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Minimum dissipative relaxed states in toroidal plasmas

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Abstract. Relaxation of toroidal discharges is described by the principle of minimum energy dissipation together with the constraint of conserved global helicity. The resulting Euler-Lagrange equation is solved in toroidal coordinates for an axisymmetric torus by expressing the solutions in terms of Chandrasekhar-Kendall (C-K) eigenfunctions analytically continued in the complex domain. The C-K eigenfunctions are obtained as hypergeometric functions that are solutions of scalar Helmholtz equation in toroidal coordinates in the large aspect-ratio approximation. Equilibria are constructed by assuming the current to vanish at the edge of plasma. For the $m = 0, n = 0$ (m and n are the poloidal and toroidal mode numbers respectively) relaxed states, the magnetic field, current, q (safety factor) and pressure profiles are calculated for a given value of aspect-ratio of the torus and for different values of the eigenvalue λr_0. The new feature of the present model is that solutions allow for both tokamak as well as RFP-like behaviour with increase in the values of λr_0, which is related directly to volt-sec in the experiment.

Keywords. Minimum dissipation; tokamak; reversed field pinch.

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1. Introduction

The relaxation model proposed by Taylor [1] is well-suited to characterize the reversed field pinch (RFP) as a self-organized state, but its application to tokamak discharges was beset with difficulties. Bhattacharjee and Kwok [2] tried to overcome these by formulating additional global invariants and constructed tokamak equilibria with zero current at the boundary which are stable to ideal and resistive modes. More recently, Kucinski and Okano [3], proposed a theory of minimum entropy production associated with minimum turbulence and tried to explain the growth of poloidal current in tokamaks by considering anisotropic resistivity. However, it is surmised by many workers [4–8] that RFP and tokamak configurations can be produced in the same toroidal discharge with the same procedure, but with few significant differences in the operation parameters. Yoshida, Uchida and Inoue [7] have attempted to explain the experimentally observed self-organised equilibrium in RFP and tokamak by a deterministic approach to incompressible dissipative magnetohydrodynamics. In an earlier work Kondoh [8] formulated an energy principle including the edge plasma effects for a slightly resistive MHD plasma and demonstrated the existence of equilibria with monotonically increasing as well decreasing pitch profiles.

In this work we propose to demonstrate that some important features of tokamak, as well as RFP discharges can indeed be obtained if these are perceived as relaxed states with minimum dissipation. The principle of minimum dissipation was first applied to the relaxation of magnetized plasma by Montgomery and Phillips [9] to understand the steady state profiles of RFP configuration under the constraint of a constant rate of supply and dissipation of helicity.

2. Euler-Lagrange equation

We consider a closed system of an incompressible, resistive magnetofluid, without any mean flow velocity, described by the standard MHD equations in presence of a small but finite resistivity η. In the absence of any externally imposed electric fields, the ohmic dissipation rate R is itself a time varying quantity. However, it can be shown [10] that helicity defined by $\int \mathbf{A}\cdot\mathbf{B}\,\mathrm{d}V$ still serves to hold as a good constraint as it decays at a time scale much slower in comparison to the decay time scale of the rate of energy dissipation. We therefore minimize the ohmic dissipation $R = \int \eta \mathbf{j}^2 \mathrm{d}V$ subject to the constraints of helicity. This leads to the following Euler-Lagrange equation that can be described as the minimum dissipation constant helicity equation [10]

$$\nabla \times \nabla \times \nabla \times \mathbf{B} = \Lambda \mathbf{B}, \tag{1}$$

where Λ is Lagrange's undetermined multiplier.

We like to emphasize that the minimum dissipation equation embraces the force-free Taylor equation as a special case. We can generate the solution of eq. (1) by extrapolating the corresponding Taylor's solutions in the complex plane and by expressing $\mathbf{B}$ as

$$\mathbf{B} = \sum_{j=0}^{2} A_j \mathbf{B}_j, \tag{2}$$

where A_j's represent constants and $\mathbf{B}_j$'s satisfy the force-free equation with complex eigenvalues

$$\nabla \times \mathbf{B}_j = \lambda \omega^j \mathbf{B}_j, \quad j = 0, 1, 2, \quad \text{and} \quad \Lambda = \lambda^3, \tag{3}$$

where ω is the complex cube root of unity.

3. Solutions of minimum dissipation equation in toroidal coordinates

In this work, we propose to obtain the solutions of the minimum dissipation equation in a torus as most plasma configurations of experimental interest are toroidal. In a torus, it is preferable to use toroidal coordinate system. The toroidal coordinates (η, ψ, ϕ) and the corresponding metric coefficients [11] are described by the following

$$x = a_0 \sinh\eta \cos\phi, \quad y = a_0 \sinh\eta \sin\phi, \quad z = a_0 \sin\psi,$$

$$h_1 = h_2 = a_0, \quad h_3 = a_0 \sinh\eta,$$

where $a_0 = a/(\cosh\eta - \cos\psi)$. The length a is related to the major and minor radius of the torus by $a^2 = R^2 - r^2$ and the aspect-ratio is given by $s = \cosh\eta = R/r$.

As the toroidally non-axisymmetric states are not so evident in experiments, we concentrate mainly on axisymmetric or ϕ independent states. The solutions of Taylor's equation in an axisymmetric torus are obtained through a representation of $\mathbf{B}_j$ in terms of the Chandrasekhar-Kendal eigenfunctions in toroidal coordinates. In a toroidally axisymmetric case, $\mathbf{B}_j$ can be represented as

$$\mathbf{B_j} = \nabla \times (\mathbf{e}_\phi \chi_j) + \frac{1}{\lambda\omega^j} \nabla \times \nabla \times (\mathbf{e}_\phi \chi_j), \quad j = 0, 1, 2. \tag{4}$$

It can be shown that the above representation satisfies eq. (3) if $\chi_j(\eta, \psi)$ satisfies the following equation

$$\frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial \eta} \left(h_3 \frac{\partial \chi_j}{\partial \eta} \right) + \frac{\partial}{\partial \psi} \left(h_3 \frac{\partial \chi_j}{\partial \psi} \right) - \frac{h_1 h_2}{h_3} \chi_j \right] + \lambda\omega^j)^2 \chi_j = 0. \tag{5}$$

This leads to a simplified form of $\mathbf{B}_j$ given by

$$\mathbf{B}_j = \nabla \times (\mathbf{e}_\phi \chi_j(\eta, \psi)) + \lambda\omega^j \chi_j(\eta, \psi) \mathbf{e}_\phi. \tag{6}$$

The following transformations are utilized to rewrite eq. (5) in terms of Π_j

$$\chi_j(\eta, \psi) = \sqrt{\cosh\eta - \cos\psi}\ \Pi_j, \quad s = \cosh\eta, \ \ \tau = \cos\psi,$$

$$\frac{\mathrm{d}}{\mathrm{d}s} \left[(s^2 - 1) \frac{\partial \Pi_j}{\partial s} \right] + \frac{\partial^2 \Pi_j}{\partial \psi^2} + \left[\frac{1}{4} - \frac{1}{s^2 - 1} \right] \Pi_j + \frac{(\lambda\omega^j)^2 a^2 \Pi_j}{(s - \tau)^2} = 0. \tag{7}$$

The above equation is not in a separable form with respect to s and ψ coordinates. An expansion of the term $(1 - \cos\psi/s)^{-2}$ in powers of $\cos\psi/s$ facilitates the separability of the equation when Π_j is expanded in a perturbative series

$$(s - \tau)^{-2} = \frac{1}{s^2} \left[1 + \frac{2\cos\psi}{s} + \frac{3\cos^2\psi}{s^2} \right], \tag{8}$$

$$\Pi_j = \Pi_{j0} + \Pi_{j1} + \Pi_{j2} \cdots. \tag{9}$$

In the expansion occurring in eq. (9), we have assumed that the successive terms decrease by powers of $1/s$. In the lowest-order in inverse-aspect-ratio, Π_{j0}, can be shown [12] to obey a hypergeometric equation. The lowest-order solutions of eq. (5), valid in the region within the torus $s_0 \le s < \infty$ (where $s = s_0$ defines the surface of the torus) are given by

$$\chi_j(\eta, \psi) = \sum_{n=0}^{\infty} A_n s^{-(n+1)} (s^2 - 1)^{1/2} {}_2F_1(\alpha_j, \beta_j, \gamma_j, 1/s^2) \cos n\psi, \tag{10}$$

where

$$\alpha_j = \frac{1}{2}[n + 2 + \sqrt{1/4 + (\lambda\omega^j)^2 a^2}],$$
$$\beta_j = \frac{1}{2}[n + 2 - \sqrt{1/4 + (\lambda\omega^j)^2 a^2}], \tag{11}$$

$$\gamma_j = n + 1.$$

With this χ_j and using eqs (2) and (6), the different magnetic field components that are solutions of eq. (1) in the lowest order of inverse-aspect-ratio can be obtained.

$$B_\phi = \lambda \sum_{j=0}^{2} \sum_{n=0}^{\infty} A_{nj} \omega^j s^{-n+1} \sqrt{s^2 - 1}\, {}_2F_1 \left(\alpha_j, \beta_j, \gamma_j, \frac{1}{s^2} \right) \cos n\psi, \quad (12)$$

$$B_\psi = \frac{1}{as} \sum_{j=0}^{2} \sum_{n=0}^{\infty} A_{nj} s^{-n} \left[(n(s^2 - 1) - 2)\, {}_2F_1 \left(\alpha_j, \beta_j, \gamma_j, \frac{1}{s^2} \right) + 2 \frac{(s^2 - 1)}{s^2} \frac{\alpha_j \beta_j}{\gamma_j} {}_2F_1 \left(\alpha_j + 1, \beta_j + 1, \gamma_j + 1, \frac{1}{s^2} \right) \right] \cos n\psi, \quad (13)$$

$$B_\eta = - \sum_{j=0}^{2} \sum_{n=0}^{\infty} A_{nj} s^{-n} (s^2 - 1)^{1/2}\, {}_2F_1 \left(\alpha_j, \beta_j, \gamma_j, \frac{1}{s^2} \right) n \sin n\psi. \quad (14)$$

As total current $\mathbf{J} \neq 0$ at the plasma edge gives rise to instabilities [2], so as our boundary condition we choose $\mathbf{J} = 0$ at the edge described by $r = r_0$. The ratio A_1/A_0, representing the ratio of the coefficients of the non-Taylor to Taylor components of $\mathbf{B}$ in a given poloidal state is obtained [10] from these boundary conditions.

4. Results

In the poloidally symmetric state defined by $n = 0$, χ_j are independent of ψ and the boundary condition is satisfied trivially. To get the eigenvalues and the eigensolutions satisfying the given boundary condition, $n = 1$ state is considered and the eigenvalues $\lambda_c r_0$ are observed to decrease with increase of aspect ratio [13]. For values of $\lambda < \lambda_c$, a continuous spectrum in λ is allowed and the symmetric state is the lowest energy dissipation rate state. The poloidal volt-sec/toroidal flux associated with the toroidal discharge can be directly interpreted in terms of the parameter λr_0. The safety factor, $q(r)$, that is an important parameter for understanding the stability of a discharge is defined as

$$q(r) = \frac{r\, B_\phi}{R\, B_\psi}. \quad (15)$$

Numerical results are obtained from the solutions of the minimum dissipation equation described by eqs (12)–(14) with $n = 0$ by scanning the values of λr_0 while keeping the aspect ratio fixed at 4.0. For this value of aspect ratio, the critical eigenvalue $\lambda_c r_0$ as seen turns out to be 4.6. The $m = 0, n = 0$ state therefore is the minimum dissipation rate state for values λr_0 up to 4.6.

In figures (1) and (2), the q as well as the corresponding toroidal magnetic field (normalized) profiles are shown for $\lambda r_0 = 1.5$ and $\lambda r_0 = 3.5$ respectively. For $\lambda r_0 = 1.5$, the q-values increase monotonically with r/r_0 which is the essential feature of a tokamak discharge. The toroidal magnetic field (B_ϕ) profile shows almost a constant behaviour. As

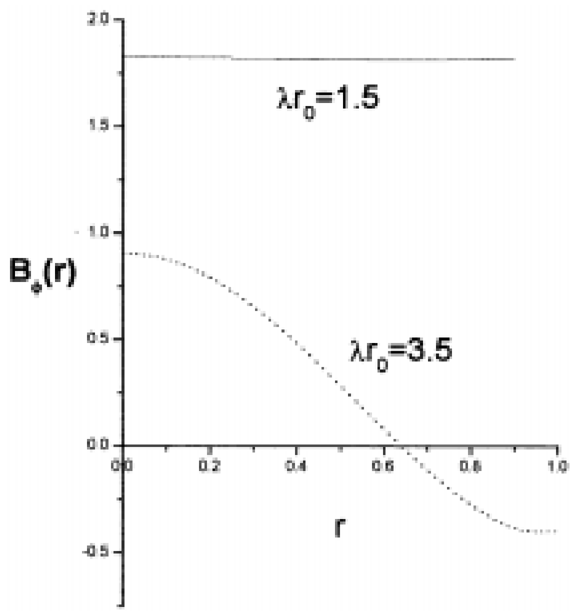

Figure 1. Figure showing the B_ϕ profiles for a torus with aspect ratio $= 4.0$ in the $m = n = 0$ state.

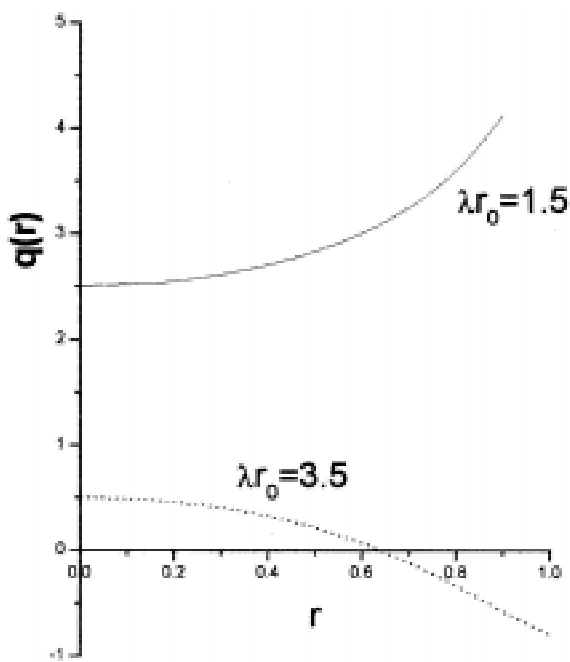

Figure 2. q-profiles showing characteristic tokamak ($\lambda r_0 = 1.5$) and RFP ($\lambda r_0 = 3.5$) features.

λr_0 is increased to around 3.5, the toroidal magnetic field reverses at the edge and the q-profile also shows a similar behaviour. For intermediate values of λr_0, q profiles with on-monotonic behaviour are obtained. As is evident from eq. (1), the solutions of minimum dissipation equation support a pressure profile unlike the Taylor equation. This work establishes that the minimum dissipation constant helicity equation has solutions that exhibit distinctly different q-profiles that are typical of tokamaks as well as reversed field pinches for different values of operational parameters.

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