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Coupled dilaton and electromagnetic field in cylindrically symmetric spacetime

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Abstract. An exact solution is obtained for coupled dilaton and electromagnetic field in a cylindrically symmetric spacetime where an axial magnetic field as well as a radial electric field both are present. Depending on the choice of the arbitrary constants our solution reduces either to dilatonic gravity with pure electric field or to that with pure magnetic field. In the first case we have a curvature singularity at a finite distance from the axis indicating the existence of the boundary of a charged cylinder which may represent the source of the electric field. For the second case we have a singularity on the axis. When the dilaton field is absent the electromagnetic field disappears in both the cases. Whereas the contrary is not true. It is further shown that light rays except for those proceeding in the radial direction are either trapped or escape to infinity depending on the magnitudes of certain constant parameters as well as on the nature of the electromagnetic field. Nature of circular geodesics is also studied in the presence of dilaton field in the cylindrically symmetric spacetime.

Keywords. Dilaton field; general relativity; cylindrically symmetric spacetime.

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1. Introduction

The dilaton black hole solutions have attracted considerable attention for the last few years and there exists fairly exhaustive literature on the subject [1–4]. When the dilaton field is a coupled system or in other words when we consider a charged dilaton sphere, the solutions are significantly modified from the ordinary black hole ones. The presence of the dilaton changes the scenario for the causal structure of the black hole and results in the formation of curvature singularities at finite radii, Horne and Horowitz [5] and Shiraishi [6] obtained charged dilaton blackhole solutions with weak rotation. Chamorro and Virbhadra [7] calculated energy associated with such charged dilaton blackhole solution. Very recently Salim *et al* [8] considered dilaton field in the context of Bianchi-I cosmologies.

In the present paper we assume an action which is part of the low energy action of string theory and study the corresponding cylindrically symmetric spacetime, where the dilaton field is coupled to an electromagnetic field. The study of such spacetime is quite interesting in view of the existing results in more special cases where the dilaton field is absent and the spacetime is due to the Einstein–Maxwell fields or due to the gravitational field alone. Our spacetime is quite general in the sense that when some of the constants appearing in our solution vanish, it either reduces to a dilatonic gravity with an axial magnetic field or to that coupled to an electric field in the radial direction for a static cylindrical charged source. The spacetime with the electric field alone has a singular surface at a finite radius which may be interpreted as the boundary of the source, while that with an axial magnetic field is found to be singluar at $r \to 0$.

It is relevant to mention that our line-element also reduces to the well-known Levi-Civita metric in the absence of the dilaton and the electromagnetic field.

It may be interesting to study the trajectories of light rays in the spacetime described above. As in the case of Levi-Civita spacetime of pure gravitational field, here also light rays directed exactly in the radial direction travel unobstructed, while photons moving along other direction in the axial plane ($z =$ constant) are either trapped or move unobstructed depending on the choices of parameters and the nature of the electromagnetic field. For the parameters chosen in this paper all the photons are trapped in the first case with axial magnetic field, while they escape to infinity in the second case with radial electric field.

2. Dilaton coupled to an electromagnetic field

The appropriate action function for the Einstein–Maxwell dilaton field (ref. [1]) is given below:

$$S = \frac{1}{16\pi}\int \left[R + 2\phi_i\phi^i - \mathrm{e}^{-2\beta\phi}F_{ij}F^{ij}\right]\sqrt{-g}\,\mathrm{d}^4x, \tag{2.1}$$

where R is the Ricci scalar, ϕ is the dilaton field and F_{ij} is the electromagnetic field tensor. β is a dimensionless free parameter which regulates the coupling between the dilaton and the electromagnetic field. $\beta = 1$ actually gives the low energy action of the string theory. Varying (2.1) with $\beta = 1$ we get

$$G_{ij} = -8\pi T_{ij}, \tag{2.2}$$

$$(\mathrm{e}^{-2\phi}F^{ij});\ i = 0 \tag{2.3}$$

and

$$\phi^{i}_{;i} = \frac{1}{2}\mathrm{e}^{-2\phi}F_{jk}F^{jk}, \tag{2.4}$$

we further consider the metric with cylindrical symmetry

$$\mathrm{d}s^2 = \mathrm{e}^{2\Psi}\mathrm{d}t^2 - \mathrm{e}^{-2\Psi}\left(\mathrm{e}^{2\gamma}\mathrm{d}r^2 + \mathrm{e}^{2\gamma}\mathrm{d}z^2 + r^2\mathrm{d}\phi^2\right), \tag{2.5}$$

where Ψ, γ are functions of the radial coordinate r alone. The metric (2.5) uses Weyl canonical coordinates, which are allowed in this case. It is not difficult to calculate the energy-momentum tensor T_{ij} from the form of the action function and this is expressed in the form

$$8\pi T_{ij} = 2\phi_i\phi_j - 2\mathrm{e}^{-2\phi}g^{k1}F_{ki}F_{lj} - g_{ij}\phi^k\phi_k + \frac{1}{2}g_{ij}\mathrm{e}^{-2\phi}F_{kl}F^{kl}. \qquad (2.6)$$

Assuming here that the surviving independent components of the electromagnetic field tensor are F_{13}, i.e, the magnetic field along z direction and F_{01}, i.e the electric field along radial direction, one can write the following field equations

$$\gamma'' - 2\Psi'' + \Psi'^2 - \frac{2}{r}\Psi' = -\phi'^2 - \frac{1}{r^2}\mathrm{e}^{-2\phi+2\Psi}(F_{13})^2 - \mathrm{e}^{-2\phi-2\Psi}(F_{01})^2, \qquad (2.7)$$

$$-\Psi'^2 + \frac{\gamma'}{r} = \phi'^2 + \frac{1}{r^2}\mathrm{e}^{-2\phi+2\Psi}(F_{13})^2 - \mathrm{e}^{-2\phi-2\Psi}(F_{01})^2, \qquad (2.8)$$

$$\Psi'^2 - \frac{\gamma'}{r} = -\phi'^2 - \frac{1}{r^2}\mathrm{e}^{-2\phi+2\Psi}(F_{13})^2 + \mathrm{e}^{-2\phi-2\Psi}(F_{01})^2, \qquad (2.9)$$

$$\gamma'' + \Psi'^2 = -\phi'^2 + \frac{1}{r^2}\mathrm{e}^{-2\phi+2\Psi}(F_{13})^2 + \mathrm{e}^{-2\phi-2\Psi}(F_{01})^2, \qquad (2.10)$$

where a prime overhead means $\partial/\partial r$. The eq. (2.3) yields directly the electromagnetic field tensor components

$$F^{13} = \frac{Q_m\mathrm{e}^{2\phi}}{(-g)^{1/2}}; \quad F^{01} = \frac{-Q_e\mathrm{e}^{2\phi}}{(-g)^{1/2}} \qquad (2.11)$$

so that the explicit form of F_{13} and F_{01} are given by

$$F_{13} = Q_m r\mathrm{e}^{2(\phi-\Psi)}; \quad F_{01} = \frac{Q_e}{r}\mathrm{e}^{2(\phi+\Psi)}, \qquad (2.12)$$

where Q_m and Q_e are integration constants to be interpreted later as the strength of the magnetic and electric fields respectively.

With the help of the field equations one readily obtains

$$\Psi'' + \frac{\Psi'}{r} = \mathrm{e}^{-2\phi-2\Psi}(F_{01})^2 + \frac{1}{r^2}\mathrm{e}^{-2\phi+2\Psi}(F_{13})^2, \qquad (2.13)$$

$$\gamma'' + \frac{\gamma'}{r} = \frac{2}{r^2}\mathrm{e}^{-2\phi+2\Psi}(F_{13})^2, \qquad (2.14)$$

$$\gamma'' - 2\Psi'' - \frac{2}{r}\Psi' + \frac{\gamma'}{r} = -2\mathrm{e}^{-2\phi-2\Psi}(F_{01})^2. \qquad (2.15)$$

The wave equation for the coupled system (2.4) may be written as

$$(r\phi')' = \frac{Q_e^2}{r}\mathrm{e}^{2\phi+2\Psi} - Q_m^2 r\mathrm{e}^{2\phi-2\Psi}. \qquad (2.16)$$

Now the three relations (2.13–2.16) connect ϕ, γ and Ψ through the following relation

$$\phi' = \Psi' - \gamma' + c_1'/r, \qquad (2.17)$$

where c_1' is a constant of integration.

For economy of space we skip all details of the intermediate steps and give the final expressions of the metric coefficients as

$$e^{2\gamma} = r^{2c_1'} \left(A_1 r^{1+d} + B_1 r^{1-d}\right)^2, \tag{2.18}$$

$$e^{2\Psi} = r \left(A_1 r^{1+d} + B_1 r^{1-d}\right) \left(C_1 r^{1+c_1} + D_1 r^{1-c_1}\right)^{-1}, \tag{2.19}$$

$$e^{2\phi} = r \left(A_1 r^{1+d} + B_1 r^{1-d}\right)^{-1} \left(C_1 r^{1+c_1} + D_1 r^{1-c_1}\right)^{-1}, \tag{2.20}$$

where $(A_1, B_1, C_1, D_1, d, c_1)$ are integration constants.

Further, a straight forward calculation shows that

$$Q_m^2 = 2A_1 B_1 d^2 \quad \text{and} \quad Q_e^2 = -2C_1 D_1 c_1^2. \tag{2.21}$$

To get a better physical insight of the metric system (2.18–2.20) we would like to truncate it into cases – one corresponding to a pure magnetic field and the other to an electric field only.

Case I (*Electric field absent*)

It may be interesting to point out that if we put either $C_1 = 0$ or $D_1 = 0$, the electric field vanishes and we get the solution of dilatonic gravity coupled to an axial magnetic field only. In such a case the dilatonic field simplifies to

$$e^{-2\phi} = (Ar^a + Br^b), \tag{2.22}$$

where A and B are new constants and

$$a = 1 - c_1 + d \quad \text{and} \quad b = 1 - c_1 - d. \tag{2.23}$$

At this stage we can express the system of metric components in a compact form as

$$e^{-2\phi} = Ar^{1-c_1-d}(r^{2d} + \alpha^2), \tag{2.24}$$

$$e^{2\Psi} = Ar^{1+c_1-d}(r^{2d} + \alpha^2), \tag{2.25}$$

$$e^{2\gamma} = A^2 r^{2(1+c+c_1-d)}(r^{2d} + \alpha^2)^2. \tag{2.26}$$

The constant c is related to c_1 and d to be shown later. Here

$$\alpha^2 = B/A. \tag{2.27}$$

Adding (2.9) and (2.10) (taking $F_{01} = 0$) and expressing the derivatives of Ψ and γ in terms of the derivatives of ϕ we arrive at the following differential equation

$$\phi'' - 2\phi'^2 + \frac{1}{2}(2c_1 - 1)\phi' - \frac{1}{r^2}(c_1^2 - 2c_1 - c) = 0, \tag{2.28}$$

which is consistent with the solution for ϕ given in (2.24) provided

$$2c = c_1^2 + d^2 - 2c_1 - 1. \tag{2.29}$$

Now omitting some trivial constants one can write the final form of the metric without loss of generality

$$\begin{aligned} ds^2 = {} & r^{1+c_1-d}(r^{2d} + \alpha^2)dt^2 - r^{(c_1^2 - c_1 + d^2 - d)} \\ & \times (r^{2d} + \alpha^2)(dr^2 + dz^2) - r^{1-c_1+d}(r^{2d} + \alpha^2)^{-1} d\phi^2, \end{aligned} \tag{2.30}$$

when we put $d = (c_1 - 1)$ and $\alpha^2 = 0$ in (2.30) both the scalar and e.m. field disappear and the metric (2.30) reduces to

$$\mathrm{d}s^2 = r^{2c_1}\mathrm{d}t^2 - r^{2c_1(c_1-1)}(\mathrm{d}r^2 + \mathrm{d}z^2) - r^{2(1-c_1)}\mathrm{d}\phi^2 \tag{2.31}$$

which is the well-known Levi-Civita metric [8] as expected with c_1 representing the mass per unit length of the cylindrical distribution. It is now easy to find ϕ'^2 from (2.22) to write

$$\phi'^2 = \frac{1}{4r^2}\left(\frac{ar^{a-b} + \alpha^2 b}{r^{a-b} + \alpha^2}\right)^2. \tag{2.32}$$

Independent of whether $2d = (a - b)$ is positive or negative we immediately come to the inference that (ϕ'^2) blows up as $r \to 0$, that is, in either case the singularity on the axis is unavoidable.

It is clear from eq. (2.4) that $\phi' = 0$ leads to the vanishing of the electromagnetic field as well. This is because the dilaton field is coupled to the electromagnetic field as is evident from the action function (2.1). The equation (2.4) has been used to arrive at the metric (2.24)–(2.26). Hence it is not possible here to remove the dilaton field leaving only the electromagnetic field to exist. When either $\alpha^2 = 0$ and $a = 0$ or $a = b = 0$ both the fields disappear and we get purely the gravitational field. In the first case the Levi-Civita metric (2.31) is recovered. In the second case, however, $d = 0$ and $c_1 = 1$ and the metric for the vacuum reduces to the form

$$\mathrm{d}s^2 = \bar{r}^2\mathrm{d}t^2 - (\mathrm{d}\bar{r}^2 + \mathrm{d}\bar{z}^2) - \mathrm{d}\bar{\phi}^2, \tag{2.33}$$

where $\bar{r} = r(1 + \alpha^2)^{1/2}$, the other coordinates being subjected to trivial transformation. The above metric represents vacuum having all the curvature scalars equal to zero. But the metric cannot be transformed to a globally Minkowskian form because $\bar{\phi}$ stands for an angular coordinate and is not a simple Cartesian coordinate.

One must note here that is of course possible to obtain a purely scalar field solution from (2.30) when the constant parameter α^2 vanishes. In that case the right hand side of (2.4) vanishes but ϕ' does not necessarily vanish. The metric in this case is given by

$$g_{00} = \mathrm{e}^{2\Psi} = r^{1+c_1+d},$$
$$g_{11} = g_{22} = -r^{c_1^2+d^2+d-c_1}$$

and

$$g_{33} = -r^2 r^{-(1+c_1+d)}. \tag{2.34}$$

Case II (*Magnetic field absent*)

The analysis to be presented here closely resembles the formalism described in Case I. So we will be very brief here. From eq. (2.21) it follows that if any of the constants $A_1 = B_1 = 0$, the magnetic field vanishes and the dilatonic field may be solved to give the following results:

$$\mathrm{e}^{-2\phi} = (Cr^a + Dr^b), \tag{2.35}$$

where again $a = 1 - c_1^* + d^*$ and $b = 1 - c_1^* - d^*$. Here C and D are arbitrary constants and c_1^*, d^* are two independent constant parameters appearing in the explicit forms of the metric

$$\mathrm{e}^{2\Psi} = r^{2(1-c_1^*)}\mathrm{e}^{2\phi}, \tag{2.36}$$

$$\mathrm{e}^{2\gamma} = r^{\{d^{*2}+(c_1^*-1)^2\}}. \tag{2.37}$$

Now putting $-\beta^2 = D/C$ the solution for the dilaton ϕ-field may be expressed as

$$\mathrm{e}^{-2\phi} = r^{1-c_1^*-d^*}(r^{2d^*} - \beta^2). \tag{2.38}$$

The spacetime singularity appears at $r^{2d^*} = \beta^2$, indicating that this is the boundary of the source. The absence of dilaton field needs either $\beta^2 = 0$ and $d^* = (c_1^* - 1)$ or $d^* = 0$ and $c_1^* = 1$. In both cases the electromagnetic field also vanishes at the same time. It is not surprising when we consider eq. (2.4). The reasons are given in the previous case for magnetic field. For the first choice that is when $\beta^2 = 0$ and $d^* = (c_1^* - 1)$ the metric reduces to

$$\mathrm{d}s^2 = r^{-2d^*}\mathrm{d}t^2 - r^{2d^*(d^*+1)}(\mathrm{d}r^2 + \mathrm{d}z^2) - r^{(1+d^*)}\mathrm{d}\phi^2.$$

This is clearly a Levi-Civita metric (2.31) with different symbols for the constant parameters (c_1 is replaced by a new symbol d^*). For the second choice on the other hand

$$\mathrm{d}s^2 = \frac{1}{1-\beta^2}\mathrm{d}t^2 - (1-\beta^2)(\mathrm{d}r^2 + \mathrm{d}z^2) - r^2(1-\beta^2)\mathrm{d}\phi^2,$$

which after some trivial coordinate transformations exactly reduces to the flat metric. Thus the spacetime exterior to a static charged cylinder [9] is not recoverable from our generalized solution, which includes a dilaton field coupled with the electric field.

On the other hand if we put only $\beta^2 = 0$, we get the spacetime for the purely dilatonic field. The solutions are

$$\mathrm{e}^{2\Psi} = r^{1-c_1^*-d^*},$$

$$\mathrm{e}^{2\gamma} = r^{\{d^{*2}+(c_1^*-1)^2\}}.$$

These solutions exactly coincide with those obtained previously in (2.34) when we replace c_1^* by $-d$ and d^* by $-c_1$.

3. The trapping of photons

It is natural to ask under what circumstances the outgoing light rays will be trapped in the spacetime obtained in each of the previous cases. This may be interesting in the context of the old result in the Levi-Civita spacetime where such thing depends on the magnitude of mass per unit length of the cylinder [10].

Case I

Here we consider the spacetime given by metric (2.30). For null geodesics in $z =$ constant plane simple calculations show

$$\dot{t} = m \cdot r^{d-c_1-1}(r^{2d}+\alpha^2)^{-1}, \tag{3.1}$$

$$\dot{\phi} = nr^{c_1-d-1}(r^{2d}+\alpha^2), \tag{3.2}$$

so that

$$\frac{\mathrm{d}t}{\mathrm{d}\phi} = \left(\frac{m}{n}\right) r^{2(d-c_1)}(r^{2d}+\alpha^2)^{-2}. \tag{3.3}$$

Using (3.3) in the metric (2.30) and remembering that the light rays are confined in $z =$ constant plane the equation of motion for such orbits reduces to

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)^2 = r^{\{1+2d-(c_1^2+d^2)\}}[r^{2d}+\alpha^2]^{-2}\left[\frac{m^2}{n^2}\,\frac{r^{2(d-c_1)}}{(r^{2d}+\alpha^2)^2} - 1\right]. \tag{3.4}$$

The relation (3.4) demands $(m^2/n^2) > Y$, where Y is written for $(r^{2d}+\alpha^2)r^{2(c_1-d)}$. Once this condition is satisfied it will remain valid for increasing r, provided Y decreases monotonically with r and the rays escape to infinity without being trapped, which needs $(\mathrm{d}r/\mathrm{d}\phi) = 0$ at some stage. On the other hand if Y increases with increasing r till it finally equals (m^2/n^2) at a certain stage, the rays will turn back and may be trapped. Here such situations depend on the magnitudes of the parameters c_1 and d, both of which are assumed to be positive. When $c_1 > d > 0, Y$ increases monotonically and the light rays are trapped. When $d > c_1 > 0$, we find the asymptotic value of Y as $r \to \infty$, which is infinity in this case and so the rays are trapped again. The same conditions may be arrived at for the spacetime due to a pure dilaton field alone $(\alpha^2 = 0)$. Again for the trajectory in the radial direction $\dot{\phi} = 0$ and so (m/n) is infinitely large. Here in general the rays travel continuously away from the axis not to turn back.

Case II

In the second example, where the dilaton field is coupled with a radial electric field generated by a charged cylinder, the null geodesics in $z =$ constant plane must satisfy the condition that $(M^2/N^2) > X$, where

$$X = \frac{r^{2(d^*-c_1^*)}}{(r^{2d^*}-\beta^2)^2}. \tag{3.5}$$

The constants M and N are related with the usual energy and angular momentum. This may be shown following the same procedure as in Case I. Omitting details of the procedure at this stage one can arrive at the conclusion that for positive magnitudes of c_1^* and d^* the outgoing light-rays in a charged dilaton cylindrically symmetric spacetime are never trapped. Contrary result may occur when the above restrictions on the parameters c_1^* and d^* are relaxed.

4. Circular geodesics

It is well-known that the circular trajectories in Levi-Civita spacetime have an interesting property that their time-like, null or space-like character depend on the mass per unit length of the source. When this is exactly equal to 1/4 all the circular geodesics are null. Similar feature is apparent for the cylindrically symmetric spacetime in the presence of the dilatonic field alone. In this case the character of the orbits depends on the relative magnitudes of the parameters c_1 and d. To demonstrate this result we write below the equations of motion for a particle orbiting along a circular track in a $z =$ constant plane. The geodesics corresponding to the radial coordinate in the spacetime given by (2.30) are as follows

$$[(1+c_1+d)r^{(c_1+d)} + \alpha^2(1+c_1-d)r^{(c_1-d)}] \\ = \frac{[(1-c_1-d)r^{3d-c_1} + \alpha^2(1-c_1+d)r^{d-c_1}]}{(r^{2d}+\alpha^2)^2} \left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^2 .$$

Again from the metric (2.30) itself there is another relation

$$\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 = r^{1+c_1-d}(r^{2d}+\alpha^2) - \frac{r^{1-c_1+d}}{(r^{2d}+\alpha^2)} \left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^2 .$$

The above two equations when combined yield the relation

$$\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 = r^{(1+c_1-d)}(r^{2d}+\alpha^2) \\ \times \left[1 - \frac{(1+c_1+d)r^{(3d-c_1)} + \alpha^2(1+c_1-d)r^{(d-c_1)}}{(1-c_1-d)r^{(3d-c_1)} + \alpha^2(1-c_1+d)r^{(d-c_1)}}\right],$$

which in turn for no electromagnetic field $(\alpha^2 = 0)$ takes a very simple form like

$$\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 = r^{(1+c_1+d)} \cdot \left[1 - \frac{1+c_1+d}{1-c_1-d}\right]. \tag{4.1}$$

The above relation (4.1) points to the immediate inference that all the circular orbits are null when $c_1 + d = 0$. Further if $d = (c_1 - 1)$ the dilaton field disappears and the condition for all null orbits is $c_1 = 1/2$ which is exactly identical with the result obtained for the Levi-Civita spacetime [11,12] (the mass per unit length is 1/4 in relativistic units).

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