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Rotating cylindrically symmetric Kaluza-Klein fluid model

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Abstract. Kaluza-Klein field equations for stationary cylindrically symmetric fluid models in standard Einstein theory are formulated and a set of physically viable solutions is reported. This set is believed to be the first such Kaluza-Klein solutions and it includes the Kaluza-Klein counterpart of Davidson's solution describing spacetime of a perfect fluid in rigid rotation about a regular axis.

Keywords. Rotating fluids; Kaluza-Klein model.

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1. Introduction

Relativistic Kaluza-Klein (KK) type theories (Applequist *et al* [1]) in five and more dimensional spacetimes have been regarded as highly significant in many contexts, especially in the quest of unification of fundamental interactions in physics. According to Kaluza-Klein theory the spacetime of the universe is expected to be five dimensional in the early stages of its evolution, the extra dimension undergoing contraction as the universe underwent expansion. The formation of large scale structures in the universe indicates that the universe must have passed through epoch of inhomogeneity during the early stages of its evolution. Several relativistic cylindrically symmetric, non-static, inhomogeneous KK fluid models admitting dimensional reduction have been reported by Patel and Dadhich [2,3].

After Godel [4] gave relativistic model of a rotating dust universe, the study of rotating fluids in the context of general relativity received considerable attention. Krasinski [5–7] and Herlt [8] have given solutions of relativistic field equations with axisymmetric rotating fluid sources for suitably chosen four-flow velocities for the fluid. Davidson [9] reported a one parameter family of solutions for a fluid admitting the equation of state $p = (2/3)\rho$, rotating about a singular axis. Since stationary Kaluza-Klein perfect fluid models in standard Einstein theory are not available in literature, it is worthwhile to obtain and study such solutions to investigate the effects of dimensionality on the various physical parameters. In this paper we have formulated the KK-field equations for cylindrically symmetric rotating distributions of perfect fluid in §2. A method for obtaining their solutions is discussed in §3. Some specific solutions of physical relevance along with their features of physical interest are discussed in §4 and some general aspects of these solutions are reported in §5.

2. The metric and field equations

A general stationary cylindrically symmetric five dimensional spacetime is represented by the metric

$$\mathrm{d}s^2 = D^2(\mathrm{d}t + H\mathrm{d}\phi)^2 - A^2\mathrm{d}r^2 - B^2\mathrm{d}z^2 - r^2C^2\mathrm{d}\phi^2 - E^2\mathrm{d}\psi^2, \tag{1}$$

where A, B, C, D and H are functions of the radial coordinate r only, r, z and ϕ are cylindrical polar coordinates, t is the time coordinate and ψ denotes the coordinate corresponding to the extra spatial dimension. Expressed with respect to the pentad

$$\theta^1 = A\mathrm{d}r, \quad \theta^2 = B\mathrm{d}z, \quad \theta^3 = rC\mathrm{d}\phi, \quad \theta^4 = E\mathrm{d}\psi, \quad \theta^5 = D(\mathrm{d}t + H\mathrm{d}\phi), \tag{2}$$

the metric (1) has the form

$$\mathrm{d}s^2 = (\theta^5)^2 - (\theta^1)^2 - (\theta^2)^2 - (\theta^3)^2 - (\theta^4)^2. \tag{3}$$

Here and in what follows the bracketed indices will indicate the components in the above pentad frame. It is a routine matter to find the pentad components $\mathbf{R}_{(ab)}$ of the Ricci tensor for the metric (1) in the frame of pentad (2).

The surviving components of $\mathbf{R}_{(ab)}$ are listed below for ready reference:

$$A^2\mathbf{R}_{(11)} = \left[\frac{B''}{B} + \frac{C''}{C} + \frac{D''}{D} + \frac{E''}{E} + \frac{2C'}{rC} - \frac{A'}{rA} - \frac{D^2H'^2}{2r^2C^2} - \frac{A'}{A}\left(\frac{B'}{B} + \frac{C'}{C} + \frac{D'}{D} + \frac{E'}{E}\right)\right], \tag{4}$$

$$A^2\mathbf{R}_{(22)} = \left[\frac{B''}{B} + \frac{B'}{B}\left(\frac{C'}{C} + \frac{D'}{D} + \frac{E'}{E} - \frac{A'}{A} + \frac{1}{r}\right)\right], \tag{5}$$

$$A^2\mathbf{R}_{(33)} = \left[\frac{C''}{C} + \frac{2C'}{rC} + \left(\frac{C'}{C} + \frac{1}{r}\right)\left(\frac{B'}{B} + \frac{D'}{D} + \frac{E'}{E} - \frac{A'}{A}\right) - \frac{D^2H'^2}{2r^2C^2}\right], \tag{6}$$

$$A^2\mathbf{R}_{(44)} = \left[\frac{E''}{E} + \frac{E'}{E}\left(\frac{B'}{B} + \frac{C'}{C} + \frac{D'}{D} - \frac{A'}{A} + \frac{1}{r}\right)\right], \tag{7}$$

$$A^2\mathbf{R}_{(55)} = -\left[\frac{D''}{D} + \frac{D'}{D}\left(\frac{B'}{B} + \frac{C'}{C} + \frac{E'}{E} - \frac{A'}{A} + \frac{1}{r}\right) + \frac{D^2H'^2}{2r^2C^2}\right], \tag{8}$$

$$A^2\mathbf{R}_{(35)} = -\frac{D}{2rC}\left[H'' + \frac{3H'D'}{D} - H'\left(\frac{A'}{A} + \frac{C'}{C} - \frac{B'}{B} - \frac{E'}{E} + \frac{1}{r}\right)\right]. \tag{9}$$

Here and in what follows an overhead prime is used to denote a differentiation with respect to r.

If the metric (1) is to represent the spacetime of a stationary perfect fluid rotating about the regular axis $r = 0$, the metric coefficients will be related to the dynamical variables through the Einstein field equations which are adopted in the form

$$\mathbf{R}_{(ab)} = -8\pi\left[(\rho + p)v_{(a)}v_{(b)} - \frac{1}{3}(\rho - p)g_{(ab)}\right], \tag{10}$$

in the pentad notation using the system of units rendering $c = G = 1$. Here ρ, p respectively denote the matter density and the fluid pressure and v_a denote components in the pentad frame of the unit time-like flow vector v^i of the fluid satisfying $v^i v_i = 1$. It is convenient to adopt a coordinate system comoving with the observer so that

$$v^a = (0, 0, 0, 0, 1) \tag{11}$$

and the field equations (10) imply the following system of equations:

$$\mathbf{R}_{(11)} = \mathbf{R}_{(22)} = \mathbf{R}_{(33)} = \mathbf{R}_{(44)} = -\frac{8\pi}{3}(\rho - p), \tag{12}$$

$$\mathbf{R}_{(55)} = -\frac{16\pi}{3}(\rho + 2p), \tag{13}$$

$$\mathbf{R}_{(35)} = 0, \tag{14}$$

where $\mathbf{R}_{(ab)}$ are as given in (4)–(9). The field equations constitute a system of six equations relating the six metric coefficients A, B, C, D, E and H and the two physical parameters ρ and p of the fluid. The system of equations (12) and (13) leads to the consistency conditions

$$\mathbf{R}_{(11)} = \mathbf{R}_{(22)} = \mathbf{R}_{(33)} = \mathbf{R}_{(44)} \tag{15}$$

together with the following expressions for ρ and p

$$16\pi\rho = -[\mathbf{R}_{(55)} + 4\mathbf{R}_{(22)}], \tag{16}$$

$$16\pi p = [2\mathbf{R}_{(22)} - \mathbf{R}_{(55)}]. \tag{17}$$

The rotation tensor

$$\omega_{ij} = (1/2)(v_{i,j} - v_{j,i}) - (1/2)(v_i f_j - v_j f_i) \tag{18}$$

for the flow field ( ) is found to have the magnitude

$$\Omega = (\omega_{ij}\omega^{ij})^{1/2} = \frac{H'D}{2rAC}, \tag{19}$$

where f_i denotes the acceleration vector which for the rotating fluid under consideration has the expression

$$f_i = \left(-\frac{D'}{D}, 0, 0, 0, 0\right). \tag{20}$$

Herein the coordinates r, z, ϕ, ψ and t are conveniently labelled as x^1, x^2, x^3, x^4 and x^5 respectively. In the next section we shall discuss a method for obtaining solutions of the above system of equations.

3. Solutions of field equations

Davidson obtained a solution of the relativistic system of field equations for a perfect fluid in rigid rotation about a regular axis. The fluid content of the spacetime of the Davidson's solution

$$\mathrm{d}s^2 = (1+k^2r^2)^{1/2}\mathrm{d}t^2 + \sqrt{23/2}kr^2(1+k^2r^2)^{1/2}\mathrm{d}\phi\mathrm{d}t$$
$$-r^2(1+k^2r^2)^{1/2}\left(1-\frac{15}{8}k^2r^2\right)\mathrm{d}\phi^2 - (1+k^2r^2)^{-3/8}(\mathrm{d}r^2+\mathrm{d}z^2), \quad (21)$$

where k is an arbitrary constant, has an equation of state $p = (2/3)\rho$. Davidson solution suggests the possibility that the system of KK-field equations (12)–(14) can be solved by assuming the following form for the metric coefficients A, B, C, D, E and H

$$A = (1+k^2r^2)^a,\ B = (1+k^2r^2)^b,\ C = (1+k^2r^2)^c,$$
$$D = (1+k^2r^2)^d,\ E = (1+k^2r^2)^e, \quad (22)$$

where a, b, c, d, e and k are constants. Expressions (22) ensure the regularity of the metric for all finite r.

Equation $\mathbf{R}_{(22)} = \mathbf{R}_{(44)}$ in (15) is then satisfied if and only if

$$b = e. \quad (23)$$

Subsequently $\mathbf{R}_{(35)} = 0$ i.e., (14) implies the two relations

$$H = \alpha r^2 \quad (24)$$

and

$$a + c = 2b + 3d, \quad (25)$$

where α is arbitrary constant of integration.

Equation $\mathbf{R}_{(11)} = \mathbf{R}_{(33)}$ contained in (15) in view of (23) and (25) reduces to the algebraic relation

$$2b^2 + 2b(1+4d) + d(1+2d) = 0. \quad (26)$$

The remaining equation $\mathbf{R}_{(33)} = \mathbf{R}_{(22)}$ contained in (15) then leads to the algebraic equations

$$c = (1+2d)/2 \quad (27)$$

and

$$a - 3c = d - (\alpha^2/k^2). \quad (28)$$

The matter density and the fluid pressure as determined by (16) and (17) are then found to have the explicit expressions

$$8\pi\rho = \frac{\alpha^2 + 2k^2(d - 4b)}{(1 + k^2r^2)^{2a+1}}, \tag{29}$$

$$8\pi p = \frac{\alpha^2 + 2k^2(d + b)}{(1 + k^2r^2)^{2a+1}}. \tag{30}$$

Thus the KK field equations are equivalent to the five algebraic relations (23)–(26) relating the seven parameters a, b, c, d, e, α and k with $H(r)$ as determined by (24). Accordingly the KK field equations will admit a two parameter family of solutions in this set-up. The stationary fluid of the family of these solutions satisfying $(2a + 1) > 0$, will have ρ and p regular everywhere and both radially decreasing outward. The rotating fluid source of this family of cylindrically symmetric solutions has the acceleration vector

$$f_i = \left(\frac{-2dk^2r}{(1 + k^2r^2)}, 0, 0, 0, 0 \right) \tag{31}$$

and magnitude of the rotation tensor

$$\Omega = \alpha(1 + k^2r^2)^{-2(b+d)}. \tag{32}$$

It is observed that the acceleration is finite, vanishing at the axis of rotation and as $r \to \infty$. The regularity of rotation is ensured for the fluid of the family of solutions satisfying $b + d \geq 0$.

We shall discuss certain specific cases of physical relevance which follow for certain particular choices of the free parameters, in the next section.

4. Specific solutions

The system of algebraic equations (23)–(26) is found to admit four simple solutions.

Solution 1. The choice

$$b = e = d = 0 \tag{33}$$

in view of the algebraic relations (23)–(26) determines

$$a = -c = -1/2, \quad \alpha^2 = 2k^2. \tag{34}$$

The spacetime of this class of solutions has the metric

$$\mathrm{d}s^2 = (\mathrm{d}t + \sqrt{2}kr^2\mathrm{d}\phi)^2 - \frac{\mathrm{d}r^2}{1 + k^2r^2} - \mathrm{d}z^2 - r^2(1 + k^2r^2)\mathrm{d}\phi^2 - \mathrm{d}\psi^2 \tag{35}$$

and it describes a stationary distribution of fluid with equation of state $p = \rho$, corresponding to a stiff fluid with uniform density and pressure

$$\rho = p = \frac{k^2}{4\pi}. \tag{36}$$

The rotation tensor also is of constant magnitude $\Omega = \alpha$ and the acceleration vector is zero vector.

Solution 2. When one chooses

$$a = b = e, \tag{37}$$

the KK field equations determine

$$a = (1 - \sqrt{5}/2), \tag{38}$$

$$c = (1 + \sqrt{5})/4, \tag{39}$$

$$d = (-1 + \sqrt{5})/4, \tag{40}$$

$$\alpha^2 = k^2(-1 + 3\sqrt{5})/2. \tag{41}$$

The rotating fluid of this family of solutions admits the equation of state $p = \mu\rho$ and has

$$8\pi(2\sqrt{5} - 3)p = 8\pi\rho = \frac{3k^2}{(1 + k^2r^2)^{3-\sqrt{5}}}. \tag{42}$$

The fluid complies with the requirements of weak energy conditions

$$\rho \geq 0, \quad p \geq 0, \quad \rho - p \geq 0 \tag{43}$$

and has both acceleration and rotation regular everywhere. The spacetime metric of this class of solutions is

$$\begin{aligned} \mathrm{d}s^2 &= (1 + k^2r^2)^{(\sqrt{5}-1)/2} \left(\mathrm{d}t + \frac{(3\sqrt{5} - 1)^{1/2}}{\sqrt{2}} r^2 k \mathrm{d}\phi \right)^2 \\ &\quad - r^2(1 + k^2r^2)^{2-\sqrt{5}}(\mathrm{d}r^2 + \mathrm{d}z^2 + \mathrm{d}\psi^2) - r^2(1 + k^2r^2)^{(\sqrt{5}+1)/2}\mathrm{d}\phi^2. \end{aligned} \tag{44}$$

This metric is five dimensional counterpart of Davidson's solution (21).

Solution 3. It follows that the KK field equations (23)–(26) are satisfied when

$$a = 0,\ b = e = \frac{\sqrt{2} - \sqrt{5}}{4\sqrt{2}},\ c = \frac{2\sqrt{2} + \sqrt{5}}{4\sqrt{2}},\ d = \frac{\sqrt{5}}{4\sqrt{2}},\ \alpha^2 = [(3 + \sqrt{5})/2]k^2. \tag{45}$$

The rotating fluid of this solution has the equation of state $p = \mu\rho$ with

$$0 \leq \mu = \frac{5\sqrt{2} + \sqrt{10} - \sqrt{5}}{5\sqrt{5} + \sqrt{10} - \sqrt{2}} \leq 1 \tag{46}$$

and matter density

$$16\pi\rho = \frac{k^2(5\sqrt{5}+\sqrt{10}-\sqrt{2})}{\sqrt{2}(1+k^2r^2)} \tag{47}$$

which vanishes only at infinity and is finite everywhere. The rotating fluid complies with the requirements of weak energy conditions stated earlier. All the kinematical parameters also have regular behaviour.

Solution 4. The algebraic relations (23)–(26) are found to admit the solution

$$a = -3/2, \quad b = c = e = 0, \quad d = -1/2, \quad \alpha^2 = k^2, \tag{48}$$

which represents an asymptotically flat five dimensional empty spacetime with metric

$$\begin{aligned} \mathrm{d}s^2 = {} & (1+k^2r^2)^{-1}(\mathrm{d}t + kr^2\mathrm{d}\phi)^2 - (1+k^2r^2)^{-3}\mathrm{d}r^2 \\ & -(\mathrm{d}z^2 + r^2\mathrm{d}\phi^2 + \mathrm{d}\psi^2). \end{aligned} \tag{49}$$

Solution 5. When

$$a = (-1/2), \quad b = e = c = -d = (1/4), \quad \alpha^2 = k^2, \tag{50}$$

the KK field equations are all satisfied and the metric

$$\begin{aligned} \mathrm{d}s^2 = {} & (1+k^2r^2)^{-1/2}(\mathrm{d}r + kr^2\mathrm{d}\phi)^2 - (1+k^2r^2)^{-1}\mathrm{d}r^2 \\ & -(1+k^2r^2)^{1/2}(\mathrm{d}r^2 + r^2\mathrm{d}\phi^2 + \mathrm{d}\psi^2) \end{aligned} \tag{51}$$

represents a five dimensional spacetime of a cylindrically symmetric stationary fluid with constant density and pressure related by the equation of state $\rho + p = 0$. If one sets $\Lambda = -(3/2)K^2$, the above metric represents a five dimensional solution of the field equations $\mathbf{R}_{ij} = \Lambda g_{ij}$, where Λ denotes the cosmological constant.

5. Discussion

The stationary fluid of the two parameter family of solutions of the KK field equations of §3, will in general have matter density and fluid pressure related by a physically respectable equation of state $p = \mu\rho$ where

$$\mu = \frac{1+b+2d}{1-5b+2d}\,, \tag{52}$$

satisfying the requirements $0 \leq \mu \leq 1$. We choose d and k^2 as free parameters. The algebraic relations (23)–(26) implied by the KK field equations then determine the remaining parameters as follows:

$$b = e = -(1+4d \pm \sqrt{12d^2+6d+1})/2, \tag{53}$$

$$c = (1 + 2d)/2, \quad a = 2(b + d) - 1/2, \quad \alpha^2 = 2k^2(d - b + 1). \tag{54}$$

The requirement $\mu \leq 1$ is fulfilled if and only if $b \geq 0$. The rotation given in (32) will be finite and regular everywhere if $0 < (b + d)$. In view of $b < 0$, this implies $d > -b > 0$, the regularity of p and ρ will be ensured everywhere if $2a + 1 \geq 0$. It is observed that if $b + d > 0$, the above condition is always fulfilled. Subsequently

$$2b + [-(1 + 4d) + \sqrt{12d^2 + 6d + 1}] < 0. \tag{55}$$

When $b = 0$, the solution represents the spacetime of a stationary fluid.

Since stationary Kaluza-Klein perfect fluid solutions within standard Einstein theory are not found to be reported in literature the spacetime metrics discussed here constitute first such set of solutions and it includes KK counterparts of some known relativistic stationary fluid solutions.

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