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Lie-optics, geometrical phase and nonlinear dynamics of self-focusing and soliton evolution in a plasma

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Abstract. It is useful to state propagation laws for a self-focusing laser beam or a soliton in group-theoretical form to be called Lie-optical form for being able to predict self-focusing dynamics conveniently and amongst other things, the geometrical phase. It is shown that the propagation of the gaussian laser beam is governed by a rotation group in a non-absorbing medium and by the Lorentz group in an absorbing medium if the additional symmetry of paraxial propagation is imposed on the laser beam. This latter symmetry, however, needs care in its implementation because the electromagnetic wave of the laser sees a different refractive index profile than the laboratory observer in this approximation. It is explained how to estimate this non-Taylor paraxial power series approximation. The group theoretical laws so-stated are used to predict the geometrical or Berry phase of the laser beam by a technique developed by one of us elsewhere. The group-theoretical Lie-optic (or ABCD) laws are also useful in predicting the laser behavior in a more complex optical arrangement like in a laser cavity etc. The nonlinear dynamical consequences of these laws for long distance (or time) predictions are also dealt with. Ergodic dynamics of an ensemble of laser beams on the torus during absorptionless self-focusing is discussed in this context. From the point of view of new physics concepts, we introduce a stroboscopic invariant torus and a stroboscopic generating function in classical mechanics that is useful for long-distance predictions of absorptionless self-focusing.

Keywords. Potential approximation in quantum mechanics; paraxial refractive index; group theory in optics; geometrical phase; methods in classical mechanics; nonlinear optics; self-focusing; solitons.

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1. Introduction

High power laser interaction with plasma in various contexts like laser-driven fusion, laser based particle accelerators, laser-plasma based x-ray lasers, laser material processing etc. apart from a host of optoelectronic and light-guiding-light applications need a very good understanding of the process of self-focusing and soliton dynamics [1–4] in the presence of saturable nonlinearities. The process of self-focusing described by nonlinear Schrödinger

(NLS) equation and its variations had, therefore, been a challenging problem for the last forty years of the 20th century.

Exact solutions of the NLS have been available only in the particular case of cubic nonlinearity that too in planar geometry due to Zakharov and Shabat in terms of the celebrated inverse scattering technique [5]. Except for studies relevant more to mathematical curiosities like exact integrability, the existence of singularities [4] etc. relevant to cubic and quintic nonlinearities, in realistic media at high powers where the nonlinearity can saturate and in other geometries, one had however, to resort to approximate theoretical analysis or computer simulations.

In this paper I will point out certain interesting approximate theoretical techniques applicable to the NLS with a saturating nonlinearity that have been largely developed by me and my collaborators over the past several years. My purpose will be to draw attention to four interesting physics aspects of this important problem. Apart from the conceptual interest in this problem they can provide useful algorithms for self-focusing which will be discussed in an accompanying paper in this special issue.

The first physics phenomenon I take up is also applicable to the linear problem of laser propagation in an inhomogeneous medium but was discovered along with my collaborators in the context of self-focusing: the refractive index perceived by the laser beam is quite different in the paraxial limit from the laboratory perception of the inhomogeneous refractive index! The cause of this phenomenon is ultimately to be found within the wave nature of light on account of which light interacts globally with the medium instead of interacting locally in regions close to the axis of the beam. Section 2 of the paper is devoted to this aspect. The consequent theory of self-focusing of a gaussian beam or a gaussian soliton then gives a paraxial beam width equation.

The second interesting physics result comes in the context of analysis of the resultant beam – width – phase front dynamics. This nonlinear dynamics, it turns out, is governed by simple groups like the rotation group in non-absorbing media (like in a collisionless plasma) and the Lorentz group in absorbing media. The latter case gives the generic result that all self-focusing beams collapse into self-trapped filaments. All these results are introduced in §3.

The third interesting physics aspect is the calculation of the geometric or Berry phase for a self-focusing gaussian beam in terms of both the focusing sphere and its projected plane of the complex beam-width [6]. The geometric phase is estimated for the self-focusing beam or a soliton exhibiting width-chirp dynamics. This is introduced in §4 taking advantage of the close similarity of the focusing problem with a two level quantum system.

The fourth interesting physics aspect is the study of resultant dynamics on the torus and the consequent analysis of the nonlinear dynamics. The former case in the absence of absorption dictates ergodic self-focusing on a torus. This case is discussed next in §5 in terms of the surface of section and the stroboscopic map. Special classical mechanics features such as new forms of generating functions have been introduced by us for this analysis.

A remaining incomplete physics aspect is that of radiation by the self-focusing beam. This will be briefly discussed in the final section, §6.

2. Perception of the laser beam for paraxial refractive index

In analyzing problems that resemble a square law medium close to the laser beam axis, it is an accepted practice in linear optics to resort to a Taylor expansion of the refractive index.

In the radial variable, *r*, in a cylindrical coordinate system, the axis of which is along the laser beam axis this Taylor expansion looks like [7–9]:

$$\varepsilon(r^2) \approx \varepsilon_0 - \varepsilon_2 r^2 \; = \; [\varepsilon]_{r=0} \; - \; \left[\frac{\mathrm{d}\varepsilon}{\mathrm{d}r^2}\right]_{r=0} r^2. \tag{1}$$

This is a suitable power series expansion near the turning point that is also used in quantum mechanical problems in the context of matching WKBJ solutions [10] but is defective. The origin of the above expansion so common in optics can be traced to the early theories of geometrical optics of lenses and the related theory of aberrations. By making an expansion of the above form to different orders, the ray equations for light are analyzed to give a very acceptable theory of lens design [7]. Similar approximations arise in the quantum theory of electron optics and in weak coupling approximations in many quantum systems including condensed matter and nuclear physics. These approximations do not hold in wave-optics or quantum mechanics as was painfully realized in the theory of self-focusing after a lot of exhaustive work by us and by others.

In wave optics (or in wave-mechanics of particles), it turns out that the wave perceives the lens or the optical element (or the potential) as a whole and constructing a paraxial approximation and systematically generalizing it to higher orders in r^2 will have to use a different power-series expansion than the one given above in eq. (1), viz. the coefficients $\varepsilon_{0,2}$ will be different in wave-optics or quantum mechanics! The power series in r^2, therefore, is not a Taylor expansion but a different expansion. What expansion?

Before we go into the technique of non-Taylor expansion of the refractive index, we trace out the history of this problem as it unfolded in the context of the theory of self-focusing.

In the 60's Akhmanov *et al* (Moscow Univ.) developed the paraxial method for self-focusing of laser beams using the Taylor expansion in eq. (1) above for the refractive index close to the beam axis primarily for weak nonlinearity and published many papers using the beam envelope equations culminating in a review of their work in 1972 [11]. This method explained the basic result anticipated originally by Askaryan [12] and later by Chiao *et al* [13] that nonlinear refraction and diffraction of the laser beam could balance out to give rise to the phenomenon of self-trapping or diffractionless laser beam propagation at high enough powers (above threshold power) and the prediction of a laser power dependent focal length. It had been a simple popular modeling of the self-focusing problem ever since in a variety of situations although the Zakharov-Shabat exact solution in 1972 for planar geometry took attention off this method in a limited number of idealisations of the NLS.

The early 70's saw a copious use of this method and its generalization to higher powers of the lasers (at which the nonlinearity saturates out) principally by Sodha *et al* (IIT, Delhi) who summarised their results in a review in 1976 [14] and a book in 1975 [15]. This method was belatedly rediscovered by Max (Lawrence Livermore Labs.) in 1976 for saturable nonlinearity in a plasma [16]. The Akhmanov-Sodha-Max method then relied on the dielectric constant expansion,

$$\varepsilon[EE^*(r^2)] \approx \varepsilon_0(E_0E_0^*) - \varepsilon_2(E_0E_0^*)r^2, \tag{2}$$

where the coefficients $\varepsilon_{0,2}$ are determined by Taylor expansion as in eq. (1). It continues to be a popular theory even today [17] although it is incorrect.

Almost immediately after this, computer simulations of the NLS were taken up with teams in the New York area as a prelude to research towards attempting soliton launching in optical fibers for communication purposes. In 1977 Lam, Lippman and Tappert

(Courant Institute and New York University) [18] published the surprising computer simulation result that the Akhmanov–Sodha–Max theories are incorrect even for the case of self-trapping. They showed also that if the moments of the electromagnetic field of the laser beam were calculated using the formalism of Vlasov *et al* [19], the resultant theory agreed qualitatively with the computer simulations. For record's sake it should be stressed here that the moments theory results of Vlasov *et al* valid for the cubic nonlinearity case agreed with the Akhmanov *et al* paraxial theory while the obvious generalization of the latter theory to the case of saturable nonlinearity by Sodha *et al*–Max ceased to agree with computer simulations or the moments theory! Lam *et al* [18] concluded that moments theory of laser beams seemed to take into account non-paraxial effects. This result for self-trapping was soon vindicated in 1979 by another approach to the problem, the variational method due to Anderson *et al* (Instt. EM Fields, Chalmers University, Sweden) [20]. (We have shown elsewhere [3,21,22] that the variational theory predicts exactly the same results as the moments theory for self-focusing also.) The moments and variational approaches had no clear clue as to how they were able to take the non-paraxial effects into account when in fact they were using the same gaussian laser beam *ansatz* as the Sodha *et al*–Max paraxial theories that was valid only in a paraxial regime.

Through a completely new approach of laser beam electromagnetic field representation in terms of the Fourier or Hankel type angular spectrum, we showed in around the same year, 1979, [22] that one could almost recover the results of the moments and variational approaches to laser beam propagation in a paraxial approximation, provided the paraxial approximation was made properly. Switching over to the more appropriate harmonic-oscillator-modes based laser beam electromagnetic field of the angular spectrum representation, we were later able to exactly recover the moments and variational theory self-trapping results improving also the self-focusing theory for faster focusing at the same time [3,23].

The message was very clear. The laser beam propagates in an appropriate momentum space. The paraxial approximations of the beam and the nonlinear refractive index constructed in this momentum space yield the correct results. Since the exact momentum space for the nonlinear refractive index will never be known, the better one can approximate the momentum space and use it in the subsequent power series estimation, the better are the self-trapping and self-focusing results. The laser beam sees the refractive index of the momentum space and hence gives the correct results. While the moments and variational theories for laser beam propagation are able to affect these approximations implicitly, our momentum space approach affects these approximations explicitly.

Computer simulation of the nonlinear Schrödinger equation (NLS):

$$-2ik\frac{\partial E}{\partial z} + \nabla_\perp^2 E - k^2\Phi(EE*) = 0, \tag{3}$$

where $\varepsilon(EE*) = \varepsilon_{00}[1 - \Phi(EE*)], k = \varepsilon_{00}(\omega/c)$ would go on to show that the gaussian,

$$E(r,z) = \frac{E_0}{q}\exp\left(-\frac{r^2}{r_0^2 g}\right) \tag{4}$$

is a reasonably good approximate solution even in the case of saturating nonlinearity [25], (a fact that is easier to verify in the special case of self-trapping when variations in the propagation variable z are zero and $q = g = 1$) so that eq. (2) is a good approximation for the response function. Note that eq. (2) has been written in the radial variable so that the transverse operator in eq. (3) will have to be chosen in cylindrical coordinates.

The method described here is also valid for solitons with a reinterpretation of the symbols. In the case of the soliton, the transverse operator is in the planar coordinate x, the variable z is then the reduced time variable. See refs [24,25] for clarity on symbols and operators in this context.

There is an iterative relationship between the field form in eq. (4), and the wave equation, eq. (3) approximated by the quadratic (nonlinear) refractive index profile dictated by eq. (2) and it should therefore be obvious that the power series expansion in eq. (2) cannot be a simple Taylor series as done in the older paraxial theories. The convergence of this iterative process, in terms of the choice of the non-Taylor series coefficients $\varepsilon_{0,2}$, is fastest when the procedure for paraxial approximation involves harmonic-oscillator-modes-based-momentum space as demonstrated in detail in ref. [3].

In the particular case of self-trapping, the refractive index expansion as in eq. (2) following the above procedure gives rise to the choice of coefficients $\varepsilon_{0,2}$ as ([3]):

$$\varepsilon_0(E_0^2) = -\frac{1}{2E_0^2}\int_0^{E_0^2} x\Phi(x)\mathrm{d}x + \frac{r_0^2}{2}\varepsilon_2;$$
$$\varepsilon_2(E_0^2) = -\frac{1}{E_0^2}\int_0^{E_0^2}\left(\frac{1}{x}\int^x y^2\frac{\mathrm{d}\Phi}{\mathrm{d}y}\mathrm{d}y\right)\mathrm{d}x \tag{5}$$

for an arbitrary nonlinearity where the nonlinear function $\Phi(EE*)$ is defined in eq. (3). This choice for self-trapping corresponds to the trivial choice of the complex beam width parameter, $g = 1$, in the paraxial field form in eq. (4). For the more general case of other choices for the complex beam width parameter, g, the above expansion is approximately valid if in the above expressions in eq. (5), E_0^2 is replaced by $E_0^2/|g|$ (see ref. [3] for more accurate expressions). Value of the beam-width parameter g during propagation of the beam is to be determined from the beam evolution equation. We present here a very simple derivation of such a beam evolution equation.

Substitution of the field form in eq. (4) into the wave equation, eq. (3) and separating out the r^2 dependent and the constant terms gives the equations for the complex beam-width, g and for the complex propagation constant β where $q = \exp(-i\beta z)$ which in turn give the equations,

$$\beta(z) = -\frac{1}{2}k\int^z\frac{\varepsilon_0(z)}{\varepsilon_{00}} - \frac{2}{kr_0^2}\int^z\frac{\mathrm{d}z}{g}. \tag{3a}$$

Here the first term is the usual dynamical phase and the second term is sometimes called the Gouy phase and depends on the evolution of the complex beam-width parameter. It should be replaced by the the geometrical phase of §4 in general. The second equation will be for the beam-width and can be put into the form,

$$\frac{\mathrm{d}^2s}{\mathrm{d}z^2} + \frac{\varepsilon_2}{\varepsilon_0}s = 0 \quad ; \quad g = \frac{2ik}{\varepsilon_2 r_0^2}\frac{1}{s}\frac{\mathrm{d}s}{\mathrm{d}z}. \tag{3b}$$

This equation gives the effective beam-width parameter solution if the solution of the second-order differential equation is known. We make a piece-wise continuous approximation in which case solutions of the above equation are obtained by assuming that for a thin strip of thickness Δz, the parameters of the equation are constant. This will turn out to be equivalent to the WKBJ solution of the beam-width parameter equation. (More accurate solutions of eq. (3b) can be attempted [35] that will give better results.) In such a case the solution of the above equation valid for a thin slice of z will be $s = A\sin\left(\int^z (\sqrt{(\varepsilon_2}/\varepsilon_0)\mathrm{d}z\right) + B\cos\left(\int^z (\sqrt{(\varepsilon_2}/\varepsilon_0)\mathrm{d}z\right)$. If the initial value at $z = 0$ of the complex beam-width parameter is $g = g_0$ then one can determine the ratio, $A/B = \bar{g}_0/i, \bar{g} = g(b^2 r_0^2/2)$ so that one gets the ABCD law:

$$\bar{g} = \frac{\bar{g}_0 \cos\left(\int^z \sqrt{\frac{\varepsilon_2}{\varepsilon_0}}\mathrm{d}z\right) - i\sin\left(\int^z \sqrt{\frac{\varepsilon_2}{\varepsilon_0}}\mathrm{d}z\right)}{-\bar{g}_0 i\sin\left(\int^z \sqrt{\frac{\varepsilon_2}{\varepsilon_0}}\mathrm{d}z\right) + \cos\left(\int^z \sqrt{\frac{\varepsilon_2}{\varepsilon_0}}\mathrm{d}z\right)}. \tag{3c}$$

We refer the reader to ref. [3] and to our paper in this special issue for more details of this self-focusing theory. Also a comparison of self-trapping thresholds from various theories and computer simulations has been reported in ref. [23] and should prove useful.

3. Lie-optics of self-focusing

The above theory lets us state self-focusing dynamics in terms of well-known simple groups of physics allowing a Lie-optics formulation of self-focusing. Lie optics for many linear optics problems is a well-established subject [29,30] and it should have been applied to paraxial optics naturally. Accordingly, the transformation operator for the electric field of the laser beam (cf. eq. (3c)) in a non-absorbing medium is the bilinear transformation $T_R : \bar{g}(z+dz) = T_R(\Delta z)\bar{g}(z), E[\bar{g}(z+dz)] = T_R(\Delta z)E[\bar{g}(z)]$ where the four elements of this bilinear (Mobius) transformation are related one to one to the four elements of the rotation (matrix) operator, $\hat{T}_R$,

$$\hat{T_R}(z) \equiv \mathrm{e}^{-i\sigma_x \int_0^z \surd(\frac{\varepsilon_2}{\varepsilon_0})\mathrm{d}z}, \tag{6}$$

where $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is one of Pauli's spin matrices. This operator can analytically be continued into the complex ε plane when the Mobius operator $T_L(z)$ naturally generalizes to correspond to the 2×2 Lorentz (matrix) operator, $\hat{T_L}(z)$,

$$\hat{T_L}(z) \equiv T_R T_l \equiv \mathrm{e}^{-i\sigma_x \int_0^z \mathrm{Real}(\surd(\frac{\varepsilon_2}{\varepsilon_0})\mathrm{d}z)} \cdot \mathrm{e}^{\sigma_x \int_0^z \mathrm{Imag}(\surd(\frac{\varepsilon_2}{\varepsilon_0})\mathrm{d}z)}. \tag{7}$$

Here T_l is the Lorentz pure boost transformation and the factoring of the operator is natural.

Note that these are finite z operators although in practice because of the inherent nonlinearity in the refractive index of the self-focusing medium, these operators are to be further factored into a series of operators each operating for a distance of Δz by piece-wise continuous approximation. This serves as a convenient algorithm for beam propagation and is exploited as such in an accompanying paper.

The asymptotic consequences of the operators on focusing can still be conjectured easily. The nature of the rotation operator that applies in a non-absorbing medium is to just rotate g in the complex g-plane. This corresponds to oscillatory focusing and defocusing physically. It will be further analyzed in §5.

The consequence of the repeated application of the Lorentz operator is to rotate as per the rotation factor of this operator. The pure Lorentz boost operator factor has the tendency to either decrease the radius of the circular orbit in the complex beam-width plane or to increase the radius depending on the sign of the resultant exponent. It is shown in an accompanying paper that the main effect is decrease the orbit ratio in the complex g-plane till the beam-width converges to the value $g = 1$ after many rotations. This stabilization of the complex parameter g near to unity value during self-focusing argued out on the basis of the Lorentz operator is equivalent to a simple Liaponov linear stability analysis. Physically this corresponds to any arbitrary beam asymptotically converging to a self-trapped state after undergoing oscillatory self-focusing for some distance. This will again be demonstrated in an accompanying paper.

4. Geometrical phase of the self-focusing beam

The complex g-plane may be regarded as a steriographically projected plane from a Riemannian sphere $\mathbf{G}$ to be called the focusing sphere [6] of unit diameter, $|G| = \frac{1}{2}$. If each point on this sphere is described by a position vector $\mathbf{G} \equiv (\mathbf{G_x}, \mathbf{G_y}, \mathbf{G_z}) \equiv (\mathbf{\frac{1}{2}}, \theta, \varphi)$, then the Hamiltonian operator for focusing will be given by

$$H(\mathbf{G}) = \mathbf{G}.\sigma = \mathbf{\frac{1}{2}} \frac{\mathbf{1}}{\mathbf{1} + |\mathbf{g}|^{\mathbf{2}}} \begin{pmatrix} \frac{1}{2}(1-|g|^2) & g^* \\ g & \frac{1}{2}(1-|g|^2) \end{pmatrix}$$
$$= \mathbf{\frac{1}{2}} \begin{pmatrix} \cos\theta & \sin\theta \mathrm{e}^{-i\varphi} \\ \sin\theta \mathrm{e}^{i\varphi} & -\cos\theta \end{pmatrix}. \tag{8}$$

The spinor focusing state $|g\rangle$ will then be an eigen-vector of this Hamiltonian. It can be visualized as a tangent unit vector to the sphere at the tip of the position vector $\mathbf{G}$. A path on the complex g-plane would correspond to a path on the focusing sphere G and this would carry the spinor $|g\rangle$ by parallel transport over the sphere. The spinor $|g'\rangle$ obtained after its transport over the closed path on the focusing sphere differs from the original spinor. It can be shown [6,3,28] that $\langle g'|g\rangle = \exp(-i\phi_{\mathrm{BF}})$ where ϕ_{BF} is equal to half the solid angle subtended by the closed contour at the center of the focusing sphere. This Berry phase for focusing can be worked out in terms of the complex beam-width parameter, $g \ (= r \ \exp(i\nu))$ to be equal to [6,3,28],

$$\phi_{\mathrm{BF}} = \int_0^{2\pi} \frac{\mathrm{d}\nu}{1 \pm r^2} - 2\pi. \tag{9}$$

This formula is valid both for the absorption-less self-focusing case when rotation operator applies ($+$ sign in denominator applies) and also when absorption is present so that a pure Lorentz boost operator applies ($-$ve sign in the denominator applies then). The electromagnetic field has to be corrected for these additional phases as the laser beam propagates (see for more details refs [3,6,28].) The axial g-dependent phase of the laser beam that appears in the last term of eq. (3a) is in fact a simplified form of this phase and is best replaced by the above expressions for ϕ_{BF} in general.

5. Ergodic self-focusing

In the absence of absorption, as pointed out in §3 above, the dynamics of the beam phase point on the complex beam-width parameter plane, the g-plane is a circle indicating oscillatory self-focusing. The dynamics is actually hamiltonian [31] because it is describable as just a rotational transformation. The dynamics of the beam is described by eq. (3b) which can be put in the form of a modified Riccati equation,

$$\frac{\mathrm{d}g}{\mathrm{d}z} = -iR(g, g*)(g^2 - 1) \quad ; \quad R = \sqrt{\frac{\varepsilon_2}{\varepsilon_0}}. \tag{10}$$

The same form of the equation can be derived for the case of soliton dynamics also. The absence of absorption is signified by the fact that $R = R^*$. This form of self-focusing oscillations have been the most commonly analyzed self-focusing dynamics and hence need some elaboration. It should be noted that when R is independent of g and $g*$, corresponding to the case of an inhomogeneous optical waveguide with quadratic refractive index profile in the transverse direction, this is an exact Riccati equation with known solutions. In the more general (absorptionless) self-focusing case, either one follows the procedure outlined in §3 or for following the qualitative features of the dynamics, one follows the nonlinear dynamical analysis given below.

It is useful to write the Riccati-like equation, eq. (10) in component form,

$$\frac{\mathrm{d}g_r}{\mathrm{d}z} = 2g_r g_i R \quad ; \quad \frac{\mathrm{d}g_i}{\mathrm{d}z} = (\bar{g}_i^2 - g_r^2)R \quad ; \quad \bar{g}_i^2 = g_i^2 + 1. \tag{10a}$$

These basic equations for absorptionless paraxial self-focusing flow can be written in the surface of section, the $(g_r, \bar{g}_i)$ plane as,

$$\frac{\mathrm{d}g_r}{\mathrm{d}g_i} = -\frac{2g_r g_i}{g_r^2 - \bar{g}_i^2} \tag{11}$$

which can immediately be integrated out to yield a circular path in the g-plane,

$$(g_r - g_{rc})^2 + g_i^2 = (g_{rc}^2 - 1). \tag{11a}$$

Here, g_{rc} is an arbitrary constant representing the center of the circle (that in turn can be related to the initial value g_0 of the complex beam-width parameter) and it automatically determines the radius of the circular phase plane path.

The actual phase-plane dynamics, however, is to be studied after taking into account the z-coordinate also so that eq. (10) is solved instead of the reduced equation in eq. (11). This is best done by taking the phase plane as the surface of a cylinder, the cross-section of which is a circle, a projection of the path given by eq. (11a). The axis of the cylinder could be along the z-axis, the propagation direction. The length of the cylinder could be arbitrarily fixed as, say, z_0 and the dynamics of self-focusing can be examined modulo z_0. This will be useful, for example, in examining whether self-focusing is periodic. It is more useful to bend the cylinder into a torus so that the two ends of the cylinder join and the dynamics is modulo z_0 automatically. We refer to this as the torus for absorptionless paraxial self-focusing flow.

The dynamics can be put into a Hamiltonian form in terms of the generalized momentum (g_i) and position (g_r) variables and the time variable that controls the rate at which the torus surface is being traversed during the self-focusing dynamics: $\tau = \int^z (\bar{g}_i^2 + g_r^2) R(g_r, g_i) \mathrm{d}z$,

$$\frac{\mathrm{d}g_r}{\mathrm{d}\tau} = -\frac{\partial H}{\partial g_i} \quad ; \quad \frac{\mathrm{d}g_i}{\mathrm{d}\tau} = \frac{\partial H}{\partial g_r} \tag{12}$$

with the Hamiltonian,

$$H = \frac{g_r}{g_r^2 + \bar{g}_i^2}. \tag{12a}$$

Phase point flow obeying these Hamiltonian equations will cover the invariant torus. We can, therefore, borrow a whole lot of results from the theory of Hamiltonian dynamics and in particular from ergodic theory for this flow. We mention for future reference that eq. (12) can be put in a vector form if the vector $\mathbf{x} = (g_r, g_i)$ is defined:

$$\delta \mathbf{x} = -\delta z \hat{\mathbf{z}} \times \nabla H. \tag{12b}$$

Any function, $f(H)$, of the Hamiltonian will be invariant over the torus and can serve as an invariant distribution function on this cylindrical (or toroidal) phase space to measure the distribution of an ensemble of self-focusing beams in steady state because the Hamiltonian form requires the Liouville theorem for an ensemble of self-focusing laser beams to hold good:

$$\frac{\mathrm{d}f}{\mathrm{d}z} = 2 g_i g_r \frac{\partial f}{\partial g_r} + (\bar{g}_i^2 - g_r^2) \frac{\partial f}{\partial g_i} = 0. \tag{13}$$

This is a statement of area preservation in phase space during hamiltonian dynamics and informs us that if at $z = 0$, the area patch filled up by an ensemble of self-focusing beams is specified, then the area measure will be preserved even if the actual shape of the patch undergoes any kind of transformation. This holds in any section of the cylinder provided all the points in the cross-section are counted after transformation through the rotational Lie transform of §3.

For a single self-focusing beam, the above analysis promises in general quasiperiodicity for self-focusing. Exact periodicity can occur only in a quadratically inhomogeneous linear optical waveguide when R can be regarded as a constant. This possibility is completely ruled out in the self-focusing case because it is difficult to guarantee that the time variable τ will ensure exact periodicity in general. We can actually go a step further borrowing from Poincaré-Birkhoff theorem to state that if at all there be any case of periodic self-focusing, it will be destroyed under the slightest perturbation (of the Hamiltonian) unleashing regions of chaos on multiple scales in the phase space and breakdown of the torus. In practice, therefore, it should be very unlikely that self-focusing be ever periodic. We can borrow further from ergodic theory [31,36] to state that the condition for periodicity is the rationality of the ratio $\lambda = \frac{\lambda_1}{\lambda_2}$ where,

$$\lambda_1 = \int\int_{\mathrm{Tor}} H R 2 g_i g_r \mathrm{d}g_r \mathrm{d}g_i \quad ; \quad \lambda_2 = \int\int_{\mathrm{Tor}} H R (\bar{g}_i^2 - g_r^2) \mathrm{d}g_r \mathrm{d}g_i$$

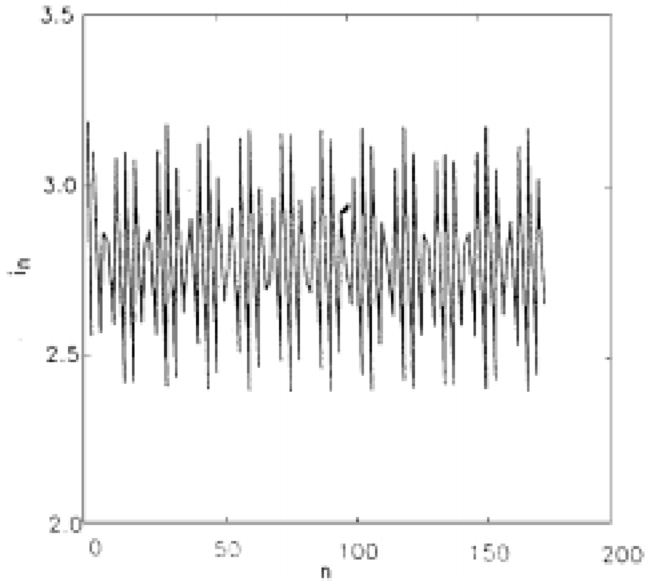

Figure 1. Time series of the rotation angle $i_n = \arg(g_n)$ in a stroboscopic map for a self-focusing laser beam in an absorption-less plasma at the discrete distances of $z_n = nz_0, z_0 = 1.750$. Note almost the pseudo-random behaviour of the complex beam-width parameter for self-focusing.

provided self-trapping conditions are not met, which is a very reasonable assumption. In all other cases when λ is irrational which is the most common kind of situation that occurs, self-focusing is ergodic in the sense that it will go on to cover the cylinder (or torus) continuously and uniformly (just as a magnetic line of force is supposed to cover a toroidal magnetic surface).

The above theory assures us that self-focusing dynamics is ergodic so that the circle given in eq. (11a) is uniformly filled out after propagation through infinite distance. Points are not filled on this circle in a continuous manner so that the actual filling up process may look like almost a random phenomenon. In practice both in computations and in experiments one has to ask the behavior and predictability of such a focusing after some finite distance of traversal of the beam. To answer this question we deviate from the standard canonical classical mechanics and ask what the behaviour of the beam is going to be stroboscopically after fixed distance intervals, $nz_0, n = 1, 2, 3, \ldots$. We also seek to find if there is any finite time invariant of the above Hamiltonian system.

To answer this, we have computed a large number of orbits and examined the pattern of evolution of the complex beam-width parameter. Since the radius is fixed for a given beam we plotted the angular rotation for each successive distance interval z_0 and found the pseudo-random behaviour of figure 1. Plotted on a cylindrical projected section, this same plot looks as in figure 2 where the successive points reached by the beam have been joined. The *envelope* of the chords of the circle is the small circle at the center to which all the chords are tangential. It is obvious that this inner *envelope* is actually an invariant of the finite time dynamics of the Hamiltonian system. If the section was of a torus that

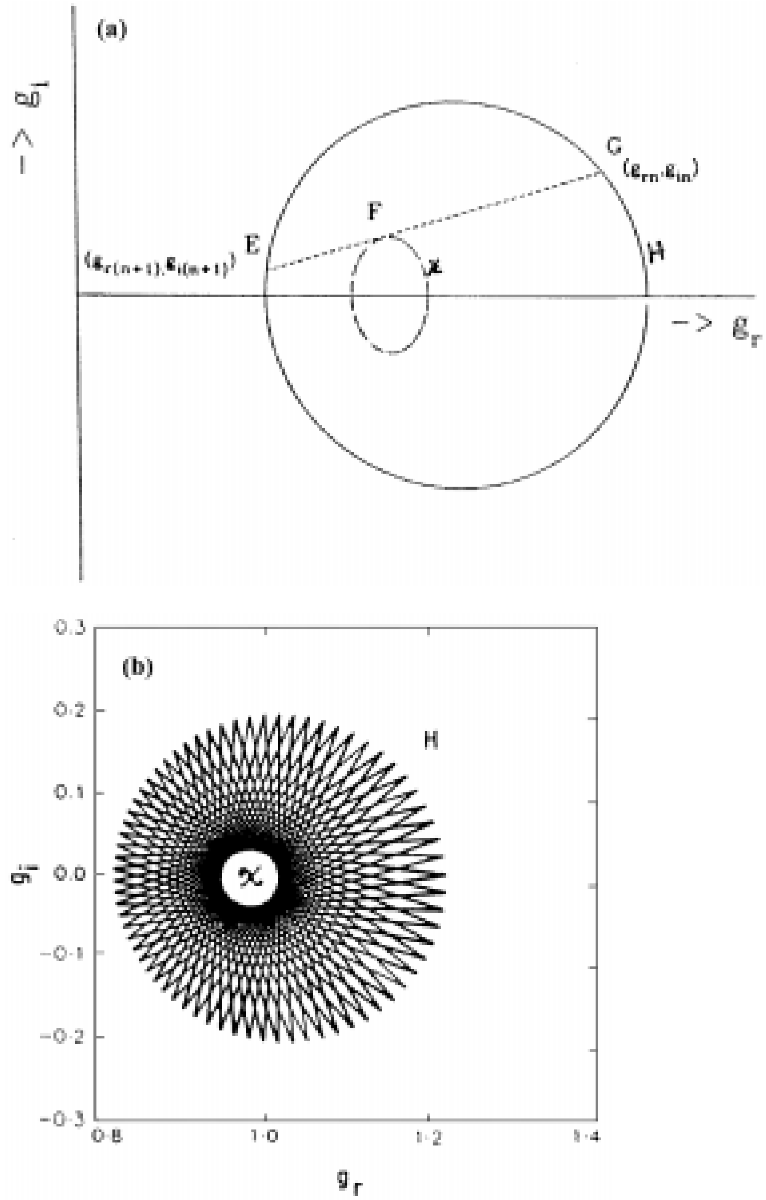

Figure 2. (**a**) Schematic of the construction of the stroboscopic invariant torus (SIT). Two successive stroboscopic points g_n and $g_{(n+1)}$ joined together are tangent to the SIT shown as an ellipse in this figure as explained by eq. (14). (**b**) The map points of figure 1 plotted on the circle map after joining successive points as described in figure (a) generate through tangency, the almost circular SIT at the center as an envelope of all the map-lines. The phase-space is a the surface of a cylinder.

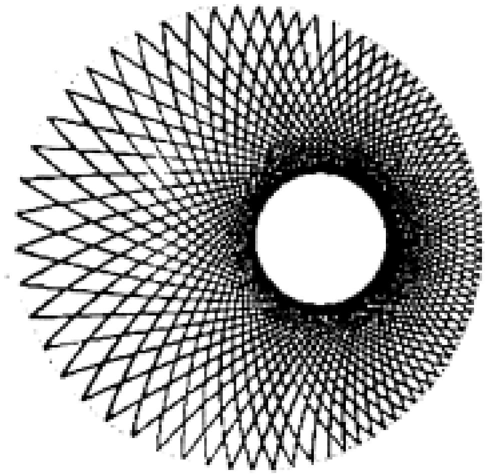

Figure 3. Same as figure 2(b) except that now the phase space is a torus obtained by bending the cylinder (that forms the phase space of figure 2) of length z_0 into a torus by joining the two ends. Note that the individual map-lines are now curved although they still create by tangency, the envelope at the center.

is formed by joining the end sections of the cylindrical phase-space of axial length z_0, this same figure gets transformed to figure 3 [34] where it is seen that each of the chords now is curved although the tangency to the stroboscopically invariant torus is still apparent.

For cylindrical phase space case of figure 2, a line joining two succesive points reached by the phase point, to be called a map-line, can be written down as

$$g_{n+1} - g_n = -z_0 \hat{\mathbf{z}} \times \nabla\chi, \tag{14}$$

where now a new kind of generating function χ gets defined that in turn defines another invariant surface, $\chi(g_r, g_i, z) = \mathrm{constant}$ to which the map-line chord is tangential. (Compare with eq. (12a) which is the same as eq. (14) but defines the map-line for infinitesimal distance of propagation.) This new generalization of Hamiltonian dynamics can be used to construct the surface χ (whenever it can be defined) that will go on to define finite distance self-focusing propagation evolution of the complex beam-width parameter.

Given this stroboscopically invariant surface (SIT), one draws a tangent to it from a suitable initial point on the invariant torus, (any of the circles of figure 1) to reach the next point in the iteration. One then repeats the procedure to reach the stroboscopic iteration to yet another point after it, and so on. The random looking but ergodic self-focusing then gets nicely predictable based on this additional invariant of the difference equation, $\chi = \mathrm{constant}$. Figure 4 is a depiction of a few such surfaces. Since there is a continuous distribution of such surfaces, it is better to depict the major, minor radii and the location of the center of the ellipse that fits the SIT to completely describe a quasi-periodic self-focusing process.

A very similar analytical representation in non-Euclidean dynamics should be possible to describe the dynamics in figure 3 describing the stroboscopically invariant torus (SIT).

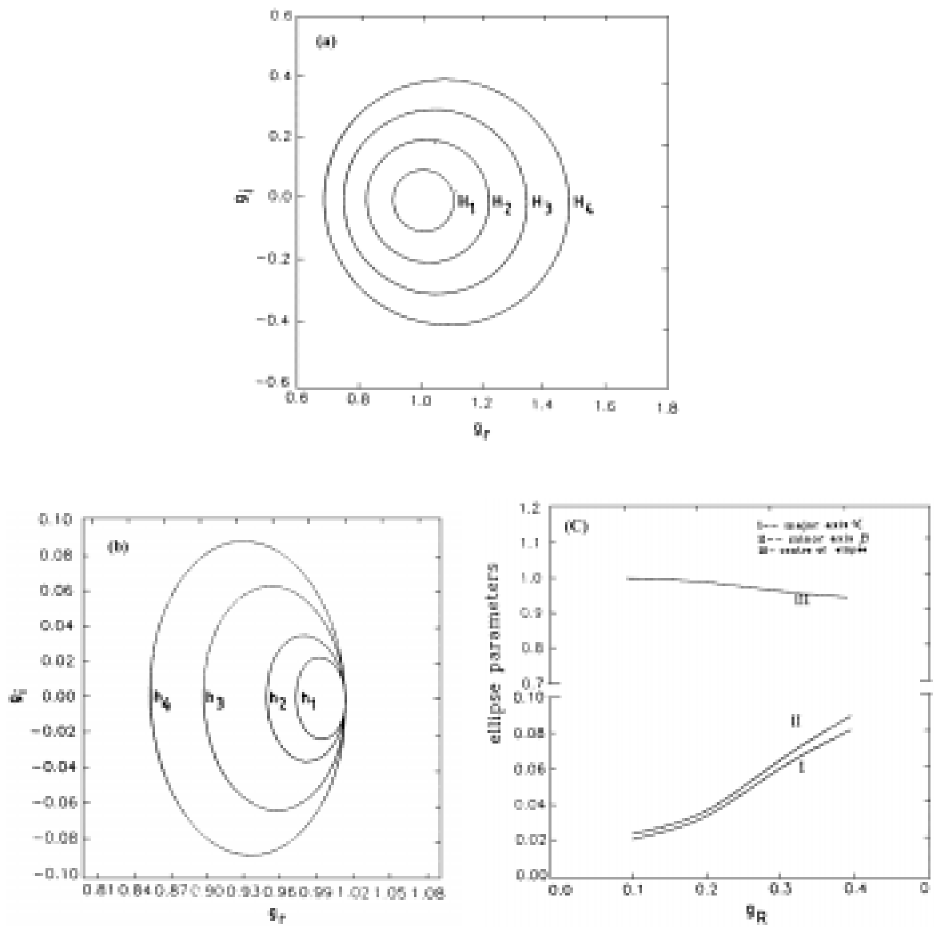

Figure 4. (**a**) Projection of the invariant torus (the $H(g, z) =$ constant surfaces) on the complex beam-width parameter plane, the g-plane for various values of the constant H corresponding to the outer circles of figure 2. (**b**) For various values of H of (a), the SIT's (stroboscopically invariant torus) projections which happen to be close to ellipses in shape. (**c**) The major axis, the minor axis and the location of the center of the ellipses depicted in (b). For obtaining the next point on the constant H figure of part (a) from a given point on it, start at that point and draw a line tangent to the SIT of (b) or an SIT generated from the data of (c), to reach the next iterative point on the constant H curve, *ad infinitum*.

6. Conclusion

A paraxial theory for self-focusing agreeing with computer simulations has been shown to be possible after explaining why this could not be done earlier. In the slow focusing limit, a simple theory of self-focusing has been presented. The results of the theory reduce to those of the moments and variational theory [3] under suitable slow-focusing approximations. The interesting Hamiltonian problem of absorptionless self-focusing has been discussed in detail including the nice nonlinear dynamical concept of SIT (stroboscopically invariant torus) through which such results can be summarised for long-distance predictions.

An important prediction is asymptotic self-trapping in the case when absorption is present using Lie-optics [35]. This is a generalization after simplification of the linear medium Lie-optics [29,30]. It may be mentioned here that the nonlinear symmetry group analysis of [26] or the renormalization group analysis [27] cannot deal with fully saturating nonlinearities as in this paper.

An important form of effective absorption that needs to be pointed out here is that of laser power depletion from the paraxial region (radiating beam) which also should lead to eventual self-trapping of the self-focusing beam as in the moments method [37]. In the present theory, any permanent depletion of radiation from paraxial region because of radiation of the self-focusing beam has to be introduced by hand. A feature present in the momentum-space paraxial approximation in this theory is, however, that energy spreads out into the higher modes of the laser only to reassemble at a farther distance in the paraxial region again, mimicking the Fermi–Pasta–Ullam energy oscillations over a superperiod [38]. This non-radiative nature of the paraxial field is similar to the Rabbi oscillations in a two-level system [39]. These aspects need further elaboration and investigation in the present theory not only using analysis but also computer simulations of the full equation using beam propagation methods [25].

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