Quantum aspects of black hole entropy

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Abstract.   This survey intends to cover recent approaches to black hole entropy which attempt to go beyond the standard semiclassical perspective. Quantum corrections to the semiclassical Bekenstein–Hawking area law for black hole entropy, obtained within the quantum geometry framework, are treated in some detail. Their ramification for the holographic entropy bound for bounded stationary spacetimes is discussed. Four dimensional supersymmetric extremal black holes in string-based $N = 2$ supergravity are also discussed, albeit more briefly.

1. Introduction

Despite the fact that gravitational collapse is a cataclysmic phenomenon wherein a multitude of physical processes (some understood, others yet to be discovered) are unleashed, the end-product – a black hole – is a pristine object. As Chandrasekhar says, '...the only elements in the construction of black holes are our concepts of space and time. They are, thus, almost by definition, the most perfect macroscopic objects there are in the universe. And since the general theory of relativity provides a unique three-parameter family of solutions for their description, they are the simplest objects as well' [1]. But this does not tell the full story. The simplicity and perfection depicted in Chandrasekhar's description were dramatically challenged in the early seventies by Jacob Bekenstein [2] and Steven Hawking [3], based on considerations that germinate from the known quantum origin of all matter (and radiation).

Bekenstein started with the laws of black hole mechanics [4], viz.,

- The zeroth law which states that the surface gravity of a black hole is a constant on the event horizon.
- The first law which, for the Kerr–Newman solution, states that

$$\mathrm{d}M \equiv \Theta \mathrm{d}\mathcal{A}_h + \Phi \mathrm{d}Q + \vec{\Omega} \cdot \mathrm{d}\vec{L}, \tag{1}$$

  where $\kappa \equiv (r_+ - r_-)/4\mathcal{A}_{\mathrm{hor}}$ is the surface gravity, $\Phi \equiv 4\pi Q r_+/\mathcal{A}_h$ is the electrostatic potential at the horizon and $\vec{\Omega} \equiv 4\pi \vec{L}/M\mathcal{A}_h$ is the angular velocity at the horizon.

- The second law which states that the area of the event horizon of a black hole can never decrease. In other words, if two black holes of horizon area $\mathcal{A}_1$ and $\mathcal{A}_2$ were to fuse, the area of the resultant black hole $\mathcal{A}_{12} > \mathcal{A}_1 + \mathcal{A}_2$.

Observing the analogy of these laws with the laws of ordinary thermodynamics, and gleaning insight from information theory, Bekenstein made the bold proposal that a black hole must have an entropy S_{bh} proportional to the area of its horizon,

$$S_{\mathrm{bh}} = \mathrm{const.} \times \mathcal{A}_h. \tag{2}$$

The constant in eq. (2) was fixed to be $1/4G$ where G is Newton's constant, by Hawking [3] who also showed that a black hole placed in a thermal background cooler than the 'temperature' given by the surface gravity (Hawking temperature) must radiate in a thermal spectrum. This remarkable result also confirmed the generalized second law of thermodynamics [2].

The black hole entropy S_{bh} arises from our lack of information about the nature of gravitational collapse. The post-collapse configuration is completely characterized by three parameters, viz., $M, Q, \vec{L}$ which encode in an unknown way the diverse set of events occurring during collapse, just as a thermodynamic system is characterized by a few quantities like pressure, volume, temperature etc. which encode the microstates of the system. The microstates responsible for the entropy should be quantum gravitational. However, a complete quantum gravity theory which serves the purpose is still not available, so the best one can do is to consider extant proposals for such a theory.

To summarize, the black hole entropy problem consists of identifying and counting the underlying quantum states in an attempt to verify if the semiclassical Bekenstein–Hawking area law (BHAL) does indeed hold. However, as eloquently argued by Carlip [5], this need not necessitate details of a proposal for quantum theory of gravitation. The semiclassical origin of BHAL might mean that one would be able to deduce it on the basis of some classical symmetry principle.

Carlip's approach basically consists of identifying a two dimensional Virasoro subalgebra of the algebra of generators of diffeomorphisms in a spacetime of any dimension with Lorentzian signature. There appear to be several possible ways in which such a subalgebra may emerge from the algebra of diffeomorphisms, given a set of boundary conditions valid at the black hole horizon. The Virasoro algebra thus obtained is then quantized by correspondence, and its primary states identified with the microstates responsible for black hole entropy. Appealing to the Cardy formula for the asymptotic degeneracy of these states, one obtains the BHAL.

The classical Poisson algebra of the spacetime diffeomorphism generators is given, for spacetimes with boundaries, by

$$\{H(\xi_1), H(\xi_2)\}_{PB} = H(\{\xi_1, \xi_2\}) + K(\xi_1, \xi_2). \tag{3}$$

For solutions of the Einstein equations (i.e. configurations for which the diffeomorphism constraints are satisfied in the bulk part of the spacetime), the non-trivial part of the algebra receives contribution only from the boundary of spacetime. It follows that the central term of the algebra (3) has contributions only from the boundary. Assuming the existence of standard fall-off conditions at asymptopia (for asymptotically flat spacetimes), one then concludes that the central term of (3) is non-trivial because of the specific boundary conditions imposed at the black hole event horizon which plays the role of the inner boundary.

Carlip's boundary conditions are ostensibly for local Killing horizons which do not appear to need a global timelike Killing vector which goes to a constant at null infinity. For such boundary conditions in any dimensional Lorentzian spacetimes, the algebra (3) can be shown to have a two dimensional Virasoro sub-algebra corresponding to conformal diffeomorphisms on the '$r - t$' plane. The central term of this algebra can be calculated in terms of the central 'charge' c.

Corresponding to this classical Virasoro algebra is a quantum Virasoro algebra; the microstates describing the horizon are identified with the primary states of this quantum Virasoro algebra. The macroscopic black hole is supposed to be described by primary states with arbitrarily high conformal weights. The boundary conditions also enable one to relate the central charge to the area (considered large in Planck units) of the horizon. One now appeals to the Cardy formula for the asymptotic density of these highly excited primary states

$$
\rho \sim \exp 2\pi \left[\frac{c}{6} \left(\Delta - \frac{c}{24} \right) \right]^{1/2}
\tag{4}
$$

with $\Delta = \mathcal{A}_h / 8\pi l_P^2, c = 3\mathcal{A}_h / 2\pi l_P^2, l_P^2 = G$. It is easy to see that

$$
S_{\mathrm{bh}} \equiv \ln \rho = S_{\mathrm{BH}},
\tag{5}
$$

where $S_{\mathrm{BH}} \equiv \mathcal{A}_h / 4 l_P^2$.

This implies that proposals for a quantum theory of gravity must have predictions for black hole entropy beyond the BHAL. In other words, there must be quantum corrections to the BHAL which cannot be deduced on the basis of semiclassical reasoning alone. On dimensional grounds, one expects that the quantum corrected black hole entropy might look like

$$
S_{\mathrm{bh}} = S_{\mathrm{BH}} + \delta_q S_{\mathrm{bh}},
\tag{6}
$$

where

$$
\delta_q S_{\mathrm{bh}} = \sum_{n=0}^{\infty} C_n \mathcal{A}_h^{-n},
\tag{7}
$$

where $\mathcal{A}_h$ is the classical horizon area and C_n are coefficients which are independent of the horizon area but dependent on the Planck length (Newton constant). However, in principle, one could expect an additional term proportional to $\ln \mathcal{A}_h$ as the leading quantum correction to the semiclassical S_{BH}. Such a term is expected on general grounds pertaining to breakdown of naive dimensional analysis due to quantum fluctuations, as is common in quantum field theories in flat spacetime and also in quantum theories of critical phenomena.

There is yet another restriction that one may want to subject these quantum corrections to. This restriction originates from the so-called holographic principle and the ensuing entropy bound [6–10]. According to this principle, the maximum entropy a black hole can have is bounded from above by the BHAL. If this restriction is taken seriously, it follows that $\delta_q S_{\mathrm{bh}} \leq 0$. In the literature there are calculations of the so-called entanglement entropy. These are based on quantum field theory in fixed classical black hole backgrounds [11] and logarithmic corrections have been obtained. But these corrections (a) appear to depend on some undetermined renormalization scale, (b) appear to be in discord with the

restriction due to the holographic entropy bound and (c) they do not include fluctuations of the background geometry. In the sequel, we consider two calculations of post-BHAL corrections where spacetime fluctuations are taken into account. One is the approach known as quantum geometry and the other based on $N = 2$ supergravity arising out of type II string theory. While preferentially dealing with the former approach in more detail, we shall also relate it to a tightening of the entropy bound arising from the holographic principle.

2. Quantum geometry

This approach, also called quantum general relativity, envisages an exact solution to the problem of quantization of standard four dimensional general relativity, in contrast to a perturbative expansion around a flat classical background.

2.1 *Classical connection dynamics*

The canonical treatment of classical general relativity, otherwise known as geometrodynamics [12], is traditionally formulated in terms of 3-metrics, i.e. restrictions of the metric tensor to three dimensional spacelike hypersurfaces ('time slices'). Canonically conjugate variables to these are then constructed, Poisson brackets between them defined and the entire set of first class constraints derived. The problem with this is that the constraints remain quite intractable.

A significant departure from this approach is to formulate canonical general relativity as a theory of 'gauge' connections, rather than 3-metrics [13]. Some of the constraints simplify markedly as a consequence, allowing exact treatment, although this is not true for all the constraints (e.g. the Hamiltonian constraint still remains difficult to analyse). The method has also undergone substantial evolution since its inception; a complex one-parameter family of connection variables is available as one's choice of the basic 'coordinate' degrees of freedom. The original Ashtekar choice [13], viz., the self-dual $\mathrm{SL}(2, C)$ connection (inspired by the work of Amitabha Sen [14]), corresponding to one member of this family, is 'geometrically and physically well-motivated' because the full tangent space group then becomes the gauge group of the canonical theory [15]. However, quantizing a theory with complex configuration degrees of freedom necessitates the imposition of subsidiary 'reality' conditions on the Hilbert space, rendering the formulation unwieldy. Also, given that the quantum wave function is a functional of the connection, one is led to dealing with complex functional analysis which is a difficult tool to use efficiently to extract physical results.

A better alternative, related to the former by canonical transformations, is to deal with the Barbero–Immirzi family of real $\mathrm{SU}(2)$ connections confined to the time-slice M

$$A_i^{(\beta)a} \equiv \epsilon^{abc}\Omega_{i\,bc} + \beta g_{ij}\Omega_{0a}^j, \tag{8}$$

labelled by a positive real number $\beta(\sim O(1))$ known as the Barbero–Immirzi (BI) parameter [15,16]. This yields the BI family of curvatures (restricted to M),

$$F_{ij}^{(\beta)a} \equiv \partial_{[i}A_{j]}^{(\beta)a} + \epsilon^{abc}A_i^{(\beta)b}A_j^{(\beta)c}. \tag{9}$$

The variables canonically conjugating to these are given by the so-called solder form

$$E_i^{(\beta)a} = \frac{1}{\beta} \epsilon^{abc} \epsilon_{ijk} e_j^b e_k^c.$$

(10)

The canonical Poisson bracket is then given by

$$\left\{ A_i^{(\beta)a}(x), E_j^{(\beta)b}(y) \right\} = \beta \delta_{ij} \delta^{ab} \delta(x, y).$$

(11)

2.2 *Quantization*

Canonical quantization in the connection representation implies that physical states are gauge invariant functionals of $A_i^{(\beta)a}(x)$ and

$$E_i^{(\beta)a} \to \hat{E}_i^{(\beta)a} \equiv \frac{\beta \hbar}{i} \frac{\delta}{\delta A_a^{(\beta)i}}.$$

(12)

A useful basis of states for solution of the quantum constraints are the 'spin network' states which generalize the loop space states used earlier [17]. A spin network consists of a collection of edges and vertices, such that, if two distinct edges meet, they do so in a vertex. It is a lattice whose edges need not be rectangular, and indeed may be non-trivially knotted. Example, the graph shown in figure 2 has 9 edges and 6 vertices.

To every edge γ_l ($l = 1, 2, \ldots, 9$) we assign a spin J_l which takes all half-integral values including zero. Thus, each edge transforms as a finite dimensional irreducible representation of $SU(2)$. In addition, one assigns to each edge a Wilson line functional of the gauge connection $h_l(A) = \mathcal{P} \exp \int_l \mathrm{d}\gamma_l^i (A \cdot \tau)_i$, where τ^a are SU(2) generators in the adjoint representation. To every vertex is assigned an SU(2) invariant tensor C^v. These assignments completely define the basis states, which form a dense set in the Hilbert space of gauge invariant functionals of $^\beta A$. The inner product of these states then induces a measure on the space of connections which can be used to define a 'loop transform' [15] of physical states, representing the same state, by diffeomorphism invariance. 'Weave' states, supported on complicated and fine meshed nets (with meshes of Planck scale size) are supposedly typical physical states. Thus, the classical spacetime continuum metamorphoses in the quantum domain into a space of 'weaves' with meshes of Planck scale size on which all curvature (and indeed all dynamics) is concentrated. The Einsteinian continuum emerges when we view the weaves from afar, and are no longer able to see the meshes.

Observables on the space of physical states (like the weaves) include geometrical operators like the area and volume operators, which typically are functionals of the canonical variables. To calculate the spectrum of these operators in the connection representation requires a technique of 'regularization' since the classical definition of these quantities translates into singular objects upon naive quantization. For example, the area operator $\hat{A}(S)$ corresponding to a two dimensional surface S intersecting a subset $\mathcal{L}$ of edges of a net, not touching any of its vertices and having no edge lying on S is formally defined as

$$\hat{A}(S)\psi_n \equiv \left(\int \mathrm{d}^2\sigma \sqrt{n_i n_j \hat{E}^{ia} \hat{E}^{jb}} \right)_{\mathrm{reg}} \psi_n.$$

(13)
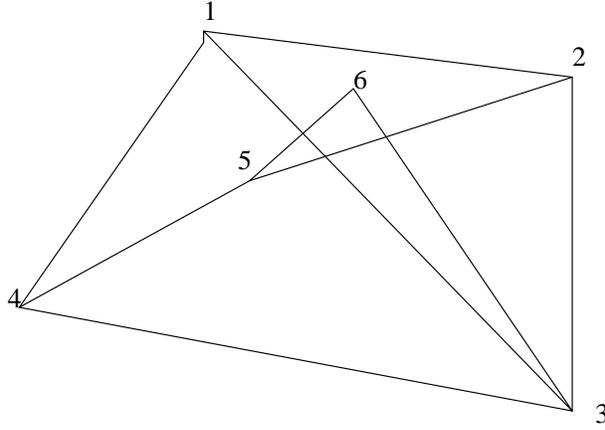
Figure 1. A spin network with 9 edges and 6 vertices.

For large areas compared to l^2_{Planck}, this reduces to [18,19]

$$\hat{A}(S)\psi_n = \beta\hbar l^2_{\text{Planck}} \sum_{l \epsilon \mathcal{L}} \sqrt{J_l(J_l + 1)}\psi_n. \tag{14}$$

The discreteness in the eigenspectrum of the area operator is of course reminiscent of discrete energy spectra associated with stationary states of familiar quantum systems. Each element of the discrete set in (14) corresponds to a particular number of intersections ('punctures') of the spin net with the boundary surface S. Diffeomorphism invariance ensures the irrelevance of the locations of punctures. This will have important ramifications later.

2.3 *Schwarzschild black hole: Boundary conditions*

We now consider the application of the foregoing formalism to the calculation of entropy of the four dimensional Schwarzschild black hole, following [18,5,20,21] and [22]. The basic idea is to concentrate on the horizon as a boundary surface of spacetime (the rest of the boundary being described by the asymptotic null infinities $\mathcal{I}^{\pm}$), on which are to be imposed boundary conditions specific to the horizon geometry of the Schwarzschild black hole vis-a-vis its symmetries etc. These boundary conditions then imply a certain description for the quantum degrees of freedom on the boundary. The entropy is calculated by counting the 'number' of boundary degrees of freedom. The region of spacetime useful for our purpose is the 'late time' part Δ of the event horizon $\mathcal{H}$ for which nothing crosses $\mathcal{H}$, and it is of constant cross-sectional area $\mathcal{A}_S$. A particular spacelike foliation of spacetime is considered, which intersects $\mathcal{H}$ (in particular Δ) in the 2-sphere S.

Standard asymptotically flat boundary conditions are imposed on $\mathcal{I}^{\pm}$; those on the event horizon essentially subsume the following: first of all, the horizon is a null surface with respect to the Schwarzschild metric; second, the black hole is an isolated one with no gravitational radiation on the horizon; thirdly, the patch Δ has two flat (angular) coordinates spanning a special 2-sphere which coincides with S, the intersection of the time-slice M

with Δ. The last requirement follows from the spherical symmetry of the Schwarzschild geometry. These boundary conditions have a crucial effect on the classical Hamiltonian structure of the theory, in that, in addition to the bulk contribution to the area tensor of phase space (the symplectic structure) arising in canonical general relativity, there is a boundary contribution. Notice that the boundary of the spacelike hypersurface M intersecting the black hole horizon is the 2-sphere S. Thus, the symplectic structure is given by

$$
\Omega|_{A^{(\beta)}, E^{(\beta)}} (\delta E^{(\beta)}, \delta A^{(\beta)} \; ; \; \delta E^{(\beta)'}, \delta A^{(\beta)'})
$$
$$
= \frac{1}{8\pi G} \int_M \mathrm{Tr}(\delta E^{(\beta)} \wedge \delta A^{(\beta)'} - \delta E^{(\beta)'} \wedge \delta A^{(\beta)})
$$
$$
- \frac{k}{2\pi} \int_{S=\partial M} \mathrm{Tr}(\delta A^{(\beta)} \wedge \delta A^{(\beta)'}),
\tag{15}
$$

where, $k \equiv (\mathcal{A}_S / 2\pi \beta G)$. The second term in (15) corresponds to the boundary contribution to the symplectic structure; it is nothing but the symplectic structure of an SU(2) level k Chern–Simons theory living on M. This is consistent with an extra term that arises due to the boundary conditions in the action, that is exactly an SU(2) level k Chern–Simons action on Δ [21]. As a consequence of the boundary Chern–Simons term, the curvature pulled back to S is proportional to the pullback (to S) of the solder form

$$
F^{(\beta)} + \frac{2\pi\beta}{\mathcal{A}_S} E^{(\beta)} = 0.
\tag{16}
$$

This is a key relation for the entropy computation.

The aforementioned boundary conditions are special cases of a larger class which define the so-called isolated horizon [23] which includes charged non-rotating back holes (including those in the extremal limit), dilatonic black holes and also cosmological horizons like de Sitter space. For the entire class, eq. (16) holds on the 2-sphere S constituting the intersection of the spatial slice M with Δ. What follows therefore certainly applies to non-rotating isolated horizons.

2.4 *Quantum entropy calculation*

In the quantum theory, we have already seen that spacetime 'in the bulk' is described by spin nets ($\{\psi_V\}$ say) at fixed time-slices. It has been shown [19] that spin network states constitute an eigenbasis for the solder form with a discrete spectrum. Now, in our case, because of the existence of the event horizon which forms a boundary of spacetime, there are additional surface states $\{\psi_S\}$ associated with Chern–Simons theory. In the canonical framework, the surface of interest is the 2-sphere S which forms the boundary of M. Thus, typically a state vector in the Hilbert space **H** would consist of tensor product states $\psi_V \otimes \psi_S$. Equation (16) would now act on such states as an operator equation. It follows that the surface states $\{\psi_S\}$ would constitute an eigenbasis for $F^{(\beta)}$ restricted to S, with a discrete spectrum. In other words, the curvature has a support on S only at a discrete set of points – punctures. These punctures are exactly the points on S which are intersected by edges of spin network 'bulk' states, in the manner discussed earlier for the definition of
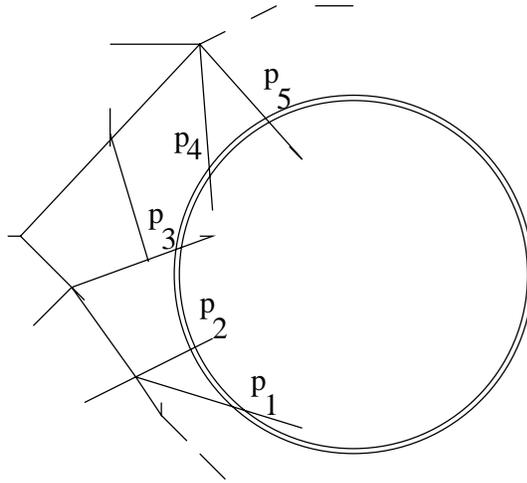
Figure 2. The bulk spin network intersecting the horizon at punctures.

the area operator. At each puncture p therefore one has a specific spin J_p corresponding to the edge which pierces S at p. The black hole can then be depicted (in an approximate sense) as shown in figure 4.

Consider now a set of punctures $\mathcal{P}_{(n)} = \{p_1, J_{p_1}; p_2, J_{P_2}; \ldots p_n, J_{p_n}\}$. For every such set, there is a subspace $\mathbf{H}_V^{\mathcal{P}}$ of $\mathbf{H}_V$ which describes the space of spin net states corresponding to the punctures. Similarly, there is a subspace $\mathbf{H}S^{\mathcal{P}}$ of $\mathbf{H}_S$ describing the boundary Chern–Simons states corresponding to the punctures in $\mathcal{P}$. The full Hilbert space is given by the direct sum, over all possible sets of punctures, of the direct product of these two Hilbert (sub)spaces, modulo internal gauge transformations and diffeomorphisms [24]. Now, given that the Hamiltonian constraint cannot be solved exactly, one assumes that there is at least one solution of the operator equation acting on the full Hilbert space, for a given set of punctures $\mathcal{P}$.

One now assumes that it is only the surface states ψ_S that constitute the microstates contributing to the entropy of the black hole S_{bh}, so that the volume states ψ_V are traced over, to yield the black hole entropy as

$$S_{\mathrm{bh}} = \ln \sum_{\mathcal{P}} \dim \mathbf{H}_S^{\mathcal{P}}. \tag{17}$$

The task has thus been reduced to computing the number of $\mathrm{SU}(2)_k$ Chern–Simons boundary states for a surface with an area that is $\mathcal{A}_S$ to within $O(l_{\mathrm{Planck}}^2)$. One now recalls a well-known correspondence between the dimensionality of the Hilbert space of the Chern–Simons theory and the number of conformal blocks of the two dimensional conformal field theory (in this case $\mathrm{SU}(2)_k$ Wess–Zumino–Witten model) 'living' on the boundary [25]. This correspondence now simplifies the problem further to the computation of the number of conformal blocks of the WZW model. Thus, the problem of counting the microstates contributing to the entropy of a 4d Schwarzschild black hole has metamorphosed into counting the number of conformal blocks for a particular 2d conformal field theory.

This number can be computed in terms of the so-called fusion matrices N_{ij}^r [26]
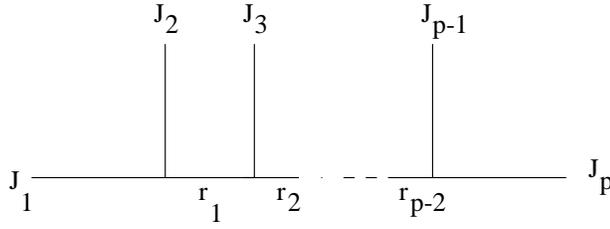
Figure 3. Diagrammatic representation of composition of spins J_i for SU(2)$_k$.

$$N^{\mathcal{P}} = \sum_{\{r_i\}} N_{j_1 j_2}^{r_1} N_{r_1 j_3}^{r_2} N_{r_2 j_4}^{r_3} \cdots N_{r_{p-2} j_{p-1}}^{j_p}. \tag{18}$$

This is very similar to the composition of angular momentum in ordinary quantum mechanics; it has been extended here to the infinite dimensional affine Lie algebra SU(2)$_k$. Diagrammatically, this can be represented as shown in figure 5 below.

Here, each matrix element N_{ij}^r is 1 or 0, depending on whether the primary field $[\phi_r]$ is allowed or not in the conformal field theory fusion algebra for the primary fields $[\phi_i]$ and $[\phi_j]$ $(i, j, r = 0, 1/2, 1, \ldots, k/2)$:

$$[\phi_i] \otimes [\phi_j] = \sum_r N_{ij}^r [\phi_r]. \tag{19}$$

Equation (18) gives the number of conformal blocks with spins $j_1, j_2, \ldots, j_p$ on p external lines and spins $r_1, r_2, \ldots, r_{p-2}$ on the internal lines.

We next take recourse to the Verlinde formula [26]

$$N_{ij}^r = \sum_s \frac{S_{is} S_{js} S_s^{\dagger r}}{S_{0s}}, \tag{20}$$

where, the unitary matrix S_{ij} diagonalizes the fusion matrix. Upon using the unitarity of the S-matrix, the algebra (18) reduces to

$$N^{\mathcal{P}} = \sum_{r=0}^{k/2} \frac{S_{j_1 r} S_{j_2 r} \ldots S_{j_p r}}{(S_{0r})^{p-2}}. \tag{21}$$

Now, the matrix elements of S_{ij} are known for the case under consideration (SU(2)$_k$ Wess–Zumino model); they are given by

$$S_{ij} = \sqrt{\frac{2}{k+2}} \sin\left(\frac{(2i+1)(2j+1)\pi}{k+2}\right), \tag{22}$$

where, i, j are the spin labels, $i, j = 0, 1/2, 1, \ldots, k/2$. Using this S-matrix, the number of conformal blocks for the set of punctures $\mathcal{P}$ is given by

$$N^{\mathcal{P}} = \frac{2}{k+2} \sum_{r=0}^{k/2} \frac{\prod_{l=1}^{p} \sin\left(\frac{(2j_l+1)(2r+1)\pi}{k+2}\right)}{\left[\sin\left(\frac{(2r+1)\pi}{k+2}\right)\right]^{p-2}}. \tag{23}$$

In the notation of [21], eq. (23) gives the dimensionality, dim $\mathcal{H}_S^{\mathcal{P}}$, for arbitrary area of the horizon k and arbitrary number of punctures. The dimensionality of the space of states $\mathcal{H}_S$ of CS theory on three-manifold with S^2 boundary is then given by summing $N^{\mathcal{P}}$ over all sets of punctures $\mathcal{P}$, $N_{\text{bh}} = \sum_{\mathcal{P}} N^{\mathcal{P}}$. Then, the entropy of the black hole is given by $S_{\text{bh}} = \log N_{\text{bh}}$.

Observe now that eq. (23) can be rewritten, with appropriate redefinition of dummy variables and recognizing that the product can be written as a multiple sum,

$$
N^{\mathcal{P}} = \left(\frac{2}{k+2}\right) \sum_{l=1}^{k+1} \sin^2 \theta_l \sum_{m_1=-j_1}^{j_1} \cdots \sum_{m_p=-j_p}^{j_p} \exp\left\{2i\left(\sum_{n=1}^{p} m_n\right)\theta_l\right\},
$$

(24)

where, $\theta_l \equiv \pi l/(k+2)$. Expanding the $\sin^2 \theta_l$ and interchanging the order of the summations, a few manipulations then yield

$$
N^{\mathcal{P}} = \sum_{m_1=-j_1}^{j_1} \cdots \sum_{m_p=-j_p}^{j_p} \left[\delta_{(\sum_{n=1}^{p} m_n),0} - \frac{1}{2}\delta_{(\sum_{n=1}^{p} m_n),1}\right.
$$
$$
\left. -\frac{1}{2}\delta_{(\sum_{n=1}^{p} m_n),-1}\right],
$$

(25)

where, we have used the standard resolution of the periodic Kronecker deltas in terms of exponentials with period $k+2$,

$$
\delta_{(\sum_{n=1}^{p} m_n),m} = \left(\frac{1}{k+2}\right) \sum_{l=0}^{k+1} \exp\left\{2i\left[\left(\sum_{n=1}^{p} m_n\right) - m\right]\theta_l\right\}.
$$

(26)

Notice that the explicit dependence on $k+2$ is no longer present in the exact formula (25).

The foregoing calculation does not assume any restrictions on k or number of punctures p. Equation (25) is thus an exact formula for the quantum entropy of a Schwarzschild black hole, or for that matter, any non-rotating isolated horizon.

Our interest focuses on the limit of large k and p, appropriate to macroscopic black holes of large area. Observe, first of all, that as $k \to \infty$, the periodic Kronecker delta's in (26) reduce to ordinary Kronecker deltas,

$$
\lim_{k \to \infty} \bar{\delta}_{m_1+m_2+\cdots+m_p,m} = \delta_{m_1+m_2+\cdots+m_p,m}.
$$

(27)

In this limit, the quantity $N^{\mathcal{P}}$ counts the number of SU(2) singlet states, rather than SU(2)$_k$ singlets states. For a given set of punctures with SU(2) representations on them, this number is larger than the corresponding number for the affine extension.

Next, recall that the eigenvalues of the area operator for the horizon, lying within one Planck area of the classical horizon area $\mathcal{A}_h$, are given by

$$
\hat{\mathcal{A}}_h \Psi_S = 8\pi\beta l_P^2 \sum_{l=1}^{p} [j_l(j_l+1)]^{\frac{1}{2}} \Psi_S,
$$

(28)

where l_P is the Planck length, j_l is the spin on the lth puncture on the 2-sphere and β is the Barbero–Immirzi parameter [15]. We consider a large fixed classical area of the horizon, and ask what the largest value of number of punctures p should be, so as to be consistent with (28); this is clearly obtained when the spin at each puncture assumes its lowest nontrivial value of $1/2$, so that, the relevant number of punctures p_0 is given by

$$p_0 = \frac{\mathcal{A}_h}{4l_P^2}\frac{\beta_0}{\beta}, \tag{29}$$

where, $\beta_0 = 1/\pi\sqrt{3}$. We are of course interested in the case of very large p_0.

Now, with the spins at all punctures set to $1/2$, the number of states for this set of punctures $\mathcal{P}_0$ is given by

$$N^{\mathcal{P}_0} = \sum_{m_1=-1/2}^{1/2} \cdots \sum_{m_{p_0}=-1/2}^{1/2} \left[\delta_{\left(\sum_{n=1}^{p_0} m_n\right),0} - \frac{1}{2} \right.$$
$$\left. z\delta_{\left(\sum_{n=1}^{p_0} m_n\right),1} - \frac{1}{2}\delta_{\left(\sum_{n=1}^{p_0} m_n\right),-1} \right]. \tag{30}$$

The summations can now be easily performed, with the result:

$$N^{\mathcal{P}_0} = \begin{pmatrix} p_0 \\ p_0/2 \end{pmatrix} - \begin{pmatrix} p_0 \\ (p_0/2 - 1) \end{pmatrix}. \tag{31}$$

There is a simple intuitive way to understand the result embodied in (31). This formula simply counts the number of ways of making SU(2) singlets from p_0 spin $1/2$ representations. The first term corresponds to the number of states with net J_3 quantum number $m = 0$ constructed by placing $m = \pm 1/2$ on the punctures. However, this term by itself overcounts the number of SU(2) singlet states, because even non-singlet states (with net integral spin, for p is an even integer) have a net $m = 0$ sector. Beside having a sector with total $m = 0$, states with net integer spin have, of course, a sector with overall $m = \pm 1$ as well. The second term basically eliminates these non-singlet states with $m = 0$, by counting the number of states with net $m = \pm 1$ constructed from $m = \pm 1/2$ on the p_0 punctures. The difference then is the net number of SU(2) singlet states that one is interested in for that particular set of punctures.

To get to the entropy from the counting of the number of conformal blocks, we need to calculate $N_{\rm bh} = \sum_{\mathcal{P}} N^{\mathcal{P}}$, where, the sum is over all sets of punctures. Then, $S_{\rm bh} = \ln N_{\rm bh}$.

The combination of terms in (31), which corresponds to counting of states in the SU(2) Chern–Simons theory, yields a formula for the quantum entropy of the black hole (isolated horizon). One can show that [19], the contribution to $N_{\rm bh}$ for this set of punctures $\mathcal{P}_0$ with all spins set to $1/2$, is by far the dominant contribution; contributions from other sets of punctures are far smaller in comparison. Thus, the entropy of an isolated horizon is given by the formula derived in ref. [27]. Here, the formula follows readily from eq. (31) and Stirling approximation for factorials of large integers. The number of punctures p_0 is rewritten in terms of area $\mathcal{A}_h$ through eq. (29) with the identification $\beta = \beta_0 \ln 2$. This allows us to write the entropy of an isolated horizon in terms of a power series in horizon area $\mathcal{A}_h$:

$$S_{\rm bh} = \frac{\mathcal{A}_h}{4l_p^2} - \frac{3}{2} \ln \left(\frac{\mathcal{A}_h}{4l_p^2} \right) - \frac{1}{2} \ln \left(\frac{\pi}{8(\ln 2)^3} \right) - O(\mathcal{A}_h^{-1}). \tag{32}$$

Notice that the corrections to the BHAL are indeed finite and negative, thereby conforming to the holographic constraint.

Inclusion of representations other than spin $1/2$ on the punctures does not affect the sign or coefficient of the logarithmic term. For example, we may place spin 1 on one or more punctures and spin $1/2$ on the rest. The number of ways singlets can be made from this set of representations can be computed in a straight forward way. Adding these new states to the ones already counted above, merely changes the constant and order $\mathcal{A}_h^{-1}$ terms in formula (32). The logarithmic corrections are therefore indeed robust. We may mention that very recently Carlip [28] has presented compelling arguments that this formula may possibly be of a universal character.

2.5 *A tighter holographic entropy bound*

The holographic principle [6–8] (HP) asserts that the maximum possible number of degrees of freedom within a macroscopic bounded region of space is given by a quarter of the area (in units of Planck area) of the boundary. This takes into account that a black hole for which this boundary is (a spatial slice of) its horizon, has an entropy which obeys the Bekenstein–Hawking area law and also the generalized second law of black hole thermodynamics [9]. Given the relation between the number of degrees of freedom and entropy, this translates into a holographic entropy bound (EB) valid generally for spacetimes with boundaries [29].

The basic idea underlying both these concepts is a network, at whose vertices are variables that take only two values ('binary', 'Boolean' or 'pixel'), much like a lattice with spin one-half variables at its sites. Assuming that the spin value at each site is independent of that at any other site (i.e., the spins are randomly distributed on the sites), the dimensionality of the space of states of such a network is simply 2^p for a network with p vertices. In the limit of arbitrarily large p, such a network can be taken to approximate the macroscopic surface alluded to above, a quarter of whose area bounds the entropy contained in it. Thus, any theory of quantum gravity in which spacetime might acquire a discrete character at length scales of the order of Planck scale, is expected to conform to this counting and hence to the HP.

Let us consider now a slightly altered situation: one in which the binary variables at the vertices of the network considered are no longer distributed randomly, but according to some other distribution. Typically, for example, one could distribute them binomially, assuming, without loss of generality, a large lattice with an even number of vertices. Consider now the number of cases for which the binary variable acquires one of its two values, at exactly $p/2$ of the p vertices. In case of a lattice of spin $1/2$ variables which can either point 'up' or 'down', this corresponds to a situation of net spin zero, i.e., an equal number of spin-ups and spin-downs. Using standard formulae of binomial distributions, this number is

$$N\left(\frac{p}{2} \middle| a \right) = 2^p \left(\begin{array}{c} p \\ p/2 \end{array} \right) [a(1-a)]^{p/2}. \tag{33}$$

Clearly, this number is maximum when the probability of occurrence $a = 1/2$; it is given by $p!/(\frac{p}{2}!)^2$. Thus, the number of degrees of freedom is now no longer 2^p but a smaller

number. This obviously leads to a lowering of the entropy. For very large p corresponding to a macroscopic boundary surface, this number is proportional to $2^p/p^{\frac{1}{2}}$. The new EB can therefore be expressed as

$$S_{\mathrm{max}} = \ln\left(\frac{\exp S_{\mathrm{BH}}}{S_{\mathrm{BH}}^{1/2}}\right),\tag{34}$$

where, recall that $S_{\mathrm{BH}} = \mathcal{A}_h/4l_P^2$ is the Bekenstein–Hawking entropy. This is a tighter bound than that of ref. [9] mentioned above. We shall argue below that, in the quantum geometry framework, it is possible to have an even tighter bound than that depicted in eq. (34).

Observe that eq. (31) has two terms of which the second is certainly negative. It follows that the number of SU(2) singlet states contributing to the entropy of the isolated horizon is bounded from above by the first term

$$N^{\mathcal{P}_0} < \binom{p_0}{p_0/2}.\tag{35}$$

It may be pointed out that the rhs of (35) also has another interpretation. It represents the counting of boundary states for an effective U(1) Chern–Simons theory. It counts the number of ways unit positive and negative U(1) charges can be placed on the punctures to yield a vanishing total charge. Upon using the Stirling approximation for large p_0 and replacing it in terms of the classical horizon area $\mathcal{A}_h$, this would then correspond to an entropy bound given by the same formula (34) above for binomial distribution of charges.

But an even tighter bound on the entropy of the black hole (isolated horizon) ensues from eq. (32). It is obvious that this yields the bound

$$S_{\mathrm{max}} = \ln\left(\frac{\exp S_{\mathrm{BH}}}{S_{\mathrm{BH}}^{3/2}}\right).\tag{36}$$

As already mentioned, this bound has a certain degree of robustness and perhaps universality.

The steps leading to the EB for any bounded spacetime now follows the standard route of deriving the Bekenstein bound (see, e.g. [10]): we assume, for simplicity that the spatial slice of the boundary of an asymptotically flat spacetime has the topology of a 2-sphere on which is induced a spherically symmetric 2-metric. Let this spacetime contain an object whose entropy exceeds the bound. Certainly, such a spacetime cannot have an isolated horizon as a boundary, since then, its entropy would have been subject to the bound. But, in that case, its energy should be less than that of a black hole which has the 2-sphere as its (isolated) horizon. Let us now add energy to the system, so that it does transform adiabatically into a black hole with the said horizon, but without affecting the entropy of the exterior. But we have already seen above that a black hole with such a horizon must respect the bound; it follows that the starting assumption that the object, to begin with, had an entropy violating the bound is not tenable.

One crucial assumption in the above arguments is that matter or radiation crosses the isolated horizon adiabatically in small enough amounts, such that the isolated character of the horizon is not seriously affected. This is perhaps not too drastic an assumption. Thus, for a large class of spacetimes, one may propose eq. (36) as the new holographic entropy bound.

3. Corrections to BHAL in 4d $N = 2$ supergravity

3.1 *Supersymmetric black holes*

Black hole solutions of four dimensional $N = 2$ supergravity have been studied exten-sively [30], following improved understanding of the Ramond–Ramond sector of string theories as the solitonic sector. Likewise, these solutions lend themselves to a solitonic interpretation, interpolating between two $N = 2$ supersymmetric ground states which are asymptotically flat geometries which behave like Bertotti–Robinson (e.g. $AdS_2 \otimes S_2$) ge-ometries at the horizon. These solutions exhibit only an $N = 1$ supersymmetry globally, so that they saturate the Bogomol'nyi–Prasad–Sommerfield (BPS) bound which relates the mass of the black hole to its charge(s):

$$M_{\mathrm{ADM/Bondi}} \geq G^{-1/2} \left(Q^2 + \tilde{Q}^2 \right)^{1/2}. \tag{37}$$

Clearly, these solutions represent extremal black holes. Almost a decade earlier, in a pioneering paper Gibbons and Hull had shown that the black hole configurations which saturated the BPS bound also led to the existence of a chiral spin-half field (Killing spinor) which satisfied the equation

$$D_\mu \epsilon(x) = \frac{1}{4} F_{\lambda\sigma} \gamma^\lambda \gamma^\sigma \gamma_\mu \epsilon(x). \tag{38}$$

Observe that the spinor in question is the zero mode corresponding to $N = 1$ supersym-metry transformations and, as such, defines the notion of a supersymmetric black hole. Gibbons and Hull also showed that, for time independent Killing spinors, the only solu-tions saturating the BPS bound are the Majumdar–Papapetrou class of solutions [31], given by

$$\begin{aligned} \mathrm{d}s^2 &= -W^2 \mathrm{d}t^2 + W^{-2} \mathrm{d}\mathbf{x}^2, \\ F &= \cos\theta F^0 + \sin\theta \tilde{F}^0, \end{aligned} \tag{39}$$

where,

$$\begin{aligned} F^0 &= \mathrm{d}W \wedge \mathrm{d}t, \\ W^{-1} &= 1 + \sum_{s=1}^{n} \frac{M_s}{|\mathbf{x} - \mathbf{x}_s|}. \end{aligned} \tag{40}$$

The solution (39) represents an assembly of n extremal Reissner–Nordstrom black holes, each having a mass M_s saturating the bound (37) and identical electric/magnetic charges. In this system, there is an exact cancellation of the gravitational attraction and electrostatic repulsion, thereby guaranteeing stability. What Gibbons and Hull's work clearly demon-strates is that the underlying reason for this stability is the unbroken $N = 1$ supersymmetry of the solution, defined as above in terms of Killing spinors.

It is easy to check that the surface gravity of the assembly of extremal black holes van-ishes at the horizon of each individual black hole. The assembly is thus stable against super-radiance and Hawking radiation. However, the solitonic character of the solution (39) becomes clear only when it is coupled to n $N = 2$ vector multiplets carrying Ramond–Ramond (electric/magnetic) charge. This coupled system interpolates between asymptot-pia and Bertotti–Robinson geometries which are $N = 2$ supersymmetric vacua.

3.2 *Entropy calculation*

The four dimensional $N = 2$ supergravity multiplet consists of a Weyl multiplet (which is itself composed of the graviton, two gravitinos), an auxiliary multiplet containing complex scalar fields $\tilde{A}$ and also superconformal gauge fields. Each $N = 2$ vector multiplet contains a gauge field, gauginos and a complex scalar field X_s. Together with $\tilde{A}$, the X_s constitute the moduli fields whose vacuum expectation values determine the moduli space of BPS-saturating supersymmetric black hole solutions. The coupling between the Weyl and vector multiplets is described by the holomorphic function $F(X_s, \tilde{A})$ which is protected from perturbative quantum corrections. As a result, it has been shown that the effective action of $N = 2$ supergravity is not renormalized for such supersymmetric black hole backgrounds.

In addition, the geometry of the moduli space exhibits a sort of 'fixed-point' behaviour: the moduli X_s evolve into constants at the horizon regardless of their value at asymptopia. Furthermore, these constants are determined as a function of the Ramond–Ramond charges parametrizing the black hole solution. The macroscopic (semiclassical) entropy of these configurations is then determined from the horizon area which is expressed in terms of the constant values of the moduli by inverting the formula relating them to the charges. For a single extremal Reissner–Nordstrom black hole, this is simply

$$S_{\text{BH}} = \frac{\pi}{16G} |C|^2, \tag{41}$$

where C is the fixed point value of the modulus field X.

Now, $N = 2$ supergravity is the low energy limit of type II superstring theory, compactified to $\mathcal{M}_4 \otimes (CY)_3$ where, $\mathcal{M}_4$ is Minkowski 4-space and $(CY)_3$ is three dimensional Calabi–Yau manifold. The black hole solution extremizes the Born–Infeld effective action corresponding to the Ramond–Ramond sector of the compactified string. This sector has a quantized description in terms of open string states terminating on D-branes propagating in a flat non-compact transverse background spacetime. The black hole is taken to correspond (in the weak coupling limit of the theory) to a set of BPS-saturating excited string states with appropriate charge quantum numbers. The logarithm of the degeneracy of these states, computed via the Cardy formula [32] is defined as the microscopic entropy of the black hole configuration, and agrees, for large values of the charges, to the BHAL as depicted in (41). This remarkable agreement is attributed to the non-renormalization of the effective action – a property of supersymmetric theories. Thus, the microscopic entropy remains unchanged as the coupling increases from weak to strong, the latter being identified with the black hole domain.

Now, one recalls that the low energy limit of string theory contains general relativity (as a coupled spin two theory in a flat background), together with corrections that can be represented as higher order curvature terms in the classical action. These 'stringy' corrections are supposed to lead to corrections in the BHAL [35]. The basic tool used here is the formalism due to Wald *et al* [36] in which black hole entropy (in any diffeo-invariant theory) is represented as a Noether charge corresponding to diffeomorphism invariance. There are two basic assumptions in the approach followed in [35]: (a) there exist BPS-saturating supersymmetric extremal black hole solutions of the string-modifed higher curvature theory and (b) the moduli space of such solutions exhibits the same fixed point behaviour as for the theory without the higher curvature terms. There is a further important caveat: Wald's formalism requires that the Killing horizon be bifurcated, a property that requires

the surface gravity at the horizon to be non-vanishing. On the other hand, extremal black holes have vanishing surface gravity and degenerate horizons. This subtlety needs to be addressed in a more careful treatment of the problem. The result obtained so far can be expressed as

$$S_{\mathrm{bh}} = S_{\mathrm{BH}} - \frac{\pi}{8}\mathrm{Im}\left(\frac{\partial F}{\partial \tilde{A}}\right). \tag{42}$$

The correction term on the rhs is to be evaluated at the 'fixed point', i.e., on the horizon. It turns out again that the result is in 'perfect' agreement with a 'microscopic' calculation based on counting of states in string theory in a flat background in weak coupling.

Several remarks regarding (42) are in order: (i) the correction has no obvious geometrical interpretation in terms of the horizon area; (ii) it is not at all clear whether the correction is positive or negative, so that its status vis-a-vis the holographic restriction is uncertain. Therefore, it is difficult to glean from this any implication for the holographic entropy bound. A more disturbing aspect is the following: the approach depends crucially on the 'special' geometry of moduli space which, in turn, is wholly dependent on the unphysical requirement of unbroken supersymmetry. This means that generic non-extremal black holes, of the type that most likely exist in the universe, are outside the purview of any of these considerations.

4. Conclusions

The common technical tool, in both the quantum geometry and the string theoretic approaches to counting black hole microstates, is two dimensional conformal field theory. Whether this implies an unknown deeper connection, or merely exhibits a technical device par excellence is yet to be ascertained. The quantum geometry approach provides the most direct route to the relevant 2d CFT which, in the foregoing, is a rational conformal field theory, viz. $SU(2)$ WZW model. The string route has more ambiguities, although there are claims that there could be a way to obtain the same answer for the quantum entropy [28]. This latter aspect needs to be better understood. It is not at all clear how two approaches so disparate in their premise can, even in principle, lead to identical results.

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