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Magnetization curves for non-elliptic cylindrical samples in a transverse field

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Abstract. Using recent results for the surface current density on cylindrical surfaces of arbitrary cross-section producing uniform interior magnetic field and an assumed set of flux-fronts, solutions of Bean's critical state model for cylindrical samples with *non-elliptic* cross-section are presented. Magnetization hysteresis loops for two cross-sections with different aspect ratios are obtained. A comparison with some exact results shows the limitations of this approach.

Keywords. Magnetization curves; Bean's model; critical state model.

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1. Introduction

The critical state model (CSM) involving just one parameter, the critical current density, J_c, was proposed by Bean [1] to describe the irreversible magnetization of hard type-II superconductors. The early solutions pertained to samples with zero demagnetization factor, viz., an infinite slab and an infinite cylinder in parallel geometry with constant J_c. The CSM has been generalized to include field dependent and more recently also the history dependent J_c with samples mostly in parallel geometry. Initial efforts to solve the CSM for samples with nonzero demagnetization factor could provide only approximate solutions. Some of the more recent efforts have mainly dealt with thin samples – discs and strips – with applied field perpendicular to thicker dimension [2,3]. Clem *et al* [4] have considered the case of field dependent J_c for thin samples and more recently an integral equation approach has been followed by Shantsev *et al* [5]. Bhagwat and Chaddah (BC) [6] have solved a re-statement of CSM for finite samples. Their results agree with those of the above cited references in the appropriate limit [7]. The solutions presented by BC are mostly analytical in nature. However, their solutions derived for samples such as elliptical cylinder and an ellipsoid are valid only for those shapes; there appears no clear way of generalizing to other shapes. In this paper we present a method that addresses to this difficulty. The problem of CSM in transverse cylindrical geometry is essentially a 2-dimensional one. To obtain the virgin curve of a zero field cooled sample one needs to determine the flux contours for different values of the applied field with the condition that the local field $\mathbf{B} \equiv 0$ within the innermost flux contour and the current density $J = J_c$ in the current carrying region. Recently we obtained an expression for surface current density on cylinders of arbitrary cross-section producing uniform interior field [8] using the method of conformal mapping in conjunction with some results from the theory of singular integral equations with a Cauchy type kernel. We believe that these results are directly applicable to the solution of CSM and the same method can be applied to study the evolution of flux contours in arbitrary cylindrical samples. The simplest of such procedures that can achieve $\mathbf{B} \equiv 0$ involves the assumption of parallel flux contours. Here we follow this procedure and present results for magnetization curves for hard superconductor cylindrical samples with *non-elliptical* cross-sections. The method is equally well applicable also to elliptical cylinders as a special case.

In the next section we present analytical formulation that relates a given volume current density to a cascade of surface currents. Flux contours are viewed as evolving from the one representing the sample surface. All these belong to one parameter family. The natural choice of the parameter would be B_a, the magnitude of applied magnetic field. But any function of it would serve equally well. We denote the parameter by ξ, such that $0 \leq \xi \leq 1$. Further, without loss of generality, we may assume that $\xi = 1$ gives the contour representing the boundary of the cylinder. Starting from the surface contour the various flux contours can be represented by their respective conformal map which maps the exterior of it to the exterior of an unit circle [9]. We then work within the approximation of parallel flux contours and present our results. We end with a discussion and possible generalization of the method adopted in this paper.

2. Analytical formulation

Consider the conformal map [9]

$$\zeta = f(\xi, u) = \xi[u - pu^{-1} - (q/3)u^{-3}]. \tag{1}$$

When $u = \exp(\iota\phi)$ describing a unit circle, we get a parametric representation of the flux contour. The parameters p and q are given by $p = a_1 + a_2$ and $q = a_1 a_2$. If we choose the numbers a_1 and a_2 such that $|a_1| < 1$, $|a_2| < 1$, the mapping (1) has the property

$$f'(u) = \xi(1 + a_1 u^{-2})(1 + a_2 u^{-2}) \neq 0 \tag{2}$$

for $|u| > 1$. Hence the mapping is *conformal* there, i.e., it maps the exterior of the unit circle to the exterior of the flux contour in the z-plane. As the parameter ξ varies from 1 to 0, we get the parallel flux-fronts, provided the parameters p and q in eq. (1) are independent of ξ. Let the current carrying region be bounded by sample surface $\xi = 1$ and a flux contour corresponding to $\xi = \xi_0$. This region can be divided into shells of thickness $\mathrm{d}\xi$, each shell viewed as a surface current density [8]:

$$\mu_0 J_s = 2\delta B_0 \xi \sin\phi / |f'(u)| \tag{3}$$

will produce a uniform field δB_0 in its interior. Such an arrangement of shells is thus capable of cancelling an uniform applied field.

To see how this can be achieved let us consider the field generated by a volume current distribution. Let the zero field cooled infinite cylindrical sample with its axis along the

z-axis and its cross-section bounded by a contour $\mathbf{L}$ be subjected to a uniform magnetic field. The sample, in response, will carry a (volume) current density $\mathbf{J}(x', y')$ parallel to the axis of the cylinder. If $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ denote unit vectors along the co-ordinate axes then the field $\mathbf{B_J}$ generated by $\mathbf{J}$ is given by [10]

$$\mathbf{B_J} = \frac{\mu_0}{2\pi} \int\int \frac{J(x', y')[-(y-y')\hat{\mathbf{i}} + (x-x')\hat{\mathbf{j}}]}{(x-x')^2 + (y-y')^2} \mathrm{d}x' \mathrm{d}y'. \tag{4}$$

The integration extends over the current carrying region. Introducing a complex function $B_J(\zeta)$, with $\zeta = x + \iota y, \zeta' = x' + \iota y'$, [10] we may write

$$B_J(\zeta) = \frac{\mu_0}{2\pi} \int\int \frac{J(x', y')}{\zeta - \zeta'} \mathrm{d}x' \mathrm{d}y'. \tag{5}$$

The components of $\mathbf{B_J}$ are obtained as $\mathbf{B_J}_x = \mathrm{Im} B_J(\zeta)$ and $\mathbf{B_J}_y = \mathrm{Re} B_J(\zeta)$. Effecting a change of variables from ζ' to $(\xi', u' = \exp(\iota\phi'))$ by writing $\zeta' = f(\xi', u')$, where f is the function defined in eq. (1), the integral in (5) can be expressed in the form

$$B(\zeta) = \frac{\mu_0}{2\pi} \int \mathrm{d}\xi' \int \frac{J(x', y')}{(\zeta - \zeta')} \frac{\chi(\xi', \phi')}{|f'(\xi', u')|} \mathrm{d}s'. \tag{6}$$

Here $\chi(\xi', \phi') \equiv x'_{\xi'} y'_{\phi'} - x'_{\phi'} y'_{\xi'}$ is the Jacobian of transformation and $\mathrm{d}s' = |\mathrm{d}\zeta'| = |f'(\xi', u')| \mathrm{d}\phi'$. The net $B(\zeta)$ can be explicitly seen as produced by a succession of current shells of thickness $\mathrm{d}\xi'$. Each current shell carries a surface current density $\mu_0 J_s = \mu_0 J\chi / |f'(\xi', u')| \mathrm{d}\xi'$. Thus the identification

$$\mu_0 J\chi \mathrm{d}\xi / |f'(u)| = 2\delta B_0 \xi \sin\phi / |f'(u)|$$

will enable us to get a volume current density that produces uniform interior field that is equal and opposite to the external field B_a. We choose δB_0 so that $J = J_c$, the critical current density, on the equator (corresponding to $\phi = \pi/2$). Hence

$$J = J_c \chi(\xi, \pi/2) \sin\phi / \chi(\xi, \phi). \tag{7}$$

Thus, to get $|J| = J_c$ we must have

$$\chi(\xi, \phi) = \chi(\xi, \pi/2) |\sin\phi|.$$

Various approximations are obtained by equating finite terms in a suitable expansion of both sides. In the present paper, however, we follow a very simple approximation by assuming parallel flux-contours.

3. The solution

With the current density given by eq. (7) we compute the field B_J which it generates within the flux-front $\xi = \xi_0$ (c.f. (6)). Since the net field in the interior of the flux-front is zero, we have the following relation between the applied field B_a and ξ_0 from

$$B_a = -B_0 = H^*(1 - p + q)(\xi_0 - 1)/4 \tag{8}$$

with

$$H^* = \mu_0 J_c a = \mu_0 J_c (1 + p - q/3). \tag{9}$$

We denote by a and b the sample dimensions at the equator and along the field direction. Virgin magnetization (magnetic moment/unit volume) of a zero field cooled sample can be obtained from the relation

$$\mathbf{m}_v = (1/2A) \int\int (\mathbf{r}' \times \mathbf{J}) \mathrm{d}x' \mathrm{d}y', \tag{10}$$

where A is the cross-sectional area of the sample and $\mathbf{r}' = \hat{\mathbf{i}}x' + \hat{\mathbf{j}}y' + \hat{\mathbf{k}}z'$. Changing the variables of integration to (ξ', ϕ') and noting that the current density is along $\hat{\mathbf{k}}$ we get

$$\mathbf{m}_v = (J_c b/2A) \int_{\xi 0}^{1} \int_{-\pi}^{\pi} \chi(\xi', \pi/2) \sin\phi' (\hat{\mathbf{i}}y' - \hat{\mathbf{j}}x') \mathrm{d}\xi' \mathrm{d}\phi'. \tag{11}$$

We have used $\mathrm{d}x'\mathrm{d}y' = \chi(\xi', u)\mathrm{d}\xi'\mathrm{d}\phi'$. The y-component of magnetization is zero by symmetry and the x-component (denoted by m_v itself) is given by

$$m_v = H^*(1 - p + q)(1 + p)(\xi_0^3 - 1)/[12(1 - p^2 - q^2/3)]. \tag{12}$$

Here we have used that $A = \pi(1 - p^2 - q^2/3)$. In the field reversal case (before full penetration) the decrease in the field changes the current density in a shell $(\xi_0', 1)$ from J to $-J$, leaving the current density in the region $\xi_0 < \xi < \xi_0'$ unchanged. The magnetization during the field reversal can be calculated and we get

$$m_\downarrow = -\mu_0 m_{\mathrm{sat}} (\xi_0^3 - 2\xi_0'^3 + 1) \tag{13}$$

and

$$B_a = -B_p(\xi_0 - 2\xi_0' + 1), \tag{14}$$

where m_{sat} and B_p are saturation magnetization and the applied field when the sample is fully penetrated i.e. when $\xi_0 = 0$ in the expression (12) and (8). Now when the field is increased beyond B_p then $B_a = B_p + B(0)$ where $B(0)$ is the field at the centre of the sample. Now having reached a sufficiently large field $B_{\max}$, let the field be reduced. By a similar procedure as for magnetization under the field reversal case, we get the field

$$B_{\mathrm{in}} = B(0) + B_p(2\xi_0' - 1) \tag{15}$$

and

$$B_A = -B_{\mathrm{in}}, \quad \mu_0 m = -\mu_0 m_{\mathrm{sat}} (1 - 2\xi'^3). \tag{16}$$

4. Results and discussion

We now present the results for two samples with different aspect ratios. The cross-section and the virgin, and, small and large hysteresis curves for sample 1 (with aspect ratio $= a/b = 6.38$) are shown in figure 1. For this sample the field for full penetration $B_p = 0.105H^*$ and the saturation magnetization $\mu_0 m_{\rm sat} = -0.1178H^*$. The corresponding exact values are respectively $0.5528H^*$ and $-0.3790H^*$. As for sample 2, the field for full penetration $B_p = 0.002484H^*$ and the saturation magnetization $\mu_0 m_{\rm sat} = -0.00955H^*$. The corresponding exact values are respectively 0.02968 H^* and $-0.4306H^*$. Thus $m_{\rm sat}/m_{\rm sat}^{\rm exact} \approx 0.3108$ for sample 1 and ≈ 0.2218 for sample 2. The same ratio for a circular cylinder can be obtained from ref. [11] and has the value ≈ 0.7854. The above comparison shows the limitations of the parallel flux-contour approximation. The detailed magnetization curves for sample 2 $(a/b = 239.7)$ are shown in figure 2. The magnetization curves in the two figures are very similar. However, change of scale on the x-axis should be noted corresponding to the large demagnetization factor for the sample 2.

In this paper we have illustrated how the formula for surface current density can be used to solve the CSM for non-elliptic cylindrical samples. We have noted that the flux contours can be described by their respective conformal maps that transform the exterior of a flux contour to the exterior of a unit circle. For simplicity, we assumed a simple form for the coefficients of the conformal mapping which results in parallel flux-contours. It should be noted that the conformal map describing the sample boundary may involve only a few coefficients (3 in the examples considered), the one describing the flux contours will, in general, require infinite number of them. Only those describing the sample boundary evolve with nonzero initial values and the rest all evolve from the value zero. The functional form of the coefficients is to be determined such that $J = J_c$ in the entire current

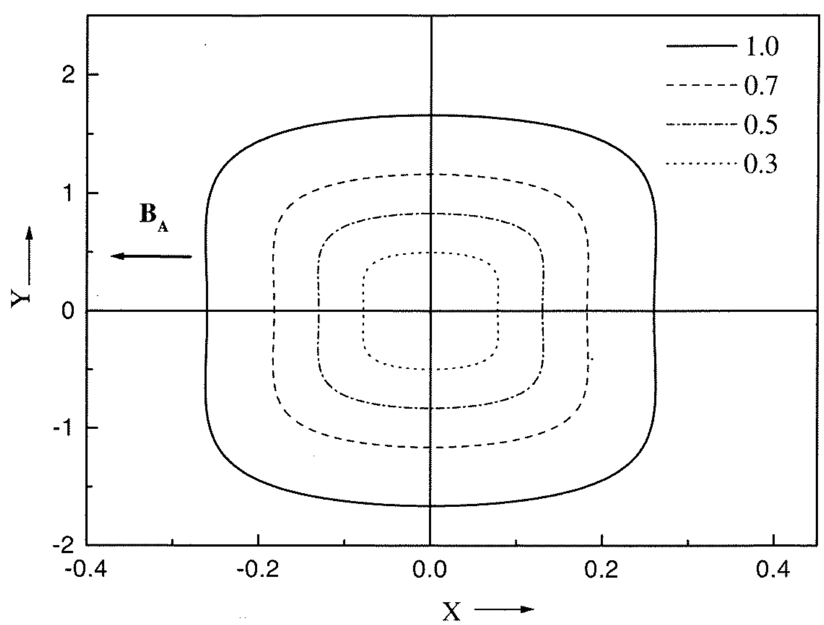

Figure 1a. Cross-section and flux fronts for sample 1 described by the equation $z = \xi_0[u - (p/u) - q/(3u^3)]$ for a set of values of $\xi_0 \cdot p = a + b, q = ab, a = 0.3$ and $b = 0.4$ are parallel and transverse sample dimensions.

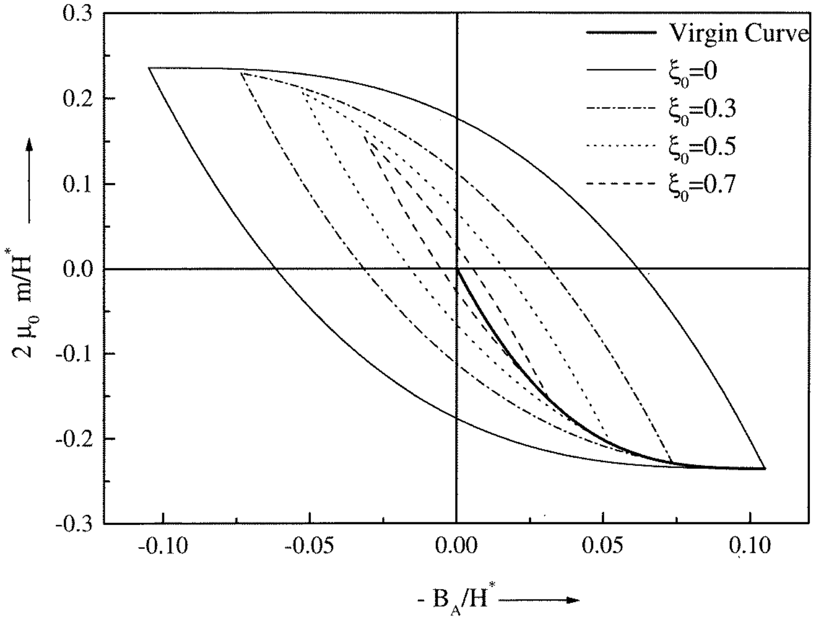

Figure 1b. Small magnetization hysteresis loops for sample 1.

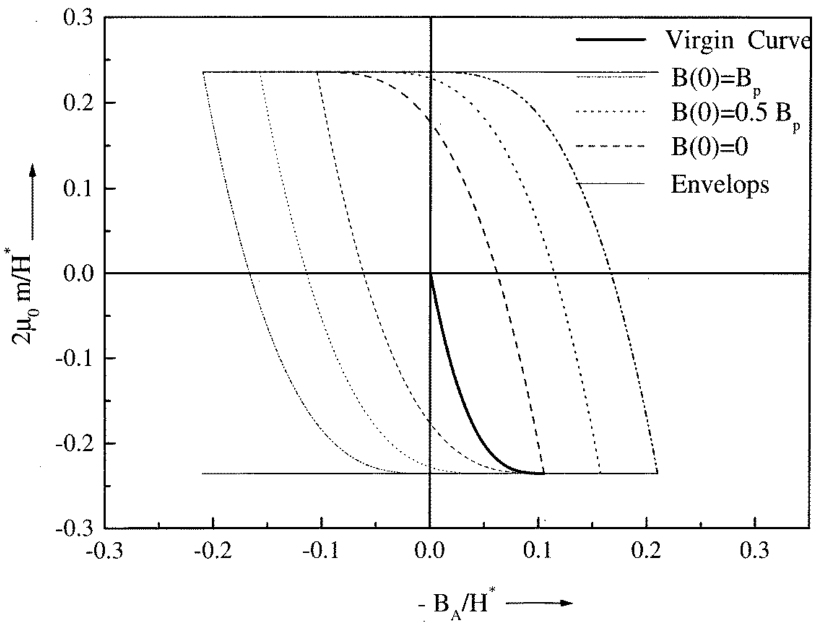

Figure 1c. Large hysteresis loops (reversal field larger than the field for full penetration B_p) for sample 1. $B(0)$ represents the field at the centre of the sample.

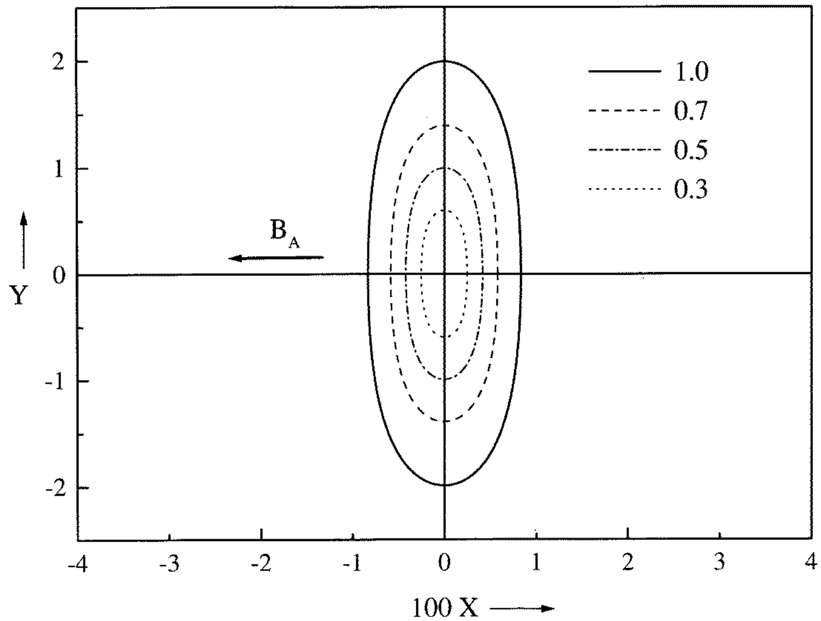

Figure 2a. Same as figure 1a, with $a = 0.99$ and $b = 0.0012589$ to give large demagnetization factor.

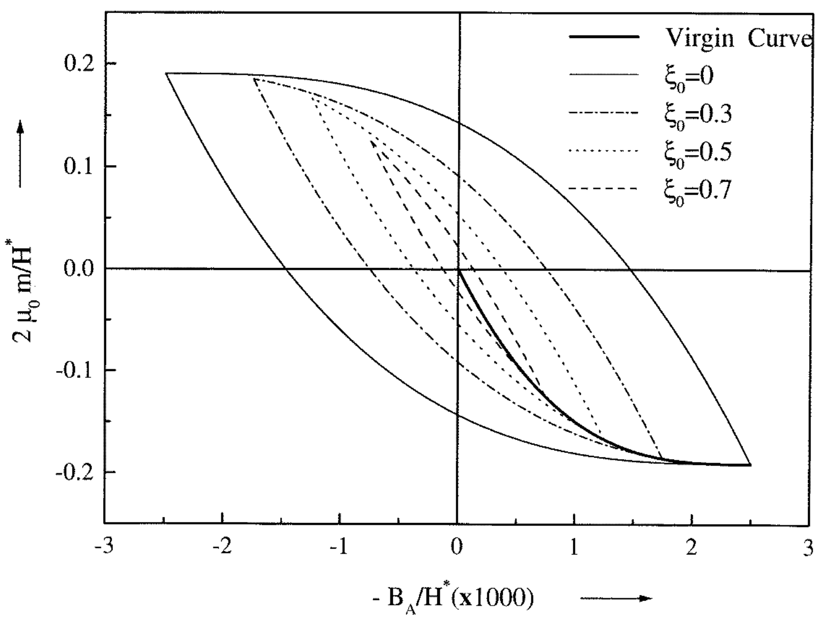

Figure 2b. Same as figure 1b, for sample 2.

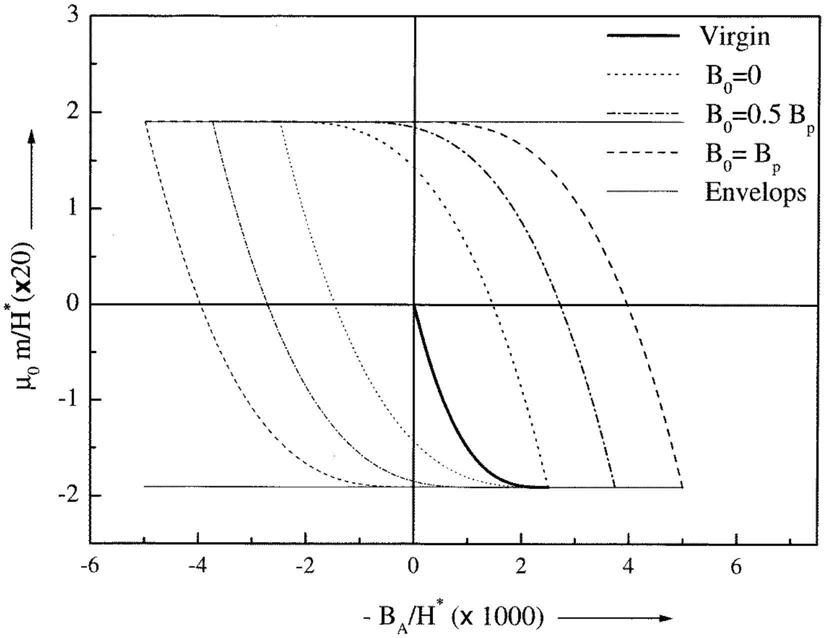

Figure 2c. Same as figure 1c, for sample 2.

carrying region. The rigorous procedure requires one to solve an infinite system of nonlinear first order ordinary differential equations. In practice the infinite system must be truncated allowing evolution of only a finite number of coefficients, the rest retaining their initial values. This corresponds to getting $J = J_c$ up to a certain order in powers of $\cos^2\phi$ (c.f. (7)).

A conformal mapping involving only the first two terms of eq. (1) describes an ellipse. If we generate a conformal map by allowing evolution of only these two coefficients for describing the flux contours we get the analytical results of Bhagwat and Chaddah [6] for cylindrical samples involving nonparallel flux contours. A similar procedure can be followed for the present samples by allowing evolution of all the three coefficients. Even for this limited generalization an analytical solution of the resulting differential equations is not possible. We do not present the details but only mention the results. If we require that $J \approx J_c$, up to terms of order $\cos^4\phi$ we need to retain three leading coefficients. These evolve as a function of ξ starting from their initial values leading to nonparallel flux contours. For this case we get $B_p = 0.2500H^*$ and $\mu_0 m_{\rm sat} = -0.2208H^*$ for sample 1 and $B_p = 0.0105H^*$ and $\mu_0 m_{\rm sat} = -0.2121H^*$. The values of $m_{\rm sat}/m_{\rm sat}^{\rm exact}$, for samples 1, 2 and a circular cylinder respectively turn out to be ≈ 0.5826, 0.4926 and 0.9425. Thus the three term approximation is not as good for general samples as for a circular cylinder. It is necessary to consider more coefficients in the conformal mapping describing the flux contours to make J closer to J_c. The results of such a calculation predicts values of B_p and $m_{\rm sat}$ close to the exact ones. In fact it slightly overestimates $m_{\rm sat}$. This aspect is being looked into. The details of the calculation will be published elsewhere.

In summary, we have generated solutions of the CSM that satisfy the requirement that

$\mathbf{B} \equiv 0$ in the current-free region of the sample, using parallel flux-contours. However, in this case $J \neq J_c$ in the current carrying region. We have presented results for magnetization curves for non-elliptical cylindrical samples. We have compared the results for the field for full penetration and saturation magnetization with the exact results. This comparison clearly brings out the limitations of this approach.

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