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Lie-optic matrix algorithm for computer simulation of paraxial self-focusing in a plasma

D SUBBARAO, R UMA, KAMAL GOYAL*, SANJEEV GOYAL** and KARUNA BATRA***

Center for Energy Studies, Indian Institute of Technology, New Delhi 110 016, India
*Presently with Tata Consultancy Services, Gurgaon, India
**Presently with the Ideal Institute of Technology, Ghaziabad, India
***Presently with the Center for Research in Cognitive, Systems, NIIT, IIT Delhi Campus, India

Abstract. Propagation algorithm for computer simulation of stationary paraxial self-focusing laser beam in a medium with saturating nonlinearity is given in Lie-optic form. Accordingly, a very natural piece-wise continuous Lie transformation that reduces to a restricted Lorentz group of the beam results. It gives rise to a matrix method for self-focusing beam propagation that is constructed and implemented. Although the results use plasma nonlinearities of saturable type, and a gaussian initial beam, these results are applicable for other media like linear optical fibers and to more general situations.

Keywords. Self-focusing; Lie-optics; laser beam propagation; computer simulation in optics; laser-plasma interaction.

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1. Introduction

The problem of self-focusing is important in high power laser-plasma interactions and needs to be modeled in a number of situations arising in this context. For arbitrary type of nonlinearities and in non-planar geometries the only method is to use semi-analytical methods to simulate the beam propagation on a computer. We wish to report a new method for this kind of beam self-focusing predictions. We describe matrix formulation of the technique and implement it based on the paraxial self-focusing of a gaussian beam formulated analytically elsewhere [1,2].

Although laser-matter interaction has taken large strides in technological applications, stationary self-focusing has remained an open subject for about 40 years [1,2] giving rise to a number of techniques to deal with it of which Lie group methods naturally arise. Notable in this context are the works of Gagnon and Winternitz [3] who performed the symmetry group analysis of the nonlinear Schrödinger equation (NLSE) in cylindrical geometry for cubic and quintic nonlinearities and gave some useful solutions. Recently Kovalev *et al* [4] tackled the cubic NLSE in cylindrical geometry by renormalization group techniques and described the evolution of an initially gaussian beam. A whole range of Lie-optic methods exists for linear optics [5,6] which have yet to be adopted for self-focusing and this paper attempts to do just that in the paraxial limit. Lie-optic methods have an in-built economy in analysis and computation and have found a natural way into computational algorithms for self-focusing even in the non-paraxial regions that has been dealt by us elsewhere [7].

Unlike the methodology of this paper and other computational schemes [7] which can deal with saturating nonlinearity, group-theoretical methods usually deal with the cubic or at most quintic nonlinearities [3,4]. An important feature of nonlinearity saturation of the refractive index with beam intensity in the NLSE for self-focusing is that beam intensity cannot shoot to infinite values because of catastrophic focusing at the self-focusing singularity because of the presence of nonlinearity saturation. Since all naturally occurring nonlinearities saturate, the existence of intensity singularities is an academic discussion arising out of mathematical curiosity more relevant to its fundamental nature in the cubic nonlinearity case. It is not encountered in the present paper since the refractive index is of the saturating type in a plasma.

Computer simulation techniques of self-focusing either in this paper or otherwise [7] involve beam representation in orthogonal spaces. The paraxial technique of this paper uses the relatively refined beam modal expansion in terms of the Laguerre-Gauss modes [1] suitable for a focusing medium like the one at hand. The problem of self-focusing being nonlinear in character, the implementation of the method is in a piece-wise linear form and the most natural manner to do it is to construct a matrix algorithm and we construct and implement that in this paper. It has been found by us through experience [1,8–10] that a better choice of the modes of the self-focusing beam gives a better choice of the momentum space for the beam which in turn lead to better results for self-trapping and self-focusing. This is because, the laser beam perceives the refractive index power series expansion in the paraxial approximation differently from the laboratory experience of the same even for a linear inhomogeneous medium and definitely for a nonlinear self-focusing medium like a plasma.

After introducing the notation and basic formulae sourced from ref. [1] in §2, we give the matrix method in §3. A lot of care is necessary in fixing the initial values of the harmonic oscillator strength (the parameter b^2) that determine the modes that are to be used, the potential that approximates self-focusing being of the harmonic oscillator type. This is discussed in §4 before giving the results in §5. Remarks on other non-paraxial beam representation methods are given in the final section.

2. Paraxial theory of self-focusing using the harmonic oscillator basis

In the problem of steady-state self-focusing, the plasma response is supposed to be so fast that it adjusts to the presence of the high power laser beam instantaneously. Polarization is neglected because the inhomogeneities present in the ambient plasma or those caused by the presence of the laser beam are both over several (large enough in number, more than ~ 10) wavelengths of the laser in scale-length. The modeling equation is the scalar wave-equation that reduces to the nonlinear Schrödinger equation (after suppressing the term $\sim \exp ikz$):

$$-2ik\frac{\partial E}{\partial z} + \frac{\partial^2 E}{\partial r^2} + \frac{1}{r}\frac{\partial E}{\partial r} + k_0^2 \varepsilon(EE^*)E = 0, \tag{1}$$

where $k = \varepsilon_{00} k_0$ ($k_0 = (\omega/c)$), the linear ambient wave-number of the laser in the plasma and the arbitrary steady-state nonlinearity is assumed to be expandable in a Taylor series in some neighborhood of the argument, EE^* so that,

$$\varepsilon(EE^*) = \varepsilon_{00}[1 - \Phi(EE^*)] \quad ; \quad \Phi(EE^*) = \sum_0^\infty a_n (EE^*)^n. \tag{2}$$

The basic assumption in the paraxial theory is that the refractive index can be expanded up to the terms in r^2 by a suitable method (not the obvious Taylor expansion) into the harmonic oscillator potential form:

$$\varepsilon(EE^*) = k^2\,(a^2 - b^4 r^2) = k^2(\varepsilon_0 - \varepsilon_2 r^2), \tag{2a}$$

where the coefficients a^2 ($= \varepsilon_0$) and b^2 ($= \sqrt{\varepsilon_2}$) are determined appropriately through eq. (6) below. The representation for the field and the refractive index are both accomplished in terms of the appropriate modes. The field representation will be:

$$E(r,z) = \sum_0^\infty E_{0m} |\psi\rangle_m \mathrm{e}^{i\beta_0(z)} \mathrm{e}^{im\gamma(z)}, \tag{3}$$

where the appropriate eigenmodes suitable for cylindrical geometry are the Laguerre-Gauss modes: $|\psi\rangle_m = L_m(b^2 r_0^2) \exp(-b^2 r_0^2)$. The coefficients E_{0m} for an initially gaussian profiled beam with waist width r_0, $E(r,0) = E_0 \exp(-r^2/r_0^2)$ will be, $E_{0m} = b^2 r_1^2 (1 - b^2 r_1^2)^m$ where r_1^2 is defined below in eq. (5). The field of the laser beam at any distance z will then be given by the self-similar gaussian,

$$E(r,z) = \frac{E_0}{q}\,\mathrm{e}^{-\frac{r^2}{r_0^2 g}} \quad ; \quad \bar{g} = \frac{1}{2}\,b^2 r_0^2 g = \frac{1 - z_{10}\mathrm{e}^{i\gamma}}{1 + z_{10}\mathrm{e}^{i\gamma}} \;;\; q = \mathrm{e}^{-i\beta_0} \frac{t}{b^2 r_1^2}, \tag{4}$$

where

$$\beta_0 = -\int^z (a^2 + b^2)\mathrm{d}z \quad ; \quad \frac{\mathrm{d}\gamma}{\mathrm{d}z} = -\frac{2b^2}{\varepsilon_0} \quad ; \quad \varepsilon_0 = (1 - (a^2 + b^2)), \tag{5}$$

$$z_{10} = (1 - b^2 r_1^2) \;;\; \frac{1}{b^2 r_1^2} = \frac{1}{2} + \frac{1}{b^2 r_0^2} \;;\; t = (1 - z_{10}\mathrm{e}^{i\gamma}) \;;$$
$$\frac{1}{p_n} = \frac{1}{2} + \frac{(2n+1)\bar{g}_r - i\bar{g}_i}{|\bar{g}|^2}.$$

Note that these formulae are in their reduced form derived from those of ref. [1] under the approximations of slow focusing. The coefficients a^2 and b^2 of the paraxial dielectric constant of eq. (2a) are given by the summation formulae (that were stated in integral form for slow focusing in ref. [1]):

$$a^2 = \frac{1}{2t}(S_1 + S_2) \quad ; \quad b^2 = -\frac{1}{t^2} S_2 \quad ; \quad S_1 = \sum_0^\infty a_n \Gamma_n \quad ; \quad S_2 = \sum_0^\infty a_n \Delta_n, \tag{6}$$

where

$$\Gamma_n = \left(\frac{E_0}{|q|}\right)^{2n} p_n \quad ; \quad \Delta_n = -\Gamma_n \frac{(1-t)}{tz_{10}}[(1-\mathrm{e}^{-i\gamma}) - (b^2 r_1^2 - p_n \mathrm{e}^{-i\gamma})].$$

The suitableness of these formulae depends on the convergence of the summations in the above formulae and will be discussed in §4.1. The nonlinearity chosen for this particular paper for demonstration is the ponderomotive nonlinearity that arises because of radiation pressure on the plasma that digs a hole in the plasma by plasma redistribution giving lower plasma number densities at the laser beam axis where there is more intensity of light. This in steady state is the well-known expression [1],

$$\varepsilon(EE^*) = 1 - \frac{\omega_p^2}{\omega^2}\mathrm{e}^{-EE^*}, \;\; EE^* = \frac{\xi\xi^*}{\xi_0\xi_0^*},$$
$$\xi_0\xi_0^* = \frac{8m\omega^2 k_B T}{e^2} \Rightarrow a_n = \frac{(-1)^n}{n!}\frac{\omega_p^2}{\omega^2\sqrt{\epsilon_{00}}}, \tag{7}$$

where the electric field of the laser has been normalized throughout as indicated by the characteristic field, ξ_0 of the ponderomotive nonlinearity. The coefficients a_n chosen for the ponderomotive nonlinearity occur in all the summation formulae above starting with eq. (2). Here ω_p is the plasma frequency with the usual meanings for all other parameters in it.

3. The matrix method for self-focusing

The starting point of the matrix formulation of the complex beam-width parameter, g is eq. (4) that gives it in terms of the parameter γ that in turn is to be evaluated by quadrature from eq. (5) in terms of a Runge-Kutta or a finite difference method. Choosing the latter method, the equation for the complex beam width and the field can be put formally into a Lie-group form, essentially in terms of the rotation or the Lorentz group.

Firstly note that eq. (4) can be rewritten in the form [1],

$$\bar{g} = \frac{\bar{g}_0 \cos\left(\int^z \sqrt{\frac{\varepsilon_2}{\varepsilon_0}}\mathrm{d}z\right) - i\sin\left(\int^z \sqrt{\frac{\varepsilon_2}{\varepsilon_0}}\mathrm{d}z\right)}{-\bar{g}_0 i \sin\left(\int^z \sqrt{\frac{\varepsilon_2}{\varepsilon_0}}\mathrm{d}z\right) + \cos\left(\int^z \sqrt{\frac{\varepsilon_2}{\varepsilon_0}}\mathrm{d}z\right)}. \tag{8}$$

Accordingly, the transformation operator for the electric field of the laser beam in a non-absorbing medium is the bilinear transformation $T_R : \bar{g}(z+\mathrm{d}z) = T_R(\Delta z)\bar{g}(z), E[\bar{g}(z+\mathrm{d}z)] = T_R(\Delta z)E[\bar{g}(z)]$ where the four elements of this bilinear (Mobius) transformation are related one to one to the four elements of the rotation (matrix) operator, $\hat{T}_R$,

$$\hat{T_R}(z) \equiv \mathrm{e}^{-i\sigma_x \int_0^z \surd(\frac{\varepsilon_2}{\varepsilon_0})\mathrm{d}z} \tag{9a}$$

where $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is one of Pauli's spin matrices. This operator can analytically be continued into the complex ε plane when the Mobius operator $T_L(z)$ naturally generalizes to correspond to the 2×2 Lorentz (matrix) operator, $\hat{T_L}(z)$,

$$\hat{T_L}(z) \equiv T_R T_l \equiv \mathrm{e}^{-i\sigma_x \int_0^z \mathrm{Real}(\surd(\frac{\varepsilon_2}{\varepsilon_0})\mathrm{d}z)} \cdot \mathrm{e}^{\sigma_x \int_0^z \mathrm{Imag}(\surd(\frac{\varepsilon_2}{\varepsilon_0})\mathrm{d}z)}. \tag{9b}$$

Here T_l is the Lorentz pure boost transformation and the factoring of the operator is natural.

Note that these are finite z operators although for computational convenience in the matrix method and because of the inherent nonlinearity in the refractive index of the self-focusing medium, these operators are to be further factored into a series of operators each operating for a small distance, Δz by piece-wise continuous approximation. From the fact that only one of Pauli's matrix is involved which commutes with itself (repeatedly) we can immediately write down the ordered operator product:

$$\begin{aligned}\hat{T}(z) &\equiv \hat{T_n}(\Delta z)\hat{T_{n-1}}(\Delta z)\ldots\hat{T_3}(\Delta z)\hat{T_2}(\Delta z)\hat{T_1}(\Delta z)\\ &\equiv e^{-i\sigma_x\surd(\frac{\varepsilon_2}{\varepsilon_0})_n\Delta z}\cdot \mathrm{e}^{-i\sigma_x\surd(\frac{\varepsilon_2}{\varepsilon_0})_{n-1}\Delta z}\ldots \mathrm{e}^{-i\sigma_x\surd(\frac{\varepsilon_2}{\varepsilon_0})_3\Delta z}\\ &\quad\cdot \mathrm{e}^{-i\sigma_x\surd(\frac{\varepsilon_2}{\varepsilon_0})_2\Delta z}\cdot \mathrm{e}^{-i\sigma_x\surd(\frac{\varepsilon_2}{\varepsilon_0})_1\Delta z}.\end{aligned} \tag{10}$$

This serves as a convenient algorithm for beam propagation and is exploited in the matrix method. We remark that having established the commutative and other properties of the operators for small slices of the plasma of thickness Δz each, it is advisable to work directly with eq. (8) for computational purposes. Corresponding to a slice of thickness Δz, the incremental value of the propagation parameter γ is $\Delta\gamma$. Hence each of the above matrices can be put into the original form of eq. (8) and one can now write it in the form:

$$\bar{g}_{i+1} = \frac{(1-\frac{(\Delta\gamma)^2}{2})\bar{g}_i - i\Delta\gamma}{-i\Delta\gamma\bar{g}_i + (1-\frac{(\Delta\gamma)^2}{2})} \tag{11}$$

which is what is actually used in our programming below.

4. Initial conditions

The Laguerre-Gauss modal expansion depends on the assumption of a basic transverse scale-length in the transverse direction, b^{-1} that determines the choice of the harmonic oscillator basis. (This in turn is the choice of the harmonic oscillator strength.) In the present theory this is chosen in a piece-wise continuous manner i.e. local values are used. One, therefore needs to determine the initial value of this parameter b given the initial power in the laser beam at $z = 0$ to proceed further.

For this purpose we write down the equation determining b^2, eq. (6), in the following form for the specific saturable plasma nonlinearity (ponderomotive nonlinearity) chosen by us:

$$\left(\frac{\omega_p r_0}{\sqrt{\epsilon_{00}}c}\right)^2 = -b^2 r_0^2 (b^2 r_1^2)^2 \frac{(\sqrt{\omega_p/\omega\epsilon_{00}})}{S_2}. \tag{12}$$

One attempt at graphically obtaining solutions for this equations is shown in figure 1. Here for various values of $b^2 r_0^2$ as parameters, the beam-width normalized to the skin-depth, $(\omega_p/\omega)k_0 r_0$ is plotted against the normalized intensity of the beam, E_0^2.

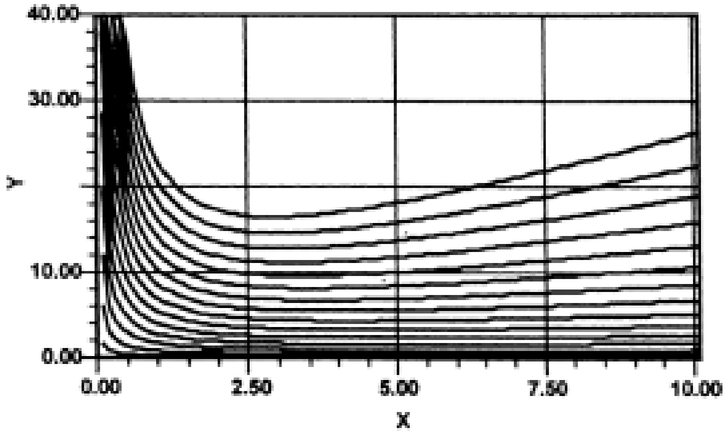

Figure 1. Plot of dimensionless initial laser beam-width, r_0 (normalized by the penetration skin depth, ω_p/c), with respect to the dimensionless laser beam intensity (normalized to the characteristic beam intensity for ponderomotive nonlinearity of the plasma, $I_0 = cE_0^2/8\pi = cm\omega^2 k_B T/\pi e^2$ (see text for meaning of symbols). The sixth curve from the top is for the value of $b^2 r_0^2 = 2$. The curves above it are for $b^2 r_0^2 > 2.0$ (from 2.2 to 3.0 in intervals of 0.2) and those below for $b^2 r_0^2 < 2$ (from 0.2 to 1.8 in intervals of 0.2 again).

The plot for $b^2 r_0^2 = 2$, corresponds to the case of the initial beam-width choice, appropriate for the normalized self-trapping beam intensity E_0^2, is shown amongst the plots for various other choices of this crucial parameter $b^2 r_0^2$. The saturation of the normalized beam width with higher normalized intensity shows that it is to be expected at high intensities for most beams to have almost identical filament width. This graph is used by us to choose a value of $(\omega_p/\omega)k_0 r_0$ for a given choice of the normalized intensity and the harmonic oscillator parameter $b^2 r_0^2$.

An equally useful graph is to replot figure 1 showing the variation of the harmonic oscillator strength, $b^2 r_0^2$ with the normalized intensity of the laser beam for a specified value of the normalized beam-width, $(\omega_p/\omega)k_0 r_0$. Figure 2 is one of such graphs for a specific value of $(\omega_p/\omega)k_0 r_0 = 10$ using MACSYMA. This graph shows how the harmonic oscillator strength peaks for a particular intensity of the laser beam indicating strongest self-focusing at that intensity. This graph in turn indicates the variation of the binding energy per photon in the self-focusing beam with the beam intensity, a measure of the attraction of photons by photons.

4.1 *Convergence of the series*

This graph of figure 2 is very sensitive to the convergence of the summations that occur in the evaluations of S_1, S_2. We find that [11] taking *odd* number of terms up to about $n = 17$ stabilizes the convergence process for the ponderomotive nonlinearity. Lesser number of terms and taking *even* number of terms can lead to very misleading results [11].

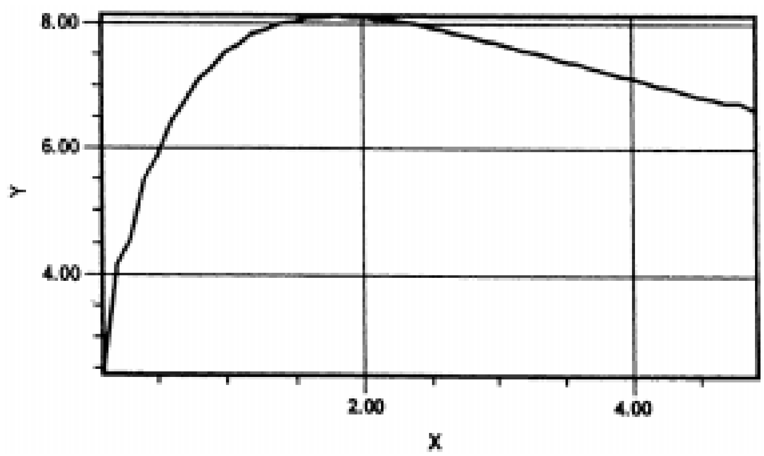

Figure 2. Same as in figure 1 except that now the ordinate is the parameter $b^2 r_0^2$ and the parameteric value of the beam initial radius is $\frac{\omega_p}{\omega} k r_0 = 10$.

This is a very important observation in view of the fact that alternative self-focusing theories dealing with cubic nonlinearity miss this point anyway but other theories that account at least for the quintic nonlinearity (that keep two terms in the summation on *n*) like the group-theoretic methods [3,4] and the moments method of Malkin [12] completely miss this aspect. These theories are in this sense really way off the mark in understanding self-focusing in a realistic (saturating nonlinearity) medium like a plasma.

Referring to eq. (6) for the summation formulae for the ponderomotive nonlinearity, absolute convergence is ensured at least if,

$$\left| \frac{E_0^2}{q^2(n+1)(2n+1)} \right| < 1 \tag{6'}$$

for large *n*. This indicates that the number of terms in the summation should be greater for larger intensities, $n > \frac{E_0}{2|q|}$ for the formulae to give consistent convergent results.

It needs to be mentioned that the summation can be written in integral forms for self-trapping and have been explicitly reported in the accompanying paper in this issue [2] . This possibility of the integral representation automatically ensures the convergence of the series at least in the case of self-trapping.

5. Results on self-focusing

The results for absorptionless self-focusing are that self-focusing is quasiperiodic. The naive portrayal of this possibility is to plot either $g_r(z) = \text{Re}(g(z))$ or $g_i(z) = \text{Im}(g(z))$ with the propagation distance z. Alternatively one can plot the related quantities of the actual beam width which is proportional to, $R(z) = |g(z)|^2/\text{Re}(g_z)$ and the beam curvature proportional to, $1/R_0 = \text{Im}(g(z))/|g(z)|^2$. We do that in figure 3. Part (c) of this figure, however, also shows a better manner of depicting this quasiperiodicity or ergodic

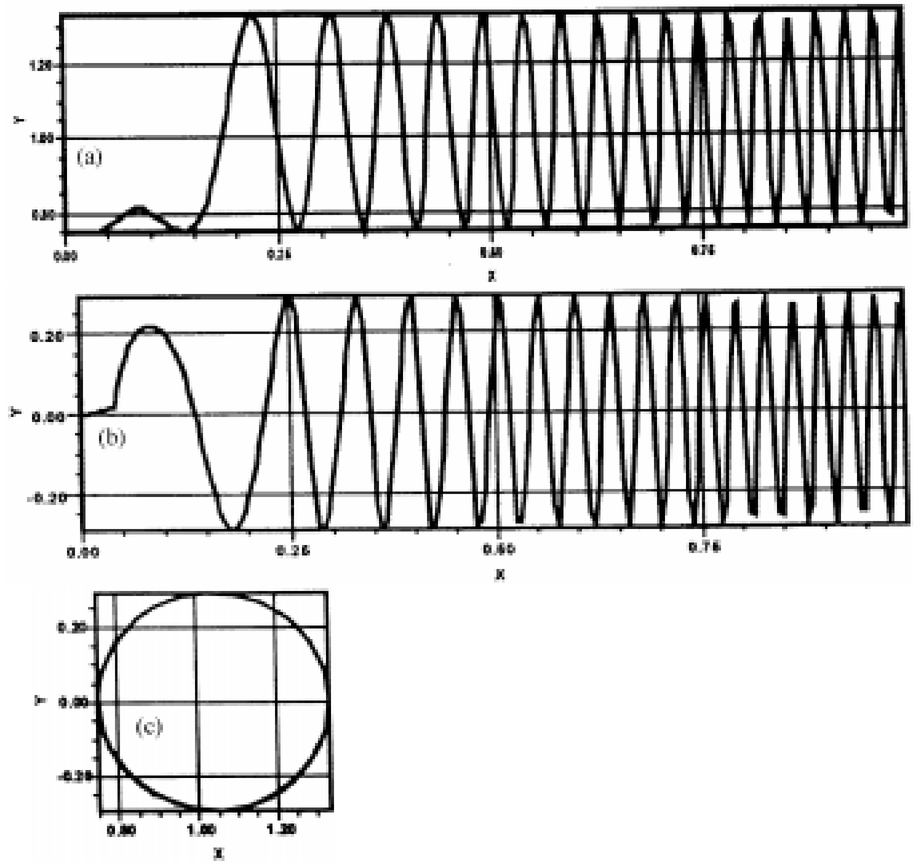

Figure 3. The dimensionless beam width, $R = |g|^2/\mathrm{Re}(g)$ of the laser beam, (**a**), and the dimensionless curvature $1/R_0$ =$\mathrm{Im}(g)/|g|^2$ of the laser beam, (**b**) plotted with respect to the propagation distance z. For absorptionless plasma for which these curves have resulted give otherwise a circle, as in (**c**), for the phase path in the g-plane for the laser beam complex beam-width and describe succinctly the quasiperiodic (ergodic) self-focusing. Here the parameters are, dimensionless intensity $I = cE_0^2/8\pi = 1$, $b^2r_0^2 = 1.5$ and $\omega_p = 0.1\omega$ for the ponderomotive nonlinearity of the plasma.

self-focusing by plotting $g_r(z)$ vs $g_i(z)$ which should be a circle anyway. This result follows from the fact that one may write $\bar{g}$ of eq. (4) or eq. (8) in the form,

$$z_{10}\mathrm{e}^{i\gamma} = \frac{1-\bar{g_0}}{1+\bar{g_0}} = \frac{1-\bar{g}}{1+\bar{g}}. \tag{13}$$

Taking absolute value of both sides one notes that this quantity is almost a constant and hence the ratio of the distance of a phase point from two fixed points $g = \pm 1$ is a constant which implies that the orbit is a circle. It can deviate from being a circle only if z_{10} varies very significantly during focusing. This result has more explicitly been discussed in ref. [3].

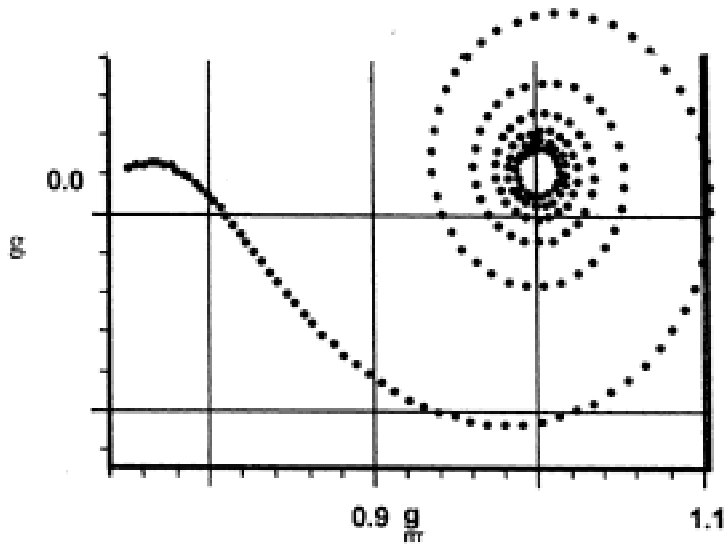

Figure 4. Self-focusing in an absorbing medium results in focusing to a self-trapped state represented by the spiral ending up at the value $g = 1$ asymptotically for almost any initial value of the complex beam-width. Parameters are the same as in figure 3 except that now the plasma is collisional with, $\nu_{\rm eff}/\omega = 0.01$.

In the presence of absorption, the present theory gives the important prediction that asymptotically, the beam collapses to a self-trapping situation as depicted in figure 4. This indicates that the beam ultimately will most often in a practical laser-plasma interaction be seen as a filament of constant self-trapped radius dictated by figure 2. Experimental correlation with self-focusing in solid active media like laser rods is very suggestive here although quantitative comparison is warranted in future.

6. Conclusions

The matrix method for paraxial self-focusing of a gaussian beam is a model problem and is best tackled by the present method. The method relies on the Lie-group representation of the complex beam-width parameter. It predicts ergodic (quasiperiodic) self-focusing in the absorptionless nonlinearity and focusing to a state of self-trapping in an absorbing plasma. It should also be possible to account phenomenologically for the radiation of the self-focusing beam that is equivalent to an absorption process because it is an energy loss mechanism. In such a case the self-focusing beam even in an absorptionless plasma will asymptotically self-trap as in figure 4.

It should be possible to generalize the Lie-optic computational method for self-focusing of this paper to non-paraxial self-focusing also. The only disadvantage is that the Laguerre-Gauss modes used here do not permit easy inversion between the configuration and momentum spaces. Fast simulation algorithms covered elsewhere for non-paraxial self-focusing, known as beam-propagation methods, [7] are based on the less accurate plane-wave angular spectrum representation of the laser beam involving the fast Hankel transformation (FHT) of the beam that in turn uses the fast Fourier transform (FFT) in its construction. The algorithm is built upon the analytical theory of self-focusing using Hankel transformation formulated by us much earlier [8,9], cast using Lie-transform techniques in a form useful for implementation on a computer.

Acknowledgements

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