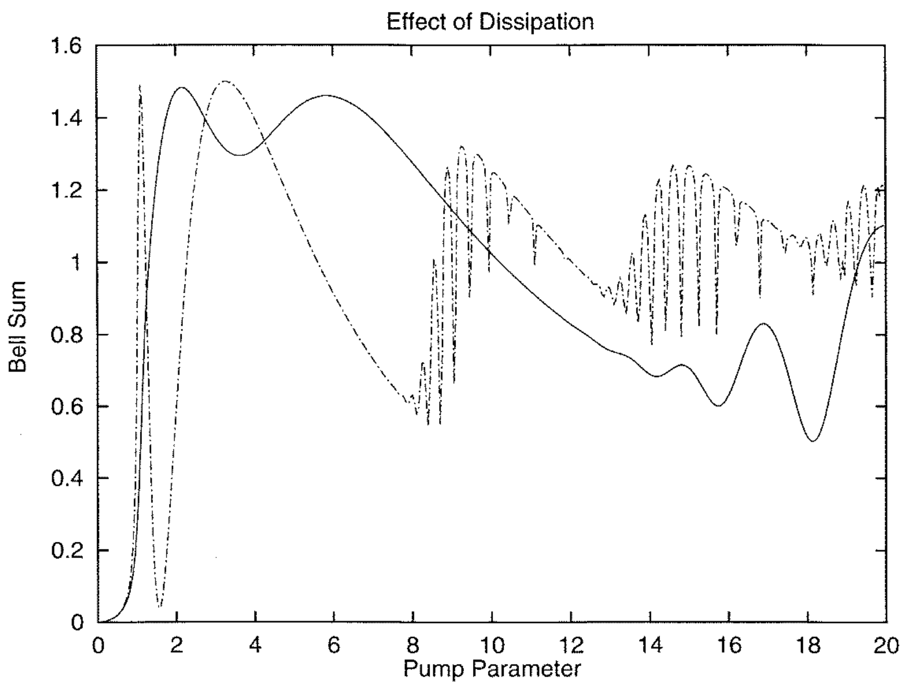

Figure 1. Plot of the Bell sum versus the pump parameter (atom-cavity interaction time) for two different values of the cavity band-width κ. The dotted curve corresponds to a higher value of κ. Dissipation brings down the magnitude of violation of Bell's inequality, except for very small interaction times. (In our units, the value of pump parameter ~ 14 corresponds to the cavity Rabi angle $\pi/2$).

the first atom. This is how the secondary correlations between pairs of experimental atoms are built up. We set the phase of the electromagnetic field to be the same as in the case of the idealised micromaser in §2 (i.e., $\phi_1 = 0, \phi_2 = \pi/3, \phi_3 = 2\pi/3$). The Bell sum is calculated for different values of the cavity pump parameter. The magnitude of violation of Bell's inequality is plotted versus the atom-cavity interaction time for two different values of the cavity bandwidth κ in figure 1. We clearly see that dissipation reduces the magnitude of violation of the Bell's inequality. However, one notices a little deviation from this for low interaction times. The reason for this is being investigated and will be reported in a future publication with other details.

In this paper we have reported the preliminary results of our investigation of violation of Bell's inequality using a one-atom maser. We find a general violation of Bell's inequality, the magnitude of which depends on the cavity parameters. However, these results are subject to the assumption we have made that the cavity field does not dissipate significantly during the period between the departure of the first atom and the arrival of the second atom. But this dissipation, however small, can be significant in the micromaser dynamics [14]. We plan to include this effect in our study. In addition, between two detector clicks, atoms may go unnoticed, which in fact, is an experimental reality. We plan to incorporate this fact in our investigation, as well. Finally, the micromaser might be useful for realization of several variants of Bell's inequality, for instance, inequalities involving more than two particles.

References

[1] J S Bell, *Physics* **1**, 195 (1964)

[2] See, for example, *Quantum mechanics versus local realism* edited by F Selleri (Plenum, New York, 1988)

[3] M Lamehi-Rachti and W Mittig, *Phys. Rev.* **D14**, 2543 (1976)
J F Clauser and A Shimony, *Rep. Prog. Phys.* **41**, 1881 (1978)
Z Y Ou and L Mandel, *Phys. Rev. Lett.* **61**, 50 (1988)
T E Kiess, Y H Shih, A V Sergienko and C O Alley, *Phys. Rev. Lett.* **71**, 3893 (1993)

[4] A Aspect, P Grangier and G Roger, *Phys. Rev. Lett.* **49**, 91 (1982)
A Aspect, J Dalibard and G Roger, *Phys. Rev. Lett.* **49**, 1804 (1982)

[5] P D Drummond, *Phys. Rev. Lett.* **50**, 1407 (1983)
P G Kwiat *et al*, *Phys. Rev. Lett.* **75**, 4337 (1995)

[6] A Datta and D Home, *Found. Phys. Lett.* **4**, 165 (1991)
R A Bartlmann and W Grimus, *Phys. Lett.* **B392**, 426 (1997)

[7] S J D Phoenix and S M Barnett, *J. Mod. Opt.* **40**, 979 (1993)
M Freyberger *et al*, *Phys. Rev.* **A53**, 1232 (1996)
E Hagley *et al*, *Phys. Rev. Lett.* **79**, 1 (1997)

[8] M Loffler, B-G Englert and H Walther, *Appl. Phys.* **B63**, 511 (1996)

[9] A Jabs, quant-ph/9811042

[10] E T Jaynes and F W Cummings, *Proc. IEEE* **51**, 89 (1963)

[11] G Rampe, F Schmidt-Kaler and H Walther, *Phys. Rev. Lett.* **64**, 2783 (1990)

[12] P Filipowicz, J Javanainen and P Meystre, *Phys. Rev.* **A34**, 3077 (1986)

[13] L A Lugiato, M O Scully and H Walther, *Phys. Rev.* **A36**, 740 (1987)

[14] N Nayak, *Opt. Commun.* **118**, 114 (1995)

[15] See, for example, W H Louisell, *Quantum statistical properties of radiation* (Wiley, New York, 1973)