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Dynamics of self-focusing and self-phase modulation of elliptic Gaussian laser beam in a Kerr-medium

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Abstract. Using a direct variational technique involving elliptic Gaussian laser beam trial function, the combined effect of non-linearity and diffraction on wave propagation of optical beam in a homogeneous bulk Kerr-medium is presented. Particular emphasis is put on the variation of beam width and longitudinal phase delay with the distance of propagation. It is observed that no stationary self-trapping is possible. The regularized phase is also seen to be always negative.

Keywords. Self-focusing; elliptic; Kerr-medium; self-phase modulation.

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1. Introduction

It was Chio *et al* [1] who first discovered the phenomenon of self-focusing an optical pulse in a Kerr-medium, which lead to considerable theoretical and experimental interest in non-linear optics. Apart from affecting high-power beam propagation and influencing most of the other non-linear phenomena, self-focusing has been identified as one of the important effects to the success of controlled laser fusion [2–5]. It is also of direct interest for space communication and molecular dynamic diagnostics [6–8].

Investigations of self-focusing of laser beams were primarily carried out for cylindrical Gaussian beams [9–16]. In a few publications, cylindrical off axis mode [3], spiral self-trapping [10], elliptic Gaussian beam [17] in anisotropic media were studied. In particular, using WKB and paraxial ray approximation, Cornolti *et al* [17] considered the self-focusing of an elliptic Gaussian beam in Kerr-non-linear medium. Originally suggested by Wagner [11], the trial function is substituted in the evolution equation, where the non-linear refractive index is Taylor expanded in the transverse direction. A generalization to include the phase was later suggested by Akhmanov *et al* [18]. The main drawback with this approach is that it lacks global pulse dynamics since it overemphasizes the importance of field closest to the pulse maximum. The moment theory [12,13] although remedies this drawback by considering the evolution of the transverse coordinate but has not been generalized to include a proper phase relationship. Karlsson *et al* [16] have used variational approach using Gaussian beam as a trial function in optical fibres. They pointed out that the paraxial ray approximation does not predict correctly the self-phase modulation of the beam propagating in bulk non-linear media. Laser systems usually generate a beam which is more nearly elliptical than circular in cross-section, it is worthwhile to study such realistic situation. This technique was also used recently for investigating self-trapping of cylindrical symmetric beams in higher order non-linear media [19]. Importance of non-paraxiality in self-focusing phenomena has been recently highlighted [20]. In this paper, we investigate the whole beam self-focusing for intense elliptic Gaussian beam in a bulk Kerr-medium using variational approach.

2. Basic formulation

In the slowly varying envelope approximation, the equation that governs the evolution of the electric field envelope in Kerr-medium is the non-linear Schrödinger equation

$$-2\iota k\frac{\partial E}{\partial z}+\frac{\partial^2 E}{\partial x^2}+\frac{\partial^2 E}{\partial y^2}+\frac{2n_2}{n_0}k^2|E|^2E=0, \tag{1}$$

where z is the longitudinal coordinate, k is the linear wave number and the refractive index n is assumed to be of the following form

$$n=n_0+n_2|E|^2. \tag{2}$$

Here n_0 is linear index of refraction of the medium and the non-linear index coefficient n_2, also called the Kerr coefficient, is a basic material parameter characterizing non-linear optical beam and pulse propagation in the medium. We investigate elliptic Gaussian beam assuming

$$E(z,r)=\varepsilon(z)\exp[-(x/\alpha)^2]\exp[-(y/\beta)^2]\exp\left[-\iota k\left(\frac{x^2}{2\alpha}\frac{\mathrm{d}\alpha}{\mathrm{d}z}+\frac{y^2}{2\beta}\frac{\mathrm{d}\beta}{\mathrm{d}z}\right)\right], \tag{3}$$

where α and β are the width parameters of the beam respectively in the x and y directions. The amplitude ε as well as beam widths α and β are all real functions of z. The second and third terms together in eq. (1) viz. $\nabla_\perp^2 E$ is spatial dispersion that spreads the beam in the transverse direction whereas the last term, the attracting non-linearity compresses the beam. The dynamics of self-focusing is determined by the relative competition between spatial dispersion and attracting non-linearity. If we consider only one transverse direction, then we obtain the one dimensional non-linear Schrödinger equation which is integrable by inverse scattering technique [21–23]. The resulting analytical solution is an exact solution and represents a special class of travelling wave called soliton. Peculiar characteristics of these solutions are that they are stable on colliding with each other and preserve their shapes, possess infinite conservation laws and exhibit phenomenon of recurrence [24,25]. Later on, it was discovered that a large class of non-linear partial differential equations exhibit soliton solutions under appropriate conditions [21]. However, eq. (1) in two dimension representing a critical case [26] is not integrable. Nonetheless, very special analytical solutions of two dimensional NLSE, called Townes solitons have been reported by Malkin [27]. Moreover, self-focusing in eq. (1) being a local phenomenon, can not be accurately captured by global estimates. Despite considerable progress, the present theory of critical self-focusing is still far from complete [26]. Some aspects of this genuinely non-linear process can be investigated by considering numerical or approximate analytical methods. We adopt here the latter approach in this paper, using a powerful variational method that has been used recently in several similar investigations [14–16,28,29]. Equation (1) can be reformulated into a variational problem corresponding to a Lagrangian L so as to make $\delta L/\delta z = 0$ equivalent to eq. (1), viz.

$$L = \left|\frac{\partial E}{\partial x}\right|^2 + \left|\frac{\partial E}{\partial y}\right|^2 - \iota k\left(E\frac{\partial E^*}{\partial z} - E^*\frac{\partial E}{\partial z}\right) - \frac{n_2}{n_0}k^2|E|^4. \tag{4}$$

Thus, the solution to the variational problem

$$\delta \iiint L\mathrm{d}x\mathrm{d}y\mathrm{d}z = 0 \tag{5}$$

also leads to the solution of the non-linear Schrödinger eq. (1). Using the ansatz, with expression (3) as a trial function into the Lagrangian L of eq. (4), we can integrate L to obtain

$$\langle L\rangle = \int_{-\infty}^{\infty} L\mathrm{d}x\mathrm{d}y. \tag{6}$$

We have arrived at a reduced variational problem. We carry out the integration in eq. (6) to get

$$\langle L\rangle = \langle L_0\rangle + \langle L_1\rangle, \tag{7}$$

where

$$\langle L_0\rangle = \frac{\pi}{16}|\varepsilon|^2\left(\frac{\beta}{\alpha} + \frac{\alpha}{\beta}\right) + \frac{k^2}{64}\varepsilon^2\alpha^2\beta\pi\frac{\mathrm{d}^2\alpha}{\mathrm{d}z^2} + \frac{k^2}{64}|\varepsilon|^2\beta^2\alpha\pi\frac{\mathrm{d}^2\beta}{\mathrm{d}z^2}$$
$$+\iota\frac{\pi}{8}k\alpha\beta\left(\varepsilon^*\frac{\partial\varepsilon}{\partial z} - \varepsilon\frac{\partial\varepsilon^*}{\partial z}\right), \tag{8}$$

$$\langle L_1\rangle = -\frac{1}{16}\frac{n_2}{n_0}\pi|\varepsilon|^4\alpha\beta k^2. \tag{9}$$

Variation with respect to $\varepsilon, \varepsilon^*, \alpha, \beta$ etc and using procedure of [14], we arrive at the following equations for α, β, ϕ :

$$\frac{\mathrm{d}^2\alpha}{\mathrm{d}\eta^2} = \frac{8}{k^4\alpha^2\beta}\left(\frac{\beta}{\alpha} - p\right), \tag{10}$$

$$\frac{\mathrm{d}^2\beta}{\mathrm{d}\eta^2} = \frac{8}{k^4\beta^2\alpha}\left(\frac{\alpha}{\beta} - p\right), \tag{11}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}\eta} = \frac{1}{k^2\alpha\beta}\left[\frac{3}{4}\left(\frac{\beta}{\alpha} + \frac{\alpha}{\beta}\right) - 2p\right], \tag{12}$$

where

$$p = \frac{n_2}{2n_0}\alpha_0\beta_0 k^2 {\varepsilon_0}^2 \tag{13}$$

and

$${\varepsilon_0}^2\alpha_0\beta_0 = \varepsilon^2\alpha\beta. \tag{14}$$

Subscripts here denote the value at $z = 0$ and $\eta = kz$ is the dimensionless variable. Equations (10) and (11) can be manipulated algebrically in the following form :

$$\frac{1}{2}\frac{\mathrm{d}^2\rho^2}{\mathrm{d}\eta^2} = \frac{8}{k^4\alpha\beta}\left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha} - 2p\right) + \left(\frac{\mathrm{d}\alpha}{\mathrm{d}z}\right)^2 + \left(\frac{\mathrm{d}\beta}{\mathrm{d}z}\right)^2, \tag{15}$$

where

$$\rho^2 = \alpha^2 + \beta^2. \tag{16}$$

For an initially parallel beam, initial conditions are

$$\frac{\mathrm{d}\alpha_0}{\mathrm{d}z} = \frac{\mathrm{d}\beta_0}{\mathrm{d}z} = 0 \tag{17}$$

implies

$$\left.\frac{\partial\rho^2}{\partial\eta}\right|_{\eta=0} = 0. \tag{18}$$

A necessary condition for stationary self-trapping is

$$\left.\frac{\mathrm{d}^2\rho^2}{\mathrm{d}\eta^2}\right|_{\eta=0} = 0. \tag{19}$$

This leads to

$$p_{\mathrm{cr}} = \frac{1}{2}\left(\frac{\beta_0}{\alpha_0} + \frac{\alpha_0}{\beta_0}\right). \tag{20}$$

3. Discussion

We have used the Runge-Kutta method to obtain the numerical solution of the equations (10), (11) and (12). While the equations (10) and (11) are ordinary differential equations describing the evolution of the beam widths, eq. (12) describes the dynamics of longitudinal phase during the beam propagation. The numerical method has been used as it is very difficult to obtain analytically the solution of these coupled equations and thereby predict the development of beam as it propagates. We, therefore, resort to numerical computation to study beam dynamics. Since k lies in the infrared region, gaseous discharged lasers based on CO_2 pumped excitation with following parameters have been chosen [30–32]

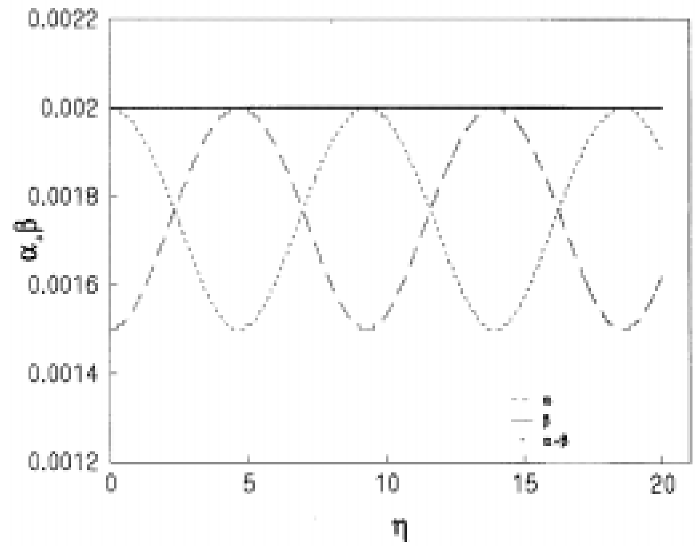

Figure 1. Variation of beam widths α and β with normalized distance of propagation η for $p = 1.04166$ with $\alpha_0 = 0.002$ cm and $\beta_0 = 0.0015$ cm.

$$\alpha_0 = 0.002\,\text{cm},\ \beta_0 = 0.0015\,\text{cm},\ k = 1.4 \times 10^3\,\text{cm}^{-1}.$$

The results are shown in the form of graphs. The parameter p is the measure of the non-linearity. It is seen that the phenomenon of cross-focusing is observed only for certain values of p. Figure 1 shows such behaviour where oscillatory self-focusing of α and β is observed for $p = 1.04166$, which is equal to the calculated value of p_{cr} obtained from expression (20). As observed in figure 1, stationary self-trapping is not possible for ellipitic beam in a Kerr-medium as α and β keep on oscillating indefinitely. On comparing these results with cylindrically symmetric Gaussian beam by substituting $\alpha = \beta$, it is observed that stationary self-trapping is possible. This is displayed in figure 1 by the horizontal line which indicates that beam propagates without convergence or divergence in a self-made waveguide. As evident from expression (20), this situation corresponds to $p_{\text{cr}} = 1$. Equations (10) and (11) further imply uniform and equal values of beam width parameters through the transit in non-linear medium. Elliptical beam however, requires slightly higher critical value of p. For $p < p_{\text{cr}}$, there is an overall expansion of the beam with non-monotonic evolution of the width parameters (figure 2). On the other hand, for $p > p_{\text{cr}}$, the overall collapse of the beam takes place as is apparent from the non-monotonic evolution of beam width parameters (figure 3). Equation (12) represents the phase change with dimensionless distance of propagation. This equation takes into account spatial diffraction as well as non-linearity and represents the wave number shift resulting from interplay between these processes. Earlier, it has been pointed out that spatial diffraction may result in spectral features qualitatively and quantitatively different from those of conventional phase modulation [9]. Presently in our case, as a result of dependence of $\mathrm{d}\phi/\mathrm{d}\eta$ on $|\epsilon|^2$, we find that longitudinal phase increases whenever p decreases. Figure 4 shows variation of longitudinal phase $\phi(\eta)$ for four different values of p. The phase may be positive or negative depending on the value of p. However, the regularized phase, ϕ_{reg} which is defined as

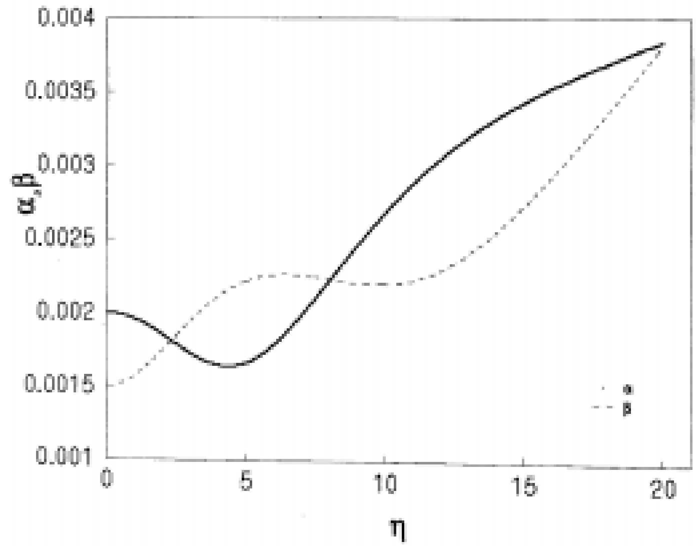

Figure 2. Variation of α and β with η in the diffraction regime for $p = 1.0$ corresponding to $p < p_{\rm cr}$. The initial widths are same as mentioned in figure 1.

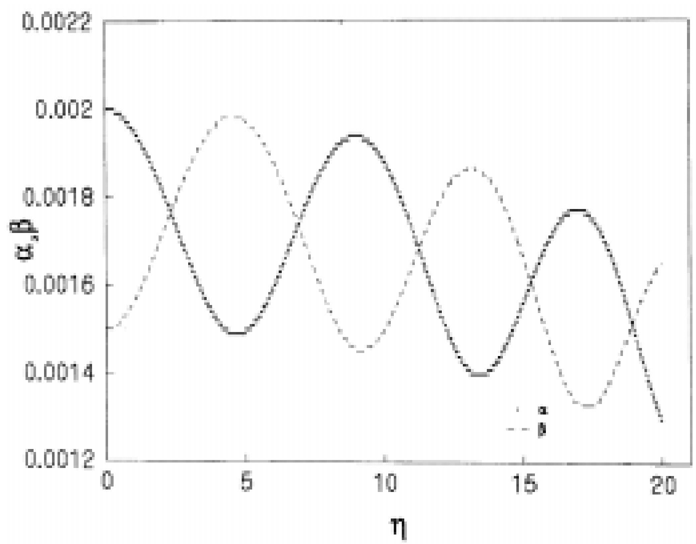

Figure 3. Beam widths α and β as a function of η showing the collapse with oscillations for $p = 1.045 > p_{\rm cr}$. The other parameters are the same as those mentioned in figure 1.

$$\phi_{\rm reg} = \phi(\eta) - \phi(\eta)|_{p=0}$$

is always negative (figure 5). This confirms the finding of Karlsson *et al* [16] and is contrary to the results of Manassah *et al* [9]. The latter predicted that under certain conditions regularized phase could change sign with distance of propagation. This is a physically unacceptable result as wave number shift with positive or negative value, may imply results contrary to experimental observations and may further lead to the erroneous conclusion

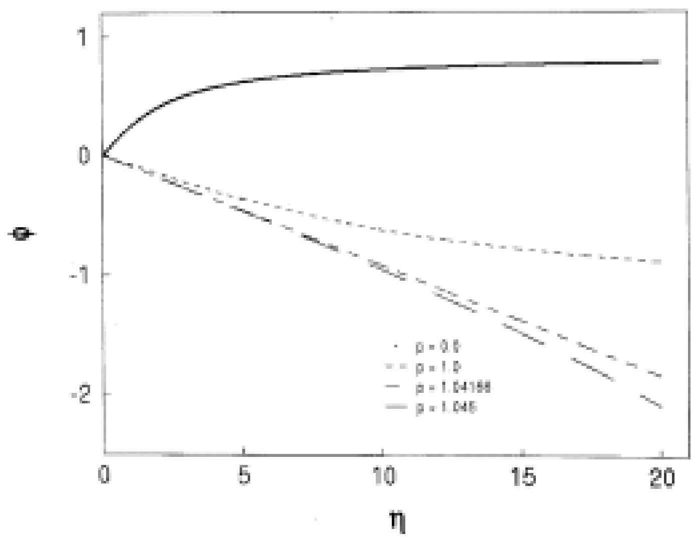

Figure 4. Plot of longitudinal phase $\phi(\eta)$ for four values of p. The initial values of α and β are as those mentioned in figure 1.

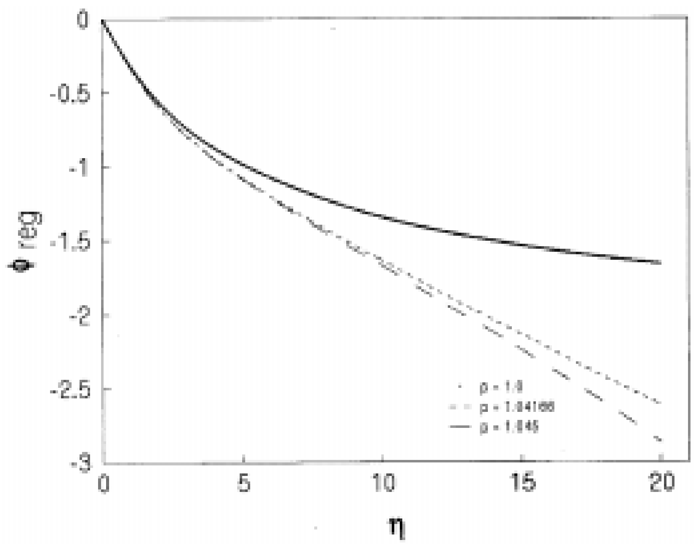

Figure 5. Plot illustrating the dependence of $\phi_{\mathrm{reg}}(\eta)$ on η for three different values of p. The initial values of α and β are the same as in figure 1.

that blue may lead the red in the supercontinuum. This inconsistent result of Manassah *et al* [9] arises due to inherent shortcoming of paraxial ray approximation as pointed by Karlsson *et al* [16]. For plane Gaussian beam, Karlsson *et al* [16] also obtained analytical results with $p = 2/3$ predicting the value of $\phi = 0$. However, no such precise value is possible for the case of elliptic Gaussian beam.

4. Conclusion

We have shown that elliptic Gaussian beam together with variational approach exhibits no stationary self-trapping. Further, the parameter p which is a function of intensity as well as non-linear refractive index, determines the fate of the beam during the propagation. It is also observed that regularized phase is always negative.

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