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Higher order corrections in perturbative quantum chromodynamics

W L VAN NEERVEN
Instituut-Lorentz, Universiteit Leiden, PO Box 9506, 2300 RA Leiden, The Netherlands

Abstract. We present some techniques which have been developed recently or in the recent past to compute Feynman graphs beyond one-loop order. These techniques are useful to compute the three-loop splitting functions in QCD and to obtain the complete second order QED corrections to Bhabha scattering.

Keywords. Quantum chromodynamics; perturbation theory; higher order.

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1. Introduction

Since the discovery of asymptotic freedom in non-abelian gauge field theories, like quantum chromodynamics (QCD), many perturbative calculations have been performed to hadron–hadron and semi-leptonic processes. Most reactions are computed up to next-to-leading order (NLO) only, except for some semi-leptonic processes. The latter is due to the simplicity of the Born approximation where the basic process is given by the interaction of the intermediate vector boson with a quark. Higher order corrections to the other reactions are still missing because of the very complicated Feynman and phase space integrals which arise in the calculations. Here we will discuss some new methods which may enable us to compute the QCD and also QED corrections beyond the NLO level.

2. Asymptotic expansions

In this section we concentrate on the computation of the coefficient functions and anomalous dimensions which show up in deep inelastic lepton–hadron scattering. The relevant quantity in this process is the structure function defined by

$$F(x,Q^2)=\frac{1}{4\pi}\int \mathrm{d}^4z\,\mathrm{e}^{iq.z}\langle p\mid[J(z),J(0)]\mid p\rangle \quad q^2=-Q^2<0 \quad x=\frac{Q^2}{2\nu}\,. \tag{1}$$

In the commutator above $J(z)$ represents the electro-weak current where we have suppressed Lorentz indices for convenience. In the Bjorken limit i.e. $Q^2\to\infty$ and $x=$ fixed the light cone region dominates the integrand so that one can make an operator product expansion (OPE)

$$[J(z), J(0)] \underset{z^2\sim 0}{=} \sum_N C^N(z^2\mu^2) O^N(\mu^2, 0)\,. \tag{2}$$

This expansion also holds for the time ordered product corresponding to forward Compton scattering

$$T(x, Q^2) = \frac{i}{4\pi^2}\int \mathrm{d}^4 z\, \mathrm{e}^{iq.z}\langle p \mid [J(z), J(0)] \mid p\rangle\,,$$
$$\mathrm{Im}\, T(x, Q^2) = \pi\, F(x, Q^2)\,. \tag{3}$$

Let us assume that we can write an unsubtracted dispersion relation in the variable $\nu = p{\cdot}q$,

$$T(\nu, Q^2) = \int_{Q^2/2}^{\infty} \mathrm{d}\nu' \frac{F(\nu', Q^2)}{\nu' - \nu}\,, \tag{4}$$

so that after substitution of the variables $\nu = \frac{Q^2}{2x}$, $\nu' = \frac{Q^2}{2x'}$ we can derive the following relation

$$T(\nu, Q^2) = x\int_0^1 \frac{\mathrm{d}x'}{x'}\frac{F(x', Q^2)}{x - x'}$$
$$= \sum_{N=0}^{\infty}\left(\frac{2p\cdot q}{Q^2}\right)^N \int_0^1 \mathrm{d}x'\, x'^{N-1} F(x', Q^2)$$
$$= \sum_{N=0}^{\infty}\left(\frac{2p\cdot q}{Q^2}\right)^N A^{(N)}(\mu^2)\mathcal{C}^{(N)}\left(\frac{Q^2}{\mu^2}\right)\,, \tag{5}$$

where the operator matrix element and the coefficient function are given by

$$A^{(N)}(\mu^2) = \langle p \mid O^N(\mu^2, 0) \mid p\rangle \tag{6}$$

and

$$\mathcal{C}^{(N)}\left(\frac{Q^2}{\mu^2}\right) = \int \mathrm{d}^4 z\, \mathrm{e}^{iq.z} C^N(z^2\mu^2)\,, \tag{7}$$

respectively. Both quantities satisfy a renormalization group equation

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + \gamma^{(N)}(g)\right] A^{(N)}(\mu^2) = 0\,,$$
$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} - \gamma^{(N)}(g)\right] \mathcal{C}^{(N)}\left(\frac{Q^2}{\mu^2}\right) = 0\,. \tag{8}$$

In the equations above the quantities $\beta(g)$ and $\gamma^{(N)}(g)$ represent the beta-function and the anomalous dimension respectively. The latter determine the Q^2-evolution of the structure function $F(x,Q^2)$ which can be measured in experiment and provides us with one of the tests of perturbative QCD. If one computes the coefficient functions in the conventional way one encounters phase space integrals and loop integrals (see e.g. [1]). However if one wants to compute the coefficient function beyond order g^4 it is more convenient to try another method which is explained in [2]. Taking ϕ_6^3-theory as an example one can expand the propagator in $T(x,Q^2)$ (3) given by $1/(k-p)^2$ as follows:

$$T(x,Q^2) \equiv T = (-ig)^2 \int \frac{\mathrm{d}^n k}{(2\pi)^n} \frac{i^3}{(k^2)^2(k-p)^2(k+q)^2}$$
$$= -ig^2 \sum_{N=0}^{\infty} \int \frac{\mathrm{d}^n k}{(2\pi)^n} \frac{(2k\cdot p)^N}{(k^2)^{3+N}(k+q)^2}\,, \tag{9}$$

where we have used n-dimensional regularization to regularize the collinear (C) and ultraviolet (UV) singularities. The asymptotic expansion of the propagator converts the Compton amplitude into a self energy type of integral. The latter is much easier to compute than expressions for box- or triangle-graphs. Another feature is that this asymptotic expansion transforms the C-divergence in T at $n=6$ into an UV singularity appearing in the expression for the self energy integral. Therefore the operator matrix element in eq. (5), containing the collinear divergence denoted by $1/\varepsilon_C$, will be replaced by the corresponding operator renormalization constant Z_{O^N},

$$Q^2 T = \sum_{N=0}^{\infty} \left(\frac{2p\cdot q}{Q^2}\right)^N A^{(N)}\left(\frac{1}{\varepsilon_C},\mu^2\right) \mathcal{C}^{(N)}\left(\frac{Q^2}{\mu^2}\right)$$
$$\to \quad \sum_{N=0}^{\infty} \left(\frac{2p\cdot q}{Q^2}\right)^N Z_{O^N}\left(\frac{1}{\varepsilon_{UV}},\mu^2\right) \mathcal{C}^{(N)}\left(\frac{Q^2}{\mu^2}\right)\,, \tag{10}$$

where Z_{O^N} depends on the UV singularity represented by $1/\varepsilon_{\mathrm{UV}}$. This transformation leaves the coefficient function unaltered. A straightforward calculation yields

$$Q^2 T = g^2 \frac{\pi^{n/2}}{(2\pi)^n} \frac{\Gamma(n/2-1)\Gamma(n/2-3)}{\Gamma(n-4)}$$
$$\times \sum_{N=0}^{\infty} \left(\frac{2p\cdot q}{Q^2}\right)^N \frac{\Gamma(4+N-n/2)}{\Gamma(3+N)} \left(\frac{Q^2}{\mu^2}\right)^{n/2-3}\,. \tag{11}$$

One can expand the gamma-functions above around $\varepsilon = n-6$ and obtain

$$Q^2 T = \frac{g^2}{64\pi^3} \sum_{N=0}^{\infty} \left(\frac{2p\cdot q}{Q^2}\right)^N \left[\frac{1}{N+1} - \frac{1}{N+2}\right]$$
$$\times \left[\frac{2}{\varepsilon} + \gamma_E - \ln 4\pi + \ln\frac{Q^2}{\mu^2} - 1 - \sum_{k=1}^{N}\frac{1}{k}\right]\,. \tag{12}$$

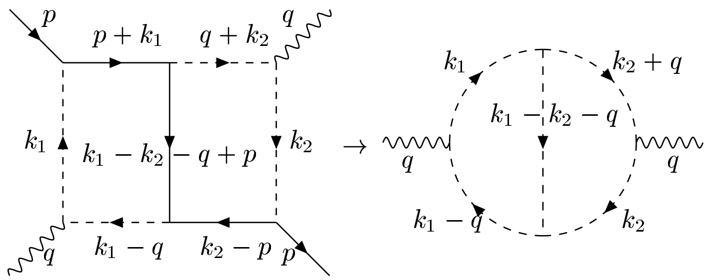

Figure 1. Reduction of a two-loop Compton graph into a self energy diagram.

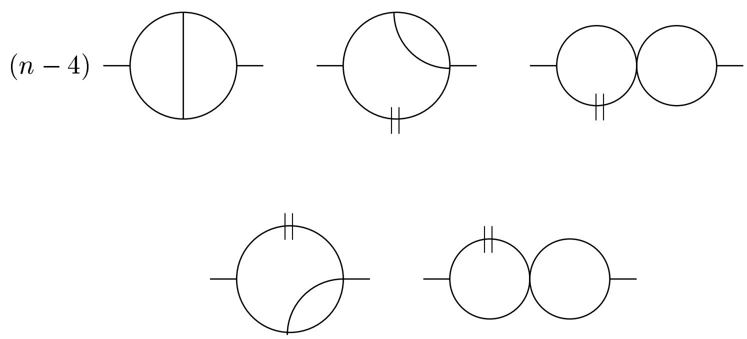

Figure 2. Diagrammatic representation of eq. (16). The symbol || means that the corresponding propagator appears twice.

In order g^2, eq. (10) can be expressed as

$$Q^2 T = \frac{g^2}{64\pi^3} \sum_{N=0}^{\infty} \left[\frac{\gamma_O^{(N)}}{2} \left\{ \frac{2}{\varepsilon} + \gamma_E - \ln 4\pi + \ln \frac{Q^2}{\mu^2} \right\} + c_q^{(N)} \right] , \tag{13}$$

so that one can read $\gamma_O^{(N)}$ and $c_q^{(N)}$. If the method is extended up to order g^4 one encounters graphs where at least three propagators, carrying the momentum p (see the solid line in figure 1), have to be expanded
This procedure leads to triple sums which are very difficult to unravel

$$\sum_i \sum_j \sum_l \left(\frac{2p \cdot k_1}{k_1^2} \right)^i \left(\frac{2p \cdot k_2}{k_2^2} \right)^j \left(\frac{2p \cdot (k_1 - k_2 - q)}{(k_1 - k_2 - q)^2} \right)^l . \tag{14}$$

To avoid this one has to remove one or more propagators, carrying the momentum p, before doing the expansion. First one can adopt the method

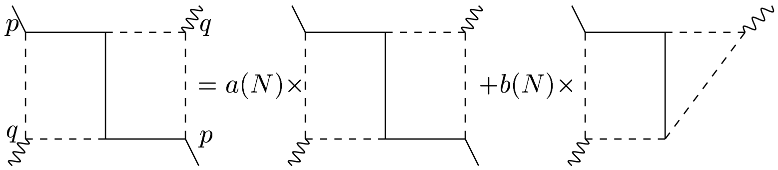

Figure 3. Difference equations.

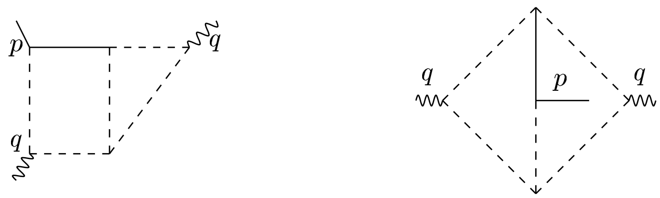

Figure 4. Building blocks.

1. *Integration by parts*

   This trick has been successfully applied to compute self energy graphs [3]. An example is

$$I = \int \frac{\mathrm{d}^n k_1}{(2\pi)^n} \int \frac{\mathrm{d}^n k_2}{(2\pi)^n} \frac{1}{k_1^2 k_2^2 (k_1+q)^2 (k_2+q)^2 (k_1-k_2)^2} \,. \tag{15}$$

   The integral above appears in the partial integration with respect to the momentum k_1 of the expression below (see figure 2)

$$0 = \int \frac{\mathrm{d}^n k_1}{(2\pi)^n} \int \frac{\mathrm{d}^n k_2}{(2\pi)^n} \frac{\partial}{\partial k_{1,\mu}} \frac{(k_1-k_2)_\mu}{k_1^2 k_2^2 (k_1+q)^2 (k_2+q)^2 (k_1-k_2)^2} \,. \tag{16}$$

   Hence I in eq. (15) can be reduced to a set of integrals each containing one propagator less than in the original expression.

2. *Recursion relations (difference equations)*

   Another method is developed in [2]. Using the method of 'difference equations' one is able to knock out one propagator (see the example in figure 3). One can show that all graphs can be reduced to two building blocks only (see figure 4). The latter contain one propagator carrying the momentum p so that one only obtains a single sum which is easy to perform. At this moment it is not clear whether the method can be extended beyond order g^4 (two-loop).

In higher order it is only possible to obtain the integral over the structure function in eq. (5) for finite moments N (see [4]). This is achieved by

$$\frac{\partial^N T}{\partial p_{\mu_1}\cdots\partial p_{\mu_N}}\Big|_{p_{\mu_i}=0} = 2^N\, N!\frac{q_{\mu_1}\cdots q_{\mu_N}}{(Q^2)^N} Z_{O^N}\left(\frac{1}{\varepsilon_{UV}},\mu^2\right)\mathcal{C}^{(N)}\left(\frac{Q^2}{\mu^2}\right). \tag{17}$$

In our example eq. (9) this procedure leads to the result

$$\frac{\partial^N T}{\partial p_{\mu_1}\cdots\partial p_{\mu_N}}\Big|_{p_{\mu_i}=0} = 2^N\, N!(-ig)^2\int\frac{\mathrm{d}^n k}{(2\pi)^n}\frac{k_{\mu_1}\cdots k_{\mu_N}}{(k^2)^{N+2}(k+q)^2} \tag{18}$$

which is again of the self-energy type.

3. Mellin–Barnes techniques

A method which turns out to be very useful to compute three-point and four-point functions even up to two-loop level is given by the Mellin–Barnes technique [5]. A simple Mellin–Barnes transformation takes the form

$$(A_1+A_2)^{-\nu} = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} A_1^{\sigma}A_2^{-\nu-\sigma}\frac{\Gamma(-\sigma)\Gamma(\nu+\sigma)}{\Gamma(\nu)}\,, \tag{19}$$

with $|\arg(A_1)-\arg(A_2)|<\pi$ and the poles in σ are located on the real axis. To illustrate this technique we take as an example the three-point function where the propagators are raised to an arbitrary power

$$\begin{aligned} J(n;\nu_1,\nu_2,\nu_3) &= \int \mathrm{d}^n k\frac{1}{[(q_1+k)^2-m_1^2]^{\nu_1}[(q_2+k)^2-m_2^2]^{\nu_2}[(q_3+k)^2-m_3^2]^{\nu_3}} \\ &= i^{1-n}\pi^{n/2}\frac{\Gamma(\sum_i\nu_i-n/2)}{\prod_i\Gamma(\nu_i)}\int\prod_i \mathrm{d}\alpha_i\,\alpha_i^{\nu_i-1}\,\delta(1-\sum_j\alpha_j) \\ &\quad\times\frac{1}{[-\sum_j\alpha_j m_j^2+\alpha_2\alpha_3p_1^2+\alpha_1\alpha_3p_2^2+\alpha_1\alpha_2p_3^2]^{\sum_k\nu_k-n/2}}\,, \end{aligned} \tag{20}$$

where $p_1=q_3-q_2$, $p_2=q_1-q_3$ and $p_3=q_2-q_1$. Next we apply the double Mellin–Barnes integral transformation given by

$$\begin{aligned} \frac{1}{(X+Y+Z)^a} &= \frac{1}{Z^a}\int_{-i\infty}^{i\infty}\frac{\mathrm{d}u}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\mathrm{d}v}{2\pi i}\,\frac{\Gamma(a+u+v)\Gamma(-u)\Gamma(-v)}{\Gamma(a)} \\ &\quad \left(\frac{X}{Z}\right)^u\left(\frac{Y}{Z}\right)^v\,, \end{aligned} \tag{21}$$

so that the integral in eq. (20) can be written as

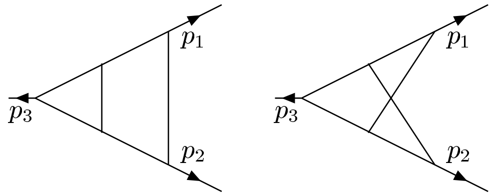

Figure 5. Two-loop triangle graphs.

$$J(n;\nu_1,\nu_2,\nu_3) = \frac{\pi^{n/2} i^{1-n}}{\prod_j \Gamma(\nu_j)} \int \prod_i \mathrm{d}\alpha_i \, \alpha_i^{\nu_i - 1} \, \delta(1 - \sum_j \alpha_j) \left(\alpha_1 \alpha_2 p_3^2\right)^{n/2 - \sum_k \nu_k}$$

$$\int_{-i\infty}^{i\infty} \frac{\mathrm{d}u}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\mathrm{d}v}{2\pi i} \Gamma\left(\sum_i \nu_i - n/2 + u + v\right)$$

$$\Gamma(-u)\Gamma(-v) \left(\frac{\alpha_3 \, p_1^2}{\alpha_1 \, p_3^2}\right)^u \left(\frac{\alpha_3 \, p_2^2}{\alpha_2 \, p_3^2}\right)^v , \tag{22}$$

where we have put for simplicity $m_i^2 = 0$. Using

$$\prod_i \int_0^1 \mathrm{d}\alpha_i \delta(1 - \sum_j \alpha_j) \alpha_1^a \alpha_2^b \alpha_3^c = \frac{\Gamma(1+a)\Gamma(1+b)\Gamma(1+c)}{\Gamma(1+a+b+c)} , \tag{23}$$

the three-point function reads

$$J(n;\nu_1,\nu_2,\nu_3) =$$

$$\pi^{n/2} i^{1-n} \int_{-i\infty}^{i\infty} \frac{\mathrm{d}u}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\mathrm{d}v}{2\pi i} \frac{\Gamma(n/2 - \nu_1 - \nu_3)\Gamma(n/2 - \nu_2 - \nu_3)\Gamma(\nu_3 + u + v)}{\prod_i \Gamma(\nu_i)\Gamma(n - \sum_j \nu_j)}$$

$$\Gamma(\sum_k \nu_k + u + v - n/2)\Gamma(-u)\Gamma(-v) \left(\frac{p_1^2}{p_3}\right)^u \left(\frac{p_2^2}{p_3}\right)^u (p_3)^{n/2 - \sum_j \nu_j} . \tag{24}$$

A special case is $\nu_1 + \nu_2 + \nu_3 = n$ (uniqueness condition). After application of the residue theorem to expression (24) one obtains

$$J(n;\nu_1,\nu_2,\nu_3) = \pi^{n/2} i^{1-n} \prod_{j=1}^{3} \frac{\Gamma(n/2 - \nu_j)}{\Gamma(\nu_j)} \left(p_j^2\right)^{\nu_j - n/2} . \tag{25}$$

The Mellin–Barnes techniques has been successfully applied to compute two-loop graphs. Examples are the three-point functions in figure 5 and the four point functions in figure 6 which are computed in [5] for $p_i^2 \neq 0$. Recent progress has been made for the two-loop box graphs in figure 6 which has been computed for all external momenta on-shell ($p_i^2 = 0$) [6,7]. This is very useful for applications to the second order corrections in Bhabha scattering and in di-jet production.

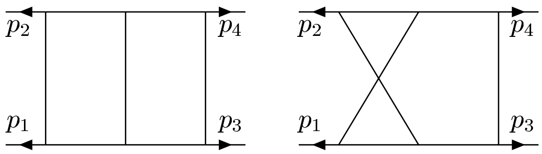

Figure 6. Two-loop box graphs.

4. Negative dimension approach

This method has been recently developed in [8,9] to express one-loop integrals containing different masses into special functions. To illustrate the technique we take the two-point function as an example. The latter is given by the integral

$$I_2^n(\nu_1,\nu_2,q^2,M_1^2,M_2^2) = \int \frac{\mathrm{d}^n k}{i\pi^{n/2}} \frac{1}{A_1^{\nu_1} A_2^{\nu_2}} \,. \tag{26}$$

Using the Schwinger representation for the propagator

$$\frac{1}{A_i^{\nu_i}} = \frac{(-1)^{\nu_i}}{\Gamma(\nu_i)} \int_0^\infty \mathrm{d}x_i\, x_i^{\nu_i-1} \, \exp\left(x_i A_i\right) \,, \tag{27}$$

expression (26) becomes equal to

$$I_2^n = \prod_{i=1}^{2} \frac{(-1)^{\nu_i}}{\Gamma(\nu_i)} \int_0^\infty \mathrm{d}x_i\, x_i^{\nu_i-1} \int \frac{\mathrm{d}^n k}{i\pi^{n/2}} \exp\left(\sum_{j=1}^{2} x_j A_j\right) \,. \tag{28}$$

The expression above can be evaluated in two different ways. The first way proceeds by expanding the exponents in a power series. This yields

$$I_2^n = \prod_{i=1}^{2} \frac{(-1)^{\nu_i}}{\Gamma(\nu_i)} \int_0^\infty \mathrm{d}x_i\, x_i^{\nu_i-1} \int \frac{\mathrm{d}^n k}{i\pi^{n/2}} \sum_{n_1=0}^{\infty} \frac{(x_1A_1)^{n_1}}{n_1!} \sum_{n_2=0}^{\infty} \frac{(x_2A_2)^{n_2}}{n_2!}$$

$$= \prod_{i=1}^{2} \frac{(-1)^{\nu_i}}{\Gamma(\nu_i)} \int_0^\infty \mathrm{d}x_i\, x_i^{\nu_i-1} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} I_2^n(-n_1,-n_2,q^2,M_1^2,M_2^2) \prod_{j=1}^{2} \frac{x_j^{n_j}}{n_j!} \,. \tag{29}$$

Using the identity

$$\frac{(-1)^{\nu_i}}{\Gamma(\nu_i)} \int_0^\infty \mathrm{d}x_i\, x_i^{\nu_i+n_j-1} = \delta_{\nu_i+n_j,0} \,, \tag{30}$$

we obtain

$$I_2^n = I_2^n(\nu_1,\nu_2,q^2,M_1^2,M_2^2) \prod_{i=1}^{2} \frac{1}{\Gamma(1-\nu_i)} \,. \tag{31}$$

There is a second way to compute eq. (28). First one shifts the momentum k which becomes $k = k' - q\, x_2/(x_1 + x_2)$. Subsequently one uses the following identities

$$\int \frac{\mathrm{d}^n k'}{i\pi^{n/2}} \left(k'^2\right)^l = l!\, \delta_{l+n/2,0} \qquad \int \frac{\mathrm{d}^n k'}{i\pi^{n/2}} \exp\left(\alpha k'^2\right) = \frac{1}{\alpha^{n/2}}\,. \tag{32}$$

Notice that since $l > 0$ one obtains $n < 0$. This the reason for the name 'negative dimension approach'. The expression in eq. (28) can now be written as

$$I_2^n = \prod_{i=1}^{2} \frac{(-1)^{\nu_i}}{\Gamma(\nu_i)} \int_0^\infty \mathrm{d}x_i\, x_i^{\nu_i - 1} \frac{1}{(x_1 + x_2)^{n/2}} \exp\left(\frac{x_1 x_2}{x_1 + x_2} q^2\right) \exp\left(-\sum_{j=1}^{2} x_j M_j^2\right)\,. \tag{33}$$

The integrand can be written as

$$\frac{1}{(x_1 + x_2)^{n/2}} \exp\left(\frac{x_1 x_2}{x_1 + x_2} q^2\right) \exp\left(-\sum_{j=1}^{2} x_j M_j^2\right)$$

$$= \sum_{l=0}^{\infty} \frac{1}{l!} (x_1 + x_2)^{-l-n/2} (x_1 x_2 q^2)^l \sum_{m=0}^{\infty} \frac{(-x_1 M_1^2 - x_2 M_2^2)^m}{m!}\,. \tag{34}$$

Using the identities

$$\left(\sum_{i=0}^{M} x_i A_i\right)^N = \Gamma(N+1) \prod_{k=1}^{M} \sum_{i_k=0}^{\infty} \frac{x_{i_k} A_{i_k}}{i_k!} \quad \text{with} \quad \sum_{k=1}^{M} i_k = N\,, \tag{35}$$

one can write

$$I_2^n = \prod_{i=1}^{2} \frac{(-1)^{\nu_i}}{\Gamma(\nu_i)} \int_0^\infty \mathrm{d}x_i\, x_i^{\nu_i - 1} \sum_{l=0}^{\infty} \frac{(x_1 x_2 q^2)^l}{l!} \Gamma(1 - l - n/2) \sum_{p_1=0}^{\infty} \sum_{p_2=0}^{\infty} \frac{x_1^{p_1}}{p_1!} \frac{x_2^{p_2}}{p_2!}$$

$$\sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{(-x_1 M_1^2)^{m_1}}{m_1!} \frac{(-x_2 M_2^2)^{m_2}}{m_2!} \quad \text{with} \quad p_1 + p_2 = -l - n/2\,. \tag{36}$$

Equating eqs (31) and (36) one obtains the result

$$I_2^n(\nu_1, \nu_2, q^2, M_1^2, M_2^2) = \sum_{l, p_i, m_i} \frac{\Gamma(1 - l - n/2)\Gamma(1 - \nu_1)\Gamma(1 - \nu_2)}{\Gamma(1+l)\Gamma(1+p_1)\Gamma(1+p_2)\Gamma(1+m_1)\Gamma(1+m_2)}$$

$$(q^2)^{l+m_1+m_2} \times \left(\frac{-M_1^2}{q^2}\right)^{m_1} \left(\frac{-M_2^2}{q^2}\right)^{m_2}\,, \tag{37}$$

with

$$p_1 + p_2 = -l - n/2 \quad p_1 = l - \nu_1 - m_1 \quad p_2 = l - \nu_2 - m_2 \,. \tag{38}$$

Choosing two independent summation indices out of five i.e. l, p_1, p_2, m_1, m_2 one can express I_2^n into the following special functions [9] given by the hyper-geometric functions ${}_2F_1$, ${}_3F_2$, the Appell functions F_i ($i = 1$–4), the Horn function H_2 and the Kampé de Fériet functions S_i ($i = 1, 2$). In the case of the two-point function in eq. (26) there are eight possibilties to express the integral into the special functions above depending on the convergence of the sums in a specific kinematic region. If we choose $|M_i^2/q^2| < 1$ one has to take m_1, m_2. Using the following identity

$$\frac{\Gamma(z-n)}{\Gamma(z)} = (-1)^n \frac{\Gamma(1-z)}{\Gamma(1-z+n)} \,, \tag{39}$$

the expression in eq. (37) can be written as

$$I_2^n(\nu_1,\nu_2,q^2,M_1^2,M_2^2) = \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty} \frac{\Gamma(1-n+\nu_1+\nu_2+m_1+m_2)}{\Gamma(n/2+1-\nu_1-\nu_2-m_1-m_2)}$$

$$\times \frac{\Gamma(1-\nu_1)\Gamma(1-\nu_2)}{\Gamma(1+m_1)\Gamma(1+m_2)}(q^2)^{n/2-\nu_1-\nu_2} \times \left(\frac{-M_1^2}{q^2}\right)^{m_1} \left(\frac{-M_2^2}{q^2}\right)^{m_2}$$

$$= (-1)^{n/2}(q^2)^{n/2-\nu_1-\nu_2} \frac{\Gamma(n/2-\nu_1)\Gamma(n/2-\nu_2)\Gamma(\nu_1+\nu_2-n/2)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(n-\nu_1-\nu_2)}$$

$$\times F_4\left(1+\nu_1+\nu_2-n, \nu_1+\nu_2-n/2, 1+\nu_1-n/2, 1+\nu_2-n/2, \frac{M_1^2}{q^2}, \frac{M_2^2}{q^2}\right) \,. \tag{40}$$

Here the fourth Appell function is given by

$$F_4(\alpha,\beta,\gamma,\gamma',x,y) = \sum_{m,n=0}^{\infty} \frac{\Gamma(\alpha+m+n)}{\Gamma(\alpha)} \frac{\Gamma(\beta+m+n)}{\Gamma(\beta)} \frac{\Gamma(\gamma)}{\Gamma(\gamma+m)} \frac{\Gamma(\gamma')}{\Gamma(\gamma'+n)}$$

$$\times \frac{x^m}{\Gamma(m+1)} \frac{y^n}{\Gamma(n+1)} \,. \tag{41}$$

We conclude that using the method above one can find different representations for a one-loop integral in different kinematic regions depending on the radius of convergence of the ratios between the scales q^2, M_1^2, M_2^2. It is unclear whether this method, which seems very complicated to us, can be used to compute two-loop integrals.

References

[1] W L van Neerven, *Nucl. Phys.* **B268**, 453 (1986)
E B Zijlstra and W L van Neerven, *Nucl. Phys.* **B383**, 525 (1992)
[2] S Moch and J A M Vermaseren, hep-ph/9912355, hep-ph/9909269
[3] K G Chetyrkin and F V Tkachov, *Nucl. Phys.* **B192**, 159 (1981)

[4] S A Larin, T van Ritbergen and J A M Vermaseren, *Nucl. Phys.* **B427**, 41 (1994)
S A Larin, P Nogueira, T van Ritbergen and J A M Vermaseren, *Nucl. Phys.* **B492**, 338
[5] N I Ussyukina and A I Davydychev, *Phys. Lett.* **B298**, 363 (1993); **B332**, 159 (1994); **B348**, 503 (1993)
[6] V A Smirnov, *Phys. Lett.* **B460**, 397 (1999)
[7] J B Tausk, *Phys. Lett.* **B469**, 225 (1999)
[8] L G Halliday and R M Ricotta, *Phys. Lett.* **B193**, 241 (1987)
A T Suzuki and A G M Schmidt, *Europhys. J.* **C5**, 175 (1998); *Phys. Rev.* **D58**, 047701 (1998)
[9] C Anastasiou, E W N Glover and C Oleari, hep-ph/9907494