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Cosmic microwave background anisotropy constraints on relict neutrinos

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Abstract. I discuss basic theory of effect of the properties of the cosmological relict neutrinos on the observations of the cosmic microwave background anisotropy.

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1. Introduction: The Einstein–Boltzmann equations

The evolution of the density perturbations in matter and radiation which we presently observe as large scale structures and cosmic microwave background radiation (CMBR) anisotropies, are determined by the Boltzmann equation which determines the phase space distribution function $f(x^\mu, p^\mu)$. The evolution of the phase space density with respect to some parameter λ (which may be time, or some function of the time coordinate) is given by

$$\frac{\mathrm{d}f(x^\mu, p^\mu)}{\mathrm{d}\lambda} = \frac{\partial f}{\partial x^\mu}\frac{\partial x^\mu}{\partial \lambda} + \frac{\partial f}{\partial p^\mu}\frac{\partial p^\mu}{\partial \lambda} = C[f]\,. \tag{1}$$

The term denoted by $C[f]$ is the collision term which is due to inter-particle scattering or decays. The trajectories of particles in the phase space are determined by the geodesic equation,

$$p^0\frac{\partial p^\mu}{\partial x^0} + \Gamma^\mu_{\alpha\beta}p^\alpha p^\beta = 0\,. \tag{2}$$

The Christoffel connections $\Gamma^\mu_{\alpha\beta}$ of the background metric are determined by solving the Einstein's equations

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu} \tag{3}$$

whose source is the energy momentum tensor $T_{\mu\nu}$ which in turn is determined by the phase space distribution $f(x^\alpha, p_\beta)$,

$$T_{\mu\nu} = \int \mathrm{d}^3 p \, (-g)^{-1/2} \frac{p_\mu p_\nu}{p_0} f(x^\alpha, p_\beta) \, . \tag{4}$$

Solution of these coupled set of equations for the photon, baryon, neutrino, dark matter fluid determines the scale of structures and the anisotropy of the observed microwave background. Some good reviews on this subject are by Ma and Bertschinger [1], Hu and Sugiyama [2] and at a more introductory level [3].

1.1 *Homogeneous relict neutrino density*

As a first approximation we solve for the distribution function of neutrinos assuming a homogeneous, isotropic Robertson–Walker metric,

$$\mathrm{d}s^2 = a^2(\eta)(\mathrm{d}\eta^2 - \gamma_{ij}\mathrm{d}x^i \mathrm{d}x^j), \tag{5}$$

where $\mathrm{d}\eta = a\mathrm{d}t$ is the conformal time variable, $a(\eta)$ is the expansion scale factor and γ_{ij} is the 3-metric of the spatial part (in the flat ($K = 0$) universe $\gamma_{ij} = \delta_{ij}$). The non-zero component of the Christoffel connection are

$$\Gamma^0_{00} = \frac{\dot{a}}{a} \, , \qquad \Gamma^0_{ij} = \frac{\dot{a}}{a} \gamma_{ij} \, , \tag{6}$$

(we denote spacetime indices by Greek and spatial indices by Latin letters). The distribution function for the neutrinos $f_0(x^0, p_i)$ is independent of the spatial position x^i, the zeroth component of p^μ is written in terms of the spatial components by using the dispersion relation

$$g^{00}(p_0)^2 = (\gamma^{ij} p_i p_j + m^2). \tag{7}$$

For particles with non-zero mass, the parameter $\lambda = \mathrm{d}s$ the proper time for time-like geodesics. The conjugate momenta p_μ to the coordinates x^μ are given by $p^\mu = m\mathrm{d}x^\mu/\mathrm{d}\lambda = mu^\mu$. For massless particles, the conjugate momenta are defined as $p^\mu = \mathrm{d}x^\mu/\mathrm{d}\lambda = u^\mu$.

For both massless and massive particles the form of the geodesic equation is identical as given in (2) and is given by

$$p^0 \frac{\mathrm{d}p^i}{\mathrm{d}\eta} + 2 \left(\frac{\dot{a}}{a} \right) p^0 p^i = 0 \, . \tag{8}$$

The Boltzmann equation for the distribution function $f(\eta, p^i)$ is then given by

$$\frac{\partial f}{\partial \eta} - 2 \left(\frac{\dot{a}}{a} \right) p_i \frac{\partial f}{\partial p_i} = 0 \, . \tag{9}$$

The collision term $C[f]$ for the case of weak interactions is $\sim G_{\mathrm{F}}^2 T^5$. Compared to the expansion term $(\dot{a}/a) \sim T^2/M_p^2$, the collision term becomes smaller after the temperature drops below 1 MeV (at the time ~ 1 s).

The Boltzmann eq. (9) is solved by a distribution function of the form $f(a^2 p_i)$. From the geodesic equation we see that the product $a^2(\eta)p_i(\eta)$ is a constant. The momenta defined with respect to the conformal time $p^i = p^0(\mathrm{d}x^i/\mathrm{d}\eta)$ red shifts as a^{-2}. The physical momenta $p^i_{\rm phys} = p^0_{\rm phys}(\mathrm{d}x^i/\mathrm{d}t)$ however redshifts as a^{-1} in the expanding universe. Consider an initial distribution function at the time of decoupling,

$$f(p) = g_s(\mathrm{e}^{\epsilon/T_d} \pm 1)^{-1} \tag{10}$$

where $\epsilon = U_\mu p^\mu$ and U_μ is the four-velocity of the center of mass of the fluid with respect to the observer. In the comoving reference frame $U_\alpha = ((g_{00})^{1/2}, 0, 0, 0)$ and

$$\epsilon = ap^0 = (\gamma_{ij}p^i p^j + m^2a^2)^{1/2}\,. \tag{11}$$

For relativistic particles the red shift in the momenta may be absorbed in the temperature parameter. The shape of the spectrum remains unchanged in time and only the temperature red shifts as $T \propto a^{-1}$.

Since the geodesic eq. (8) and consequently the Boltzmann eq. (9) are independent of mass, an initial relativistic distribution

$$f(p) = g_s\left(\mathrm{e}^{p/T} \pm 1\right)^{-1}$$

will retain the same form in time even when the particles have become non-relativistic. The distribution function at the present temperature T_0 is obtained by scaling the distribution function at the decoupling temperature T_d,

$$f(p, T_0) = f\left(p, T_d\left(\frac{a_d}{a_0}\right)\right). \tag{12}$$

If the neutrinos were relativistic at the time of decoupling $(T_d \sim 1\,\mathrm{MeV})$ then their number density at present is related to their number density at the time of decoupling,

$$\begin{aligned} n(T_0) &= g_\nu \int (-g)^{-\frac{1}{2}} p^2 \mathrm{d}p\, \mathrm{d}\Omega\,\left[\exp\left(-\frac{p}{T_d\cdot(a_d/a_0)}\right)+1\right]^{-1} \\ &= \frac{3\zeta(3)}{4\pi^2} g_\nu T_d^3 \left(\frac{a_d}{a_0}\right)^3 \\ &= \frac{3\zeta(3)}{4\pi^2} g_\nu T_0^3. \end{aligned} \tag{13}$$

The present neutrino temperature is $T_0 = (4/11)^{1/3}T_\gamma$ owing to the photon temperature $(T_\gamma = 2.73\,\mathrm{K})$ increasing after $e^+\ e^-$ recombination relative to the temperature of the decoupled particles. The neutrino number density at present is $n(T_0) = 113\,\mathrm{cm}^{-3}$ per neutrino species. Using this one can relate the neutrino masses to the present energy density of the universe in the form of neutrinos $\rho_\nu = \sum m_\nu n(T_0)$. This relation is given by [4]

$$\sum_i m_{\nu_i} \le 93\ h_0^2 \Omega_\nu\ \mathrm{eV}, \tag{14}$$

where $h_0 = H_0/100\ \mathrm{kms^{-1}Mpc^{-1}}$ and $\Omega_\nu = \rho_\nu/\Omega_{\rm crit}$. From observations of large scale structures it is found that the best fit value for $\Omega_\nu = 0.3$. This result however changes in case there is a large vacuum density contribution to the total energy density.

2. The Einstein–Boltzmann equations in a perturbed metric

If the energy momentum density of the universe has an initial perturbation (owing to some seeding mechanism like inflation or cosmic strings) then its subsequent growth is governed by the coupled set of Einstein (3) and Boltzmann (1) equations. The dominant perturbations over the Robertson-Walker metric may be expressed in terms of two scalars functions. The perturbed metric may be written as

$$\mathrm{d}s^2 = a^2(\eta)\left[(1+2\psi)\mathrm{d}\eta^2 - (1-2\phi)\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j\right], \tag{15}$$

where we have used the conformal Newtonian gauge. Here $\psi(x)$ may be interpreted as a local Newtonian potential and $\phi(x)$ is the perturbation in the spatial curvature. The non-zero components of the Christoffel connections which govern the geodesic equations are

$$\begin{aligned}
\Gamma^0_{00} &= \left(\frac{\dot a}{a}\right) + \dot\psi, \\
\Gamma^0_{0i} &= \psi_{,i}, \\
\Gamma^i_{00} &= \psi^{,i}, \\
\Gamma^0_{ij} &= \gamma_{ij}\left(\frac{\dot a}{a}(1-2\phi-2\psi) - \dot\phi\right), \\
\Gamma^i_{jk} &= \left(\phi^{,i}\gamma_{jk} - \phi_{,k}\delta^i_j - \phi_{,j}\delta^i_k\right),
\end{aligned} \tag{16}$$

where over dots denote (conformal) time derivatives and commas denote partial derivatives with respect to spatial variables. The phase space distribution function is expressed as the zeroth order distribution (10) plus an inhomogeneous perturbation,

$$f(\vec x,\vec p,\eta) = f_0(p)\left(1+\Theta(\vec x,\vec p,\eta)\right), \tag{17}$$

where $p = \sqrt{\gamma_{ij}p^ip^j}$ is the magnitude of the spatial momentum. Writing $p^i = pn^i$ where n^i are the direction cosines along the particle trajectories, the Boltzmann equation for $f(\vec x, p, \vec n, \eta)$ is given by

$$\frac{\partial f}{\partial\eta} + \frac{\partial x^i}{\partial\eta}\frac{\partial f}{\partial x^i} + \frac{\partial p}{\partial\eta}\frac{\partial f}{\partial p} + \frac{\partial n^i}{\partial\eta}\frac{\partial f}{\partial n^i} = C[f]. \tag{18}$$

Using the geodesic equations we have

$$\frac{\partial p}{\partial\eta} = p\dot\phi - ap^0 n_i\psi_{,i}. \tag{19}$$

The last term in the lhs of (18) is second order in perturbation and may be dropped. Subtracting out the Boltzmann equation for the unperturbed distribution from (18) we obtain the equation for the perturbation to the distribution $\Theta(\vec x, p, \vec n, \eta)$,

$$\frac{\partial\Theta}{\partial\eta} + \frac{p}{ap^0}n^i\frac{\partial\Theta}{\partial x^i} + \frac{p}{f_0}\frac{\partial f_0}{\partial p}\left[\dot\phi - \frac{ap^0}{p}n^i\frac{\partial\psi}{\partial x^i}\right] = c[f]. \tag{20}$$

The components of the energy momentum tensor can also be written as a zeroth order plus an inhomogeneous piece,

$$
\begin{aligned}
T^0{}_0 &= \int (-g)^{-\frac{1}{2}} d^3p\, \epsilon(p) f_0(p)(1+\Theta) \\
&= \bar\rho + \delta\rho, \\
T^i{}_0 &= \int (-g)^{-\frac{1}{2}} d^3p\, n_i f_0(p)\Theta \\
&= (\bar\rho + \bar p) v^i, \\
T^i{}_j &= \int (-g)^{-\frac{1}{2}} d^3p\, \frac{n^i n_j}{\epsilon(p)} f_0(p)(1+\Theta) \\
&= -(\bar p + \delta p)\delta^i_j.
\end{aligned} \tag{21}
$$

Using (21), the Einstein's equation (3) for the perturbation turn out to be of the form

$$
\nabla^2\psi + 3\left(\ddot\phi + \frac{\dot a}{a}(\dot\phi + \dot\psi)\right) - 6\left[\left(\frac{\dot a}{a}\right)^2 - \frac{\ddot a}{a}\right]\psi = 4\pi G a^2 \bar\rho (1 + 3C_s^2)\delta, \tag{22}
$$

$$
\dot\phi_{,i} + \left(\frac{\dot a}{a}\right)\psi_{,i} = -4\pi G a^2 \bar\rho (1+\chi) v_i, \tag{23}
$$

$$
\begin{aligned}
&\ddot\phi - \nabla^2\phi - \frac{1}{3}\nabla^2(\phi - \psi) + \left(\frac{\dot a}{a}\right)(5\dot\phi + \dot\psi) + 2\left(\frac{\ddot a}{a} + \left(\frac{\dot a}{a}\right)^2\right) \\
&\quad \psi = -4\pi G a^2 \bar\rho (1 - C_s^2)\delta,
\end{aligned} \tag{24}
$$

$$
(\phi - \psi)_{,ij} - \frac{1}{3}\gamma_{ij}\nabla^2(\phi - \psi) = 0 \qquad (i \neq j), \tag{25}
$$

where

$$
\delta \equiv \frac{\rho - \bar\rho}{\bar\rho}, \qquad C_s^2 \equiv \frac{\partial p}{\partial \rho}, \qquad \chi \equiv \frac{\bar p}{\bar \rho}. \tag{26}
$$

From (25) we see that $\phi \simeq \psi$ to the leading order in perturbation.

The combined set of Einstein–Boltzmann equations (22–25,20) for the baryons, photons, CDM and HDM (neutrinos) are solved numerically [5] to determine the evolution of perturbations of density and metric. Some broad features can however be seen analytically.

Super horizon modes: For perturbations whose wavelengths are longer than the horizon one can neglect the spatial gradients with respect to the $\partial/\partial\eta$ terms. Einstein's equations (22–25) can then be combined to give

$$
\ddot\psi + (1 + C_s^2)\left(\frac{\dot a}{a}\right)\dot\psi + 3(C_s^2 - \chi)\left(\frac{\dot a}{a}\right)\psi = 0\,. \tag{27}
$$

This equation has no growing solutions even for vacuum dominated universe $(\chi = -1)$. This implies that the perturbation variables δ and ψ remain constant outside the horizon. The variables δ and ψ are related via the equation

$$\nabla^2\psi - 3\left(\frac{\dot{a}}{a}\right)\left(\dot{\psi} + \frac{\dot{a}}{a}\psi\right) = 4\pi G a^2 \bar{\rho}\delta \tag{28}$$

which is obtained by combining equations (22) and (24), and is analogue of Poisson's equation.

Sub-horizon perturbation: Perturbations δ and ψ can only grow when the horizon size becomes larger than their wavelengths. This can be seen from Einstein's equations where now $\partial/\partial\eta$ terms are neglected with respect to the spatial gradients. Einstein's equations can then be combined to yield

$$\nabla^2\psi = 4\pi G a^2 \bar{\rho}\delta, \qquad \text{(Poisson's equation)} \tag{29}$$

$$\dot{\delta} + (1+\chi)\nabla\cdot\vec{v} + 3(C_s^2 - \chi)\left(\frac{\dot{a}}{a}\right)\delta = 0, \qquad \text{(Continuity equation)} \tag{30}$$

$$\dot{\vec{v}} + (1-3\chi)\left(\frac{\dot{a}}{a}\right)\vec{v} = -\frac{C_s^2}{1+\chi}\nabla\delta - \nabla\psi. \qquad \text{(Euler's equation)} \tag{31}$$

Equations (30) and (31) may be combined to give

$$\ddot{\delta} + 2\left(\frac{\dot{a}}{a}\right)\dot{\delta} = 4\pi G\bar{\rho}\delta + \frac{C_s^2}{a^2}\nabla^2\delta. \tag{32}$$

Equation (32) implies that only modes of δ with wavelengths larger than $\lambda_J = C_s/a\,(\pi/G\bar{\rho})^{1/2}$ (Jeans length) can grow. Modes with wavelength shorter than λ_J oscillate. The smallest structures which can be formed have a mass $M_J = 4/3\pi(a\lambda_J)^3\bar{\rho}$. If δ is dominated by neutrinos (HDM) then $M_J \simeq 10^{17} M_\odot$ which is the size of superclusters.

3. Microwave background anisotropy

The perturbations in the photon phase space distribution are observed as anisotropy of the cosmic microwave background temperature. The photon temperature perturbation $\Delta \equiv (\Delta T/T)$ is defined by the distribution function

$$f(\vec{x}, p, \vec{n}, \eta) = f_0\left(\frac{p}{(1+\Delta)T}\right), \tag{33}$$

where f_0 is the Bose–Einstein distribution function at temperature $T(1+\Delta)$. Comparing expressions (33) and (17) we see that the perturbation of temperature $\Delta(\vec{x}, p, \vec{n}, \eta)$ is related to the perturbation in the distribution function $\Theta(\vec{x}, p, \vec{n}, \eta)$ as

$$\Delta(\vec{x}, p, \vec{n}, \eta) = -\left(\frac{p}{f_0}\frac{\partial f_0}{\partial p}\right)^{-1}\Theta(\vec{x}, p, \vec{n}, \eta)\,. \tag{34}$$

Using (34) and the Boltzmann equation for photon distribution (20) with $p^0 = p$ we see that after the photon decouples, the temperature anisotropy obeys the Boltzmann equation

$$\dot{\Delta} + n^i \frac{\partial}{\partial x^i}(\Delta - \psi) + \dot{n}_i \frac{\partial}{\partial n_i}\Delta + \dot{\phi} = 0. \tag{35}$$

The evolution of Δ does not depend upon the photon momentum p. We expand the temperature anisotropy $\Delta(\vec{x}, \hat{n}, \eta)$ in terms of spherical harmonics as

$$\Delta(\vec{x}, \hat{n}, \eta) = \int \mathrm{d}^3x\ \mathrm{e}^{i\vec{k}\cdot\vec{x}}\Delta(\vec{k}, \hat{n}, \eta)$$

$$\Delta(\vec{k}, \hat{n}, \eta) = \sum_{l=0}^{\infty}(-i)^l(2l+1)\Delta_l(\vec{k}, \eta)P_l(\vec{k}\cdot\hat{n}). \tag{36}$$

The angular power spectrum is

$$\langle\Delta(\hat{n}_1)\ \ \Delta(\hat{n}_2)\rangle = \frac{1}{4\pi}\sum_{l=0}^{\infty}(2l+1)C_l P_l(\hat{n}_1\cdot\hat{n}_2). \tag{37}$$

C_l and Δ_l are related by

$$C_l = 4\pi\int \mathrm{d}^3k\ \rho(k)\Delta_l^2(k, \eta), \tag{38}$$

where $\rho(k)$ is the initial amplitude spectrum of the temperature anisotropy whose subsequent growth is governed by the Boltzmann equation (35). For anisotropies at large angular scales (low l's) from the combined Boltzmann–Einstein's equation we obtain

$$\Delta(\hat{n}, \eta_0) = \frac{1}{3}\psi(\vec{x} = \vec{n}(\eta_0 - \eta_{\mathrm{rec}}), \eta_{\mathrm{rec}}) \tag{39}$$

and

$$\Delta_l(k, \eta_0) = \frac{1}{3}j_l(k(\eta_0 - \eta_{\mathrm{rec}})), \tag{40}$$

where η_{rec} is the time of photon decoupling. The relation (39) is called the Sach's Wolfe effect [6]. Assuming the initial density spectrum to be

$$P(k) = A(\eta_0 - \eta_{\mathrm{rec}})^3(k(\eta_0 - \eta_{\mathrm{rec}}))^{n-4}. \tag{41}$$

Then from (38), (40) and (41) we see that

$$C_l = \frac{2^n\pi^3}{9}\ A\ \frac{\Gamma(3-n)\Gamma\left(\frac{2l+n-1}{2}\right)}{\Gamma^2\left(\frac{4-n}{2}\right)\Gamma\left(\frac{2l+5-n}{2}\right)} \tag{42}$$

for $n = 1$ (which is the prediction of many inflation models)

$$C_l = \frac{8\pi^2}{9}\frac{A}{l(l+1)}. \tag{43}$$

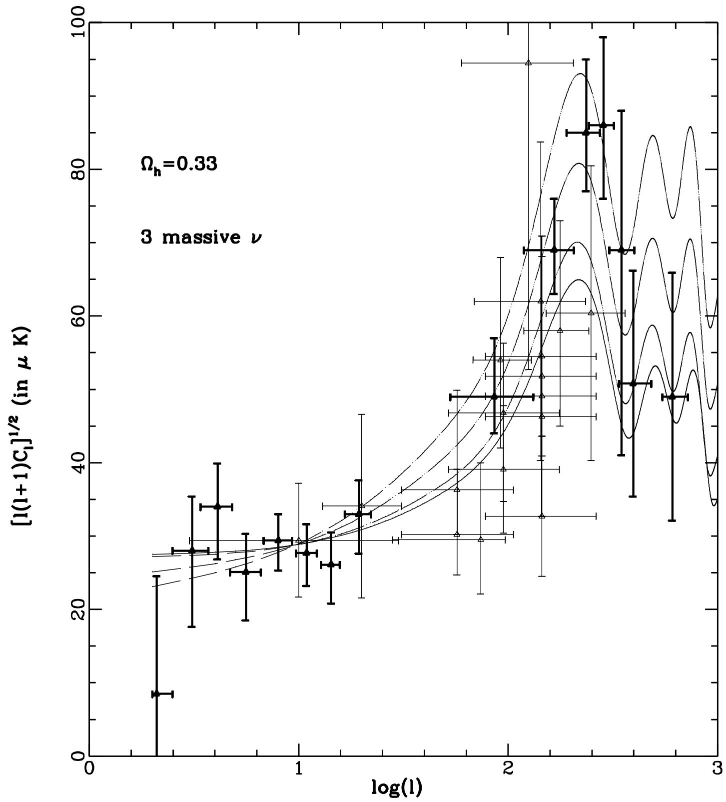

Figure 1. A standard neutrino model with different n values. C_l curves from bottom to top correspond to $n = 1, 1.1, 1.2, 1, 3$. Bold data points refer to COBE (Tegmark 1996), CAT (Scott *et al*, 1996), Saskatoon (Netterfield *et al*, 1997); see Scott *et al* (1995) for a summary of all the experiments.

The Cobe measurement [7] of $\Delta T/T$ at large angular scales determined the amplitude of the primordial density spectrum. The present observations at all angular scales is shown in figure 1. Observations from MAP and PLANCK experiments will considerably lower the error bars and will be instrumental in constraining theoretical models.

References

[1] C Ma and E Bertschinger, *Ap. J.* **455**, 7 (1995)
[2] W Hu and N Sugiyama, *Ap. J.* **471**, 542 (1996)
[3] James G Bartlett, astro-ph/9903260
[4] R Cowsik and J Mcclelland, *Phys. Rev. Lett.* **29**, 669 (1972)
[5] Uros Seljak, *Ap. J.* **469**, 437 (1996)
[6] R K Sachs and A M Wolfe, *Ap. J.* **147**, 73 (1967)
[7] G F Smoot *et al*, *Ap. J.* **396**, L1 (1992)