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Density contrast indicators in cosmological dust models

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Abstract. We discuss ways of quantifying structuration in relativistic cosmological settings, by employing a family of covariant density constrast indicators. We study the evolution of these indicators with time in the context of inhomogeneous Szekeres models. We find that different observers (having either different spatial locations or different indicators) see different evolutions for the density contrast, which may or may not be monotonically increasing with time. We also find that monotonicity seems to be related to the initial conditions of the model, which may be of potential interest in connection with debates regarding gravitational entropy and the arrow of time.

1. Introduction

An important question in cosmology is the origin of the great deal of structure observed in the universe. It is usually assumed that this structuration is caused by gravitational instability and that the resulting lumpiness increases with time. If true, this would not only be important in connection with structure formation in the universe, but could also provide a gravitational arrow of time, pointing in the direction of increasing lumpiness. There are both similarities and differences here with the notion of entropy in statistical mechanics. Whereas they both postulate monotonicity, there is an important difference, in that in statistical mechanics the entropy, which essentially measures disorder, increases with time, whereas gravitational structuration seems to show increasing order with time. Ideas regarding gravitational entropy go back to a hypothesis by Penrose [11], the so called Weyl hypothesis, according to which gravitational entropy is related to the integral of the quadratic invariant constructed out of the Weyl tensor over an appropriate hyperspace. A great deal of work has gone into checking the applicability of this conjecture, but unfortunately without overall success [1,7].

Whatever the fate of this debate, an important question in this regard is whether the assumption of monotonicity in structuration is a justified assumption in relativistic cosmological settings, particularly in the context of recollapsing models. An answer to this question would be useful in informing the debates both on structure formation in the universe as well as the question of gravitational entropy.

Now in order to quantify structuration one would require some measure of inhomogeneity. It is known that mathematical and physical measures of inhomogeneity do not always agree. Mathematically a $1+3$ space-time is defined as homogeneous if it possesses a three dimensional simply transitive group of motions acting on the space-like surfaces [9]. Physically, on the other hand, inhomogeneity is usually expressed in terms of quantities which are in principle observable, such as the matter-density ρ and/or the pressure p of the cosmological fluid. This may suggest that a spacetime is homogeneous if both ρ and p are functions of time only. This definition is, however, gauge dependent and not satisfactory, as one can always find examples where $\rho = \rho(t)$ and $p = p(t)$, while the space-time is not mathematically homogemeous [3].

Thus simple contrasts in ρ and p are not suitable, as they are not gauge independent. Furthermore, all observations involve coarse graining, which means that in order to be able to make meaningful comparisons between theoretical and observational quantities, gauge invariant *averages* need to be defined [14]. It would thus be useful from an observational point of view to have, for example, a measure of the matter-density contrast with these properties which could allow comparisons to be made on different spatial and temporal scales.

Here as a first step in quantification of structuration and testing the hypothesis of its monotonical increase with time, we give a brief account of a recent study of the evolution of density contrast in the relativistic cosmological settings, using a family of density contrast measures introduced in [10].

2. Density contrast indicators

Density contrast indicators are among the most useful indicators to measure the degree of inhomogeneity of cosmological models. Over the years a large number of such proposals have been put forward (see [10]), many of which however have the undesirable properties of being either local or non-covariant.

One of the main problems in defining covariant density contrast measures is to ensure that the underlying hypersurface remains space-like. If a uniquely defined preferred time-like direction (such as the 4-velocity u^a of the fluid vector flow) exists, then the problem is resolved. In general, however, this is not the case. Here we consider $1+3$ spacetimes, such that the space-like surfaces remain orthogonal to the time-like vector u^a. Interestingly, from an observational point of view, this can always be done for a dust source.

From a physical point of view, it would be desirable to have a covariant analogue of a global spatial variability index for the density ρ given by

$$\int \frac{|\rho - \rho_0|}{\rho_0} \mathrm{d}V, \tag{1}$$

where ρ_0 is the mean density defined appropriately and dV is the comoving 3-volume element. This can readily be found in the form

$$\int_\Sigma |\chi_a| \, \mathrm{d}V, \tag{2}$$

where $\chi_a = \frac{h_a^b}{\rho}\frac{\partial \rho}{\partial x^b}$ is the fractional density gradient introduced by Ellis and Bruni [4], $h_{ab} = g_{ab} + u_a u_b$ projects orthogonal to u^a and the integration is over a 3-surface Σ or part thereof. Despite its covariance, this indicator is not unique. Now given our interest in checking the monotonicity hypothesis as well, it is not a priori clear what are the indicators that best codify such structuration and in particular what their monotonic properties may be. Therefore, instead of concentrating on a single indicator, we shall consider a two parameter family of possible covariant *density contrast indicators* S_{IK}, recently put forward in [10], in the form

$$S_{IK} = \int_{\Sigma} \left| \frac{h^{ab}}{\rho^I} \frac{\partial \rho}{\partial x^a} \frac{\partial \rho}{\partial x^b} \right|^K \mathrm{d}V, \quad I \in \Re,\ K \in \Re \setminus \{0\}. \tag{3}$$

An important feature of this family of indicators is that they may be treated as local or global depending upon the size of Σ. It also includes as special cases the indicator given by Tavakol and Ellis [13], for which $I = 2$ and $K = 1/2$, and its pointwise version given by Bonnor [2].

In the next section we will give a brief summary of results obtained in [10] regarding the evolution of density contrast indicators for dust cosmological models and briefly discuss some of their consequences.

3. Asymptotic evolution of the density contrast

In order to study the evolution of density contrast in inhomogeneous relativitic settings, we calculated the indicators S_{IK} for Szekeres models [12]. The form of the metric employed here was given by Goode and Wainwright [5], which can be visualized as exact perturbations of FLRW models in the form

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + R^2 \mathrm{e}^{2\nu}(\mathrm{d}x^2 + \mathrm{d}y^2) + R^2 H^2 W^2 \mathrm{d}z^2, \tag{4}$$

where $W = W(z)$, $H = H(x,y,z,t)$ and $\nu = \nu(x,y,z)$. These models can be divided into two classes, depending upon whether $\frac{\partial(\mathrm{Re}^{\nu})}{\partial z} \neq 0$ (class I) or $\frac{\partial(\mathrm{Re}^{\nu})}{\partial z} = 0$ (class II). For both classes, the function $R = R(z,t)$ satisfies the evolution equation

$$\dot{R}^2 = -k + 2\frac{M}{R}, \tag{5}$$

where $M = M(z)$ is a positive arbitrary function for the class I models and a positive constant for class II models. The matter-density is in both cases given by

$$\rho(x,y,z,t) = \frac{3MA}{4\pi R^3 H}, \tag{6}$$

where $A = A(x,y,z)$ is an arbitrary function. The Szekeres solutions have in total six arbitrary functions but one of them can be specified using the coordinate freedom. Denoting the time of the initial singularity by $T = T(z)$, the class II models are given by $T =$ constant. In the following we shall, in line with other authors (see e.g. [2]), also make this assumption in the case of class I models, in order to make the initial singularity simultaneous and hence avoid white holes. We recall that class I Szekeres models include the Lemaitre–Tolman–Bondi (LTB) spherical symmetric inhomogeneous dust models [8] as special cases, which in turn contain all the FLRW dust models as subcases and have therefore been used widely in cosmological studies [6]. The class II Szekeres models generalize both Kantowski–Sachs and Einstein–de Sitter models [6].

Table 1. Asymptotic evolution of S_{IK} for the class I and class II Szekeres models, where we have set $\psi(z,t) = (t-T)/M - 2\pi$.

	Class II	Classes I and II	
	$k=0$	$k=-1$	$k=+1$
S_{IK}	$(t-T)^{\frac{8}{3}IK-\frac{20}{3}K+2}$	$(t-T)^{3IK-8K+3}$	$\psi^{IK-\frac{10}{3}K+1}$

Table 2. Constraints on I and K in order to ensure $\frac{\partial S_{IK}}{\partial t} > 0$ asymptotically in class I and class II Szekeres models.

Models	Class I	Class II
$k=0$	–	$I > \frac{5}{2} - \frac{1}{K}$
$k=-1$	$I > \frac{8}{3} - \frac{1}{K}$	$I > \frac{8}{3} - \frac{1}{K}$
$k=+1$	$I < \frac{10}{3} - \frac{1}{K}$	$I < \frac{10}{3} - \frac{1}{K}$

Now, given the form of the expression for the matter-density (6) in these models, it is clear that in general there are no analytic solutions to the integrals in (3). We were, however, able to calculate analytically the late time asymptotic evolution of these indicators in subcases with $k=0$. This is of interest, since they give useful information regarding the asymptotic monotonicity of these indicators. We also made some numerical studies for particular LTB [10] and Szekeres [15] models. Our results are summarized in the tables. Table 1 gives the asymptotic expressions for the indicators S_{IK} as a function of time, whereas table 2 summarizes the conditions on I and K such that the indicators S_{IK} increase monotonically.

These results indicate that for both ever-expanding ($k=0$ and $k=-1$) and recollapsing ($k=+1$) Szekeres models, we can always choose I and K such that the S_{IK} is asymptotically increasing both separately and simultaneously. In particular, this is true for $I=2$ and $K=1$. However, there are cases of interest, such as $I=2$ and $K=1/2$, for which no such intervals can be found in which S_{IK} has this property. Also as can be seen from table 1 different indicators can, for different values of I and K, give different predictions concerning the asymptotic homogenization of these models. We have also calculated the condition for monotonicity around the initial singularity which turned out to be $I > 2 - 1/K$.

4. Conclusions

We have discussed the use of density contrast indicators to quantify the structuration of gravitational systems. We have considered the family of such indicators which are covariant and scale dependent and have studied their evolution with time in the context of the cosmological dust models of Szekeres. We have found that if the decaying modes of density perturbations are present, no such indicators exist which grow monotonically in time. In class I Szekeres metrics the vanishing of the decaying modes implies that the initial singularity is simultaneous [2] and therefore monotonicity of density contrast seems to be related to a particular choice of initial conditions. For a simultaneous Big-Bang it is easier to find monotonicity for ever-expanding than for recollapsing models. We also note that due to the spatial dependence of the turning point in the recollapsing cases, one can find different regions where the indicators may either increase or decrease monotonically. This indicates that, in such universe's, different observers can see different density contrast arrows at a given time, for certain values of I and K.

The study of evolution of density contrasts is of potential importance regarding the debates on structure formation as well as the questions concerning the arrow of time and the notion of gravitational entropy in cosmology. We hope our results here make a positive contribution in this direction.

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