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Starobinsky model in Schrödinger description

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Abstract. In the Starobinsky inflationary model inflation is driven by quantum corrections to the vacuum Einstein equation. We reduce the Wheeler–DeWitt equation corresponding to the Starobinsky model to a Schrödinger form containing time. The Schrödinger equation is solved with a Gaussian ansatz. Using the prescription for the normalization constant of the wave function given in our previous work, we show that the Gaussian ansatz demands Hawking type initial conditions for the wave function of the universe. The wormholes induce randomness in initial states suggesting a basis for time-contained description of the Wheeler–DeWitt equation.

Keywords. Fourth-order gravity; inflation; curvature fluctuation; decoherence; wave function of the universe.

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1. Introduction

Inflation is an essential ingredient in modern cosmology to solve the horizon, the flatness and the monopole problems. In order to ascertain whether inflation is natural, there have been several attempts to study the curvature squared cosmology along with the Einstein curvature term. Even before Guth's [1] proposal of inflationary scenario, Starobinsky [2] proposed an inflationary model taking one loop quantum corrections to the classical Einstein Lagrangian. It has been shown that this model is equivalent to the curvature squared model in which the universe tunnels into the de Sitter phase that becomes unstable and finally emerges into the Friedmann era.

In order to show that the inflation is quite a natural phenomena for any initial conditions (called no hair conjecture), several authors [3] attempted to show that, for some models, inflation is really an attractor. The works done by Mijic, Moris and Suen [4], Maeda [5] and Vilenkin [6,7] refer to the early universe situation of inflationary cosmology. Vilenkin deals with a quantum view point to study the Starobinsky model and considers the tunneling phenomenon in the de Sitter phase through solution of the Wheeler–DeWitt equation [7] in quantum cosmology. In this approach one requires a specific ansatz for the initial conditions, that are currently divided into two ways; the Vilenkin proposal [8] and the

Hartle–Hawking proposal [9]. In the approach (without phase transition), the de Sitter solution is obtained as self-consistent solution of the vacuum Einstein equations modified by one loop quantum corrections due to quantized conformal matter fields or through the solution of the Wheeler–DeWitt equation with appropriate boundary conditions leading to inflationary de Sitter modes. The other approach is related to phase transitions. Here the inflation is driven by a false vacuum energy density and is related to the inflaton field in the model. The no hair conjecture is found to exist in both type of approaches as revealed by the works of Starobinsky and Schmidt [10]. In Maeda [5] as well as in Mijic *et al* [4], the analysis of the inflation is looked upon in a classical way. Using Wald's [11] strong and dominant energy condition, it has been shown that in R^2-type cosmology inflation is quite natural but acts as a transient attractor only. In [4], the curvature squared theory has been shown to be equivalent to the Einstein gravity theory with a scalar field but without any potential form for the scalar field. Though Maeda succeeds in obtaining a potential form in their work with a long plateau region, it is not clear how the universe from a region of Planck region arrives at the plateau region simply by losing energy to roll over the flat region depending on initial conditions. We believe, the answer should come through the quantum behaviour, which is absent in their model [4,5]. The quantum analysis carried out by Vilenkin [7] looks nice. However the initial condition of tunneling behaviour with respect to no hair conjecture has not been tackled with justice. However Starobinsky analysed his R^2 type theory to show that there is an unstable de Sitter solution followed by the present Friedmann era after sufficient inflation. Though Vilenkin tried to answer some apsects of stability of de Sitter solution in the Starobinsky model, it is not clear even now how the self consistent de Sitter solution in the Starobinsky model exits to the Friedmann era, though it is unstable in both past as well as in the future. We try to understand this aspect from the energy conservation.

Moreover the curvature fluctuation term has been introduced in an adhoc way. In calculating the curvature fluctuations, Vilenkin treats the wave function as if it is normalized but it is known that the normalization of Vilenkin wave function is awkward. Whereas the instability is related to the curvature fluctuations and which in turn is related to the randomness in the initial states it is therefore necessary to study the origin of randomness. As the Starobinsky solution emerges due to the quantum corrections, the quantum superposition principle remains inherent in this description. Hence the emergence of the classical de Sitter universe should be answered in a natural way from the study of the quantum wave function of the universe.

In this paper we try to answer this problem at least being successful to point out the direction in which the investigation has to be persued to get a final answer. Our aim is to study (i) the normalization aspect of the wave function, (ii) the tunneling probability and (iii) the randomness and the curvature fluctuation in the Starobinsky description. It is shown that the energy momentum conservation forces a de Sitter universe to tunnel into a realistic spacetime. The tunneling from the Euclidean to the Lorentzian sector also demands a Wheeler–DeWitt type equation for the quantum wave function (referred as instanton) to settle the initial conditions which in turn is related to the two proposals of the quantum wave function of the universe. Here we find that the wormhole connected universe greatly modifies the initial conditions to be imposed on the Euclidean–de Sitter instanton universe. We report in this paper the main results.

The quantum instability of the de Sitter spacetime has been discussed at length by Mottola [12] and many others to understand the nature of Im $L_{\rm eff}$ with respect to the de Sitter spacetime. It may be pointed out that the Im $L_{\rm eff}$ is a measure of the instability. From these works and from the work of Nevelli [13], it is clear that the tunneling paths in the Euclidean sector are crucial to understand the instability of the de Sitter vacuum. But there are no successful attempts to include such a contribution in the Starobinsky type model. In §2 we discuss the Starobinsky model with some known results, barely needed in our follow up discussion and also discuss the tunneling of the universe in the de Sitter space, fixed by the energy momentum conservation. This treatment allows us to persue the quantum evolution of the universe in the context of the Wheeler–DeWitt equation and study the evolution of the universe in R^2 cosmology. In §3 we introduce a time variable to reduce the Wheeler–DeWitt equation to the Schrödinger form (containing time) of quantum mechanics. Section 4 deals with the calculations of the wave function in the Starobinsky model. Section 5 ends with a discussion.

2. The Starobinsky model

In the Starobinsky model inflation is driven by the quantum corrections to the vacuum Einstein equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -8\pi G\langle T_{\mu\nu}\rangle, \tag{1}$$

where $\langle T_{\mu\nu}\rangle$ is the quantum corrections and is given by

$$\langle T_{\mu\nu}\rangle = \frac{\alpha}{6}(1)_{H\mu\nu} + \beta(3)_{H\mu\nu}, \tag{2}$$

where

$$(1)_{H\mu\nu} = R_{;\mu;\nu} - g_{\mu\nu}R_{;\sigma}^{;\sigma} + RR_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R^2 \tag{3}$$

and

$$(3)_{H\mu\nu} = R_{\mu}^{\sigma}R_{\nu\sigma} - \frac{2}{3}RR_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R^{\sigma\tau}R_{\sigma\tau} + \frac{1}{4}g_{\mu\nu}R^2. \tag{4}$$

We use the following conventions

$$R^{\alpha}_{\beta\gamma\delta} = \partial_\delta\Gamma^{\alpha}_{\beta\gamma} - \partial_\gamma\Gamma^{\alpha}_{\beta\delta} - \Gamma^{\alpha}_{\mu\gamma}\Gamma^{\mu}_{\beta\delta} + \Gamma^{\alpha}_{\mu\delta}\Gamma^{\mu}_{\beta\gamma}, \tag{5}$$

$$R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu} \tag{6}$$

and the Ricci identity as

$$A^{\alpha}_{;\nu;\mu} - A^{\alpha}_{;\mu;\nu} = A^{\beta}R^{\alpha}_{\beta\mu\nu}, \tag{7}$$

with $R^{\alpha}_{\beta\gamma\delta}$ antisymmetric in last two indices. The metric is assumed to be of the Robertson–Walker form,

$$\mathrm{d}s^2 = C(\eta)\left[\mathrm{d}\eta^2 - \mathrm{d}\Sigma_k^2(r,\theta,\phi)\right], \tag{8}$$

where $d\Sigma_k^2$ is the metric on a 3-sphere, 3-plane and 3-hyperboloid for closed $(k = +1)$, flat $(k = 0)$ and open $(k = -1)$ metric respectively. As we would need some properties of $(1)_{H\mu\nu}$ and $(3)_{H\mu\nu}$ in our description, we mention here some salient features. The tensor $(1)_{H\mu\nu}$ is identically conserved. Using the Bianchi identity

$$\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)_{;\nu} = 0, \tag{9}$$

we get from (3)

$$(1)H^{\nu}_{\mu;\nu} = R^{;\nu}_{;\mu;\nu} - g^{\nu}_{\mu}R^{;\sigma}_{;\sigma;\nu} + R_{;\nu}R^{\nu}_{\mu}. \tag{10}$$

Defining $R^{;\nu} = A^{\nu}$ and using Ricci identity eq. (7), we get

$$(1)_{H^{\nu}_{\mu;\nu}} = 0. \tag{11}$$

The tensor $(3)_{H\mu\nu}$, on the other hand is conserved only in the conformally flat spacetime and can be written as

$$(3)_{H\mu\nu} = \frac{1}{12}R^2 g_{\mu\nu} - R^{\rho\sigma}R_{\rho\mu\sigma\nu}. \tag{12}$$

The trace of the above two tensors is given by

$$(1)_{H^{\mu}_{\mu}} = -3R^{;\nu}_{;\nu} \tag{13}$$

and

$$(3)_{H^{\mu}_{\mu}} = \frac{1}{3}R^2 - R_{\alpha\beta}R^{\alpha\beta}. \tag{14}$$

The sum of the eqs (13) and (14) gives the so called trace anomaly i.e., the regularization carried out in n-dimension keeps a footprint when put back in 4-dimension and thus breaks the conformal invariance of the original action. In the RW spacetime since we have a single variable $C(\eta)$, it is sufficient to consider the time–time components of eq. (1); all other components will be linearly related. We write $(1)_{H_{00}}$ and $(3)_{H_{00}}$ explicitly.

$$(1)_{H_{00}} = \frac{1}{2}C^{-1}\left[-9\ddot{D}D + \frac{9}{2}\dot{D}^2 + \frac{27}{8}D^4 + 9kD^2 - 18k^2\right], \tag{15}$$

$$(3)_{H_{00}} = C^{-1}\left[\frac{3}{16}D^4 + \frac{3}{2}kD^2 + 3k^2\right], \tag{16}$$

$$R_{00} - \frac{1}{2}g_{00}R = -3\left(\frac{D^2}{4} + k\right). \tag{17}$$

Here $D = \dot{C}/C$ and the dot refers to the derivative with respect to η. Introducing

$$d\eta = \frac{dt}{a(t)}, \tag{18}$$

we get the '00' component of (1) as

$$\frac{a'^2+k}{a^2} = \frac{1}{H_0^2}\left[\frac{a'^2+k}{a^2}\right]^2 - \frac{1}{M_0^2}\left[\frac{2a'a'''}{a^2} - \frac{a''^2}{a^2} + \frac{2a''a'^2}{a^3} -3\left(\frac{a'}{a}\right)^4 - 2k\frac{a'^2}{a^2} + \frac{k^2}{a^4}\right], \tag{19}$$

where all constants β, α and G are incorporated in H_0 and M_0. Equation (19) has solutions

$$a(t) = H_0^{-1}\cosh H_0 t, \quad k=+1 \tag{20}$$

$$a(t) = a_0\exp(H_0 t), \quad k=0 \tag{21}$$

$$a(t) = H_0^{-1}\sinh H_0 t, \quad k=-1 \tag{22}$$

Equations (1) and (19) refer to the Starobinsky description. In quantum language it is said that the universe tunnels quantum mechanically from 'nothing' to the de Sitter inflationary mode (eqs (20)–(22)). Here 'nothing' means no classical spacetime. At the moment of nucleation, $t=0$ the universe has a size $a(0)=H^{-1}$, zero velocity i.e., $a'(0)=0$. This is the beginning of time. In quantum theory, tunneling is mainly due to vacuum fluctuations. The quantum cosmology deals with the tunneling of universe from the Euclidean region $a<H^{-1}$ usually termed as the classically forbidden region to a classically allowed region $a>H^{-1}$. There are several modes of inflationary models, among which the de Sitter mode of evolution plays a key role. If the de Sitter phase plays any previledged role with signature change occurring at the moment of nucleation, one has to know the behaviour of quantum wave function of the universe in the Lorentzian regime from a path integral approach or from the tunneling approach.

To decide some of these aspects and also the stability of the Starobinsky phase we note the following exercise. Let us write $(1)_{H\mu\nu}$ term in two parts

$$(1)_{H\mu\nu} = \left[R_{;\mu;\nu} - g_{\mu\nu}R^{;\alpha}_{;\alpha} - RR_{\mu\nu} + \frac{g_{\mu\nu}R^2}{4}\right] + \left[2R\left(R_{\mu\nu} - \frac{g_{\mu\nu}R}{4}\right)\right] = (1)_{\tilde{H}\mu\nu} + (1)_{S\mu\nu} \tag{23}$$

and neglect the second term to construct $\langle T_{\mu\nu}\rangle$. The $\langle T_{\mu\nu}\rangle$ so constructed is not conserved as is evident from eqs (10) and (11). Neglecting $(1)_{S\mu\nu}$ term we get

$$2^{(1)}H_{00} = \frac{1}{C}\left(-9D\ddot{D} + \frac{45D^4}{8} - \frac{9\dot{D}^2}{2} + 27kD^2C + 18k^2\right). \tag{24}$$

Using eqs (24) and (16) in (1) we get for the '00' component of the Einstein equation as

$$\frac{a'^2+k}{a^2} = \frac{1}{H_0^2}\left[\frac{a'^2+k}{a^2}\right]^2 - \frac{1}{M_0^2}\left[\frac{2a'a'''}{a^2} + \frac{a''^2}{a^2} + \frac{2a''a'^2}{a^3} - \frac{5a'^4}{a^4} - 6k\frac{a'^2}{a^4} - \frac{k^2}{a^4}\right], \tag{25}$$

with $a' = \partial a/\partial t$, d$\eta$ =d$t/a(t)$. All constants β, α and G are incorporated in H_0 and M_0. Equation (25) has a characteristic feature that, it has also the same set of solutions (20)–(22) as well as the Euclidean–de Sitter solution for $k = +1, 0, -1$ but $\langle T_\mu^{\ \nu} \rangle_{;\nu} \neq 0$, as is evident from the expression of $(1)_{S\mu\nu}$. To show it, we convert eq. (25) setting $H = a'/a$ as

$$H^2(H^2 - H_0^2) = \frac{H_0^2}{M_0^2}(2HH^{''} + 10H^2H' + H'^2). \tag{26}$$

This shows that $H = H_0$ is a solution of (26). This result seems quite interesting. In realistic spacetime we know $T^\nu_{\mu;\nu} = 0$. So emergence in the de Sitter mode is possible only if $(1)_{S\mu\nu} = 0$. From the expression given in eq. (23), we find that for the de Sitter type of solution $(1)_{S\mu\nu}$ exactly vanishes. This explains the emergence of de Sitter phase in the Starobinsky description. As soon as the universe evolves into the Starobinsky type de Sitter phase we have $t = 0$ i.e., the universe starts at $a = H^{-1}$. As we observe the de Sitter mode of solution even with $\langle T_\mu^{\ \nu} \rangle_{;\nu} \neq 0$, the existence of such a solution is permitted provided the violation obeys

$$(\Delta T_{00})(\Delta \tau) \gg 1, \tag{27}$$

where $(\Delta \tau) \ll$ the Planck time. Equation (27) then implies $\Delta T_{00} \gg 10^{19}$ GeV i.e., we have a hot big bang scenario with temperature $\approx 10^{17}$ GeV.

It is quite probable that at such a short scale (10^{-33} cm) higher order corrections like R^3, R^4 etc. begin to compensate the violation. However, whatever be the magnitude of compensation, the violation is there and the only way the universe tunnels to a realistic spacetime is through the de Sitter mode of evolution making $(1)_{S\mu\nu} = 0$ and there is a signature change at the moment of nucleation. The universe tunnels from the Euclidean–de Sitter to the Lorentzian de Sitter solution. Using the expression for $(1)_{S\mu\nu}$, we get

$$|\Delta T_{00}| \approx \frac{k^2}{C} \approx \frac{k^2}{a^2(0)}, \tag{28}$$

taking $a'(0) = 0$ around the nucleation point. Now $a(0) = H^{-1}$, so

$$|\Delta T_{00}| \approx k^2 H^2. \tag{29}$$

At this point one may be tempted to argue that the tunneling to the Starobinsky phase occurs with higher probability when H gets smaller. But we then have to abandon the inflationary scenario and its fruits related to the horizon and the flatness problem. However there is a dynamical zeroing of $|\Delta T_{00}|$ leading to de Sitter mode causing $(1)_{S\mu\nu} = 0$. In order to incorporate this dynamical aspect in the framework of the quantum cosmology we lean towards the calculation of tunneling probability to the Starobinsky phase. The treatment will help us answer about the nature of the boundary conditions that would satisfy the solution of the Wheeler–DeWitt equation.

It should be pointed out that while subtracting a part from the energy momentum tensor, as if we spoiled the energy momentum conservation still getting the de Sitter solution with the truncated T_{00}. What we observe that only for the de Sitter solution, $(1)_{S\mu\nu}$ term is automatically zero and the energy conservation is restored. For any other solution energy momentum conservation would be violated. This is the reason of spontaneous nucleation of the Starobinsky phase into the de Sitter phase, hitherto not pointed out in any previous work. One should not confuse that we get de Sitter solution even violating the energy conservation. This would, if true, then leads to horizon size fluctuations at the onset of the FRW hot big bang destroying the benefits of inflation. What we like to point out is that the uncertainty principle plays a decisive role in setting the quantum character of the universe as well as the initial conditions.

3. Time in quantum gravity and the Schrödinger equation

The discussion in §2 reveals that the quantum corrections to the vacuum Einstein's equation drive the inflation in the Starobinsky model and there is a transition from the Euclidean to the Lorentzian region. Moreover, the transition to the classical spacetime occurs for $T^{\nu}_{\mu;\nu} = 0$, with a de Sitter inflation. There occurs thus a quantum to classical transition in which energy conservation plays a deciding role as if it acts as a boundary condition for the transition.

In the classical description, the time and the Hamiltonian are related through the Hamilton's equation whereas the Schrödinger equation plays the same role in the quantum description. But in quantum gravity, because of the constraint relation $H = 0$, quantization of the gravitational field faces a conceptual problem in that no time parameter appears at a fundamental level. This is the problem of time in quantum gravity and recently the matter is under serious investigation by various authors [14–20].

The energy conservation, the Schrödinger equation (the Wheeler–DeWitt equation in quantum gravity) and the interpretational framework require the introduciton of an external time parameter to understand in a coherent way the quantum to classical transition and the tunneling problem in the quantum gravitational description. We will be mainly concerned with these two aspects in the Starobinsky description. The emergence of universe in classical region requires the solution of the Wheeler–DeWitt equation and the effectiveness i.e., the interpretational framework will be discussed in studying the decoherence mechanism in quantum gravity.

It has been shown by Santamato [21] how to arrive at the Schrödinger equation starting from classical description using Madelung–Bohm [22,23] and Feynes–Nelson [24,25] approaches bearing epistemological content of traditional quantum mechanics. He starts with a function $S(q,t)$ satisfying the Hamilton–Jacobi equation of the classical mechanics

$$\frac{\partial S}{\partial t} + H(q, \nabla S, t) = 0, \tag{30}$$

where $H(q, p = \nabla S, t)$ is the classical Hamiltonian such that

$$v^i(q,t) = \frac{\partial H(q, \nabla S, t)}{\partial p_i} \tag{31}$$

and following Madelung–Bohm and Feynes–Nelson approaches defines a probability density $\bar{\rho}(q,t)$ satisfying the continuity equation

$$\partial_t \rho + \partial_i(\bar{\rho} v^i) = 0. \tag{32}$$

It has been then shown that the function

$$\psi(q,t) = \sqrt{\rho(q,t)}\exp\left[\frac{i}{\hbar}S(q,t)\right] \tag{33}$$

satisfies the Schrödinger equation for a given Hamiltonian combining (30) and (32) with gradient of scalar curvature of space (the Weyl space) introducing randomness on initial positions.

As there is no time variable in quantum gravity we demand $S(q,t) = S(q)$ and it satisfies the Hamilton–Jacobi equation but source free. Defining a time parameter, we call it WKB time, it is obvious that $\sqrt{\rho(q,t)} \propto \psi(q,t)$ then satisfies the Schrödinger equation when $S(q,t)$ is time independent. Though not identical, the same view is also expressed by Kiefer [18,19], in defining a time operator d/dt starting from a minisuperspace Wheeler–DeWitt equation. We discuss briefly the outline of our approach with minimally coupled scalar field in gravitational background. The details is placed elsewhere [26]. In the next section the approach will be elucidated for the Starobinsky description.

We start with an action

$$I = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{-R}{16\pi G} - \frac{1}{2\pi^2}\left(\frac{1}{2}\phi_{,\mu}\phi^{,\mu} + V(\phi)\right)\right] - \frac{1}{8\pi G}\int_\Sigma \mathrm{d}^3x\sqrt{h}K \tag{34}$$

in the FRW model

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right]. \tag{35}$$

The Hamiltonian constraint is

$$-\frac{1}{2Ma}P_a^2 + \frac{1}{2a^3}P_\phi^2 - \frac{M}{2}ka + a^3V(\phi) = 0. \tag{36}$$

The dynamical equations are

$$\frac{\ddot{a}}{a} = -\left[\frac{\dot{a}^2}{2a^2} + \frac{k}{2a^2} + \frac{3}{M}\left\{\frac{1}{2}\dot{\phi}^2 - V(\phi)\right\}\right] \tag{37}$$

and

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + \frac{\partial V}{\partial\phi} = 0. \tag{38}$$

In (34) to (37), $M = (3\pi/2G) = (3\pi m_{\mathrm{Planck}}^2/2), k = 0, \pm 1$ and

$$P_a = -Ma\dot{a}, \quad P_\phi = a^3\dot{\phi}. \tag{39}$$

The prime denotes derivative with respect to time and K is the trace of extrinsic curvature. Identifying $P_a = (\partial S_0/\partial a)$, $P_\phi = (\partial S_0/\partial\phi)$, the Hamilton–Jacobi equation is

$$-\frac{1}{2M}\left(\frac{\partial S_0}{\partial a}\right)^2 + \frac{1}{2a^2}\left(\frac{\partial S_0}{\partial\phi}\right)^2 - \frac{1}{2}MKa^2 + a^4V(\phi) = 0. \tag{40}$$

We define a time operator (a directional derivative)

$$\frac{\partial}{\partial t} = \sum_i \left(\frac{\partial H}{\partial P_i} \frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial}{\partial P_i} \right). \tag{41}$$

Going to the FRW minisuperspace, we get the Wheeler–DeWitt equation $H\psi = 0$ i.e.,

$$\left[\frac{\hbar^2}{2M} \left(\frac{\partial^2}{\partial a^2} + \frac{p}{a} \frac{\partial}{\partial a} \right) + \frac{\hbar^2}{2a^2} \frac{\partial^2}{\partial \phi^2} - \frac{Mka^2}{2} + a^4 V(\phi) \right] \psi = 0. \tag{42}$$

Identifying $P_a = -i\hbar(\partial/\partial a)$ and $P_\phi = -i\hbar(\partial/\partial \phi)$ and substituting

$$\psi(a, \phi) = \exp \left[i \sum_{n=0}^{\infty} A_n M^{1-n} \right] \Phi(a, \phi) \tag{43}$$

in (42) we find in M^2- order $(\partial A_0/\partial \phi) = 0$, and in M^1- order

$$\frac{a^2}{2} \left(\frac{\partial A_0}{\partial a} \right)^2 + \frac{1}{2} k a^4 = 0. \tag{44}$$

Identifying $S_0 = MA_0$, eq. (44) gives the source free Hamilton–Jacobi equation (see (40)):

$$\frac{a^2}{2M} P_a^2 + \frac{M}{2} k a^4 = 0. \tag{45}$$

Using this condition in (41) we get

$$\frac{\mathrm{d}}{\mathrm{d}t} = -\frac{1}{Ma} \frac{\partial S_0}{\partial a} \frac{\partial}{\partial a}, \tag{46}$$

since $S_0 = S_0(a)$ only. Thus writing

$$\psi(a, \phi) = \mathrm{e}^{iA_0(a)M} \Phi(a, \phi) \tag{47}$$

and substituting (47) in (42) and using (44) and (46) we get

$$i\hbar \frac{\partial \Phi}{\partial t} \simeq H_\phi(a, \phi) \Phi \tag{48}$$

neglecting $\hbar^2-$ order term $(\partial^2 \Phi/\partial a^2)$. As mentioned in the beginning of this section, we obtain the reduction (48) using (41), instead of the continuity equation. In the next section we take up the Starobinsky model to evaluate $|\psi^2|$ using a Gaussian ansatz for $\Phi(a, \phi)$.

One might point out various drawbacks in obtaining (48) using (41), (43) and (46), which are now being persued by many authors in the framework: canonical quantization of gravity [27,28]. Recently we have been able to obtain (48) using the prescription of 'time before quantization' [26]. One may consider (48) from a different angle. Suppose we have (48) valid in the semiclassical region, we construct H_ϕ for the Starobinsky description, adopt a boundary condition for ϕ (of course valid in the large scale factor region), investigate whether the boundary condition is anyhow related to the boundary condition at the small scale factor region and reproduces the wave function of the universe suiting a given boundary condition proposal.

4. Wave functions in the Starobinsky model

The main problem in effecting canonical quantization in the Starobinsky description in minisuperspace formalism is the absence of an action in closed form. However it has been shown that [7] for $M_0^2 \ll H_0^2$ an action in closed form is obtained as

$$S = 2\pi^2 \int L(R) a^3 \mathrm{d}t \tag{49}$$

with

$$L(R) = \frac{1}{16\pi G}\left(R + \frac{R^2}{6M_0^2} + \frac{R^2}{R_0}\ln\frac{R}{R_0}\right), \tag{50}$$

where $R_0 = 12H_0^2$. Evaluating the action it is found that it is maximum at $R = R_0$ so that the semiclassical tunneling probability is given by

$$P \propto \exp(-|S_0|) = \exp\left(-\frac{4\pi}{GM_0^2}\right). \tag{51}$$

The curvature fluctuation is obtained by expanding $S(R)$ around $R = R_0$,

$$S(R) = S_0 + \frac{1}{2}S''(R_0)(\delta R)^2. \tag{52}$$

Using (49) and (50), $S(R)$ is given by

$$S(R) = \frac{24\pi}{G}\left(\frac{1}{R} + \frac{1}{6M_0^2} + \frac{1}{R_0^2}\ln\frac{R}{R_0}\right), \tag{53}$$

so that the curvature fluctuation

$$\left(\frac{\delta R}{R_0}\right)^2 \sim \frac{GH_0^2}{\pi}. \tag{54}$$

It is interesting to note that this curvature fluctuation is proportional to $|\Delta T_{00}|$ as is found in (28). However the result (54) is classical. An informative description for the nucleation of the universe can be obtained by solving the Wheeler–DeWitt equation for the wave function of the universe. Vilenkin [7] attempted this problem solving (42) for the Starobinsky model with the tunneling boundary condition to obtain a curvature fluctuation of the order of (54). Our aim is to study these aspects with the formalism just mentioned above to explore:

(i) the normalization aspect of the wave function,
(ii) the tunneling probability,
(iii) the randomness and the curvature fluctuation,
(iv) the randomness and the decoherence and
(v) the back reaction and the wormholes contribution to the solution of the Wheeler–DeWitt equation. In this paper we take up the first three point.

Using the method outlined in [7], the Wheeler–DeWitt equation reads

$$\left[\frac{\partial^2}{\partial q^2} - \frac{1}{q^2}\frac{\partial^2}{\partial x^2} - V(q,x)\right]\Psi(q,x) = 0, \tag{55}$$

where $q = H_0 a(L'/L'_0)^{1/2}$ with $L' = L'(R)$ and $L'_0 = L'(R_0)$, and

$$x = \frac{1}{2}\ln\left(\frac{R}{R_0}\right), \tag{56}$$

$$V(q,x) = \lambda^{-2} q^2 (1 - q^2 + \mu^2(x) q^2), \tag{57}$$

$$\mu^2(x) = \frac{M_0^2}{2H_0^2}(2x + e^{-2x} - 1), \tag{58}$$

and

$$\lambda = \frac{GM_0^2}{6\pi}. \tag{59}$$

In deriving (55) it is assumed that $M_0^2 \ll H_0^2$ and terms $\sim (M_0/H_0)^4$ are neglected. Assuming $|x| < 1$ and substituting $Q = q/\sqrt{\lambda}$, we find using (55) and (57)

$$\left[\frac{\partial^2}{\partial Q^2} - \frac{1}{Q^2}\frac{\partial^2}{\partial x^2} - Q^2(1 - Q^2 U(x))\right]\Psi = 0, \tag{60}$$

where

$$U(x) = \lambda(1 - m^2 x^2) \tag{61}$$

and

$$m^2 = \frac{M_0^2}{H_0^2}. \tag{62}$$

As discussed in the previous section, we convert (55) in the Schrödinger form substituting the WKB form

$$\Psi = \exp\left(i\sum_{n=0}^{\infty} M^{1-n} A_n\right)\psi(Q,x). \tag{63}$$

Collecting terms in different order in M we obtain for M^2- order $(\partial A_0/\partial x) = 0$, which implies that A_0 is purely a functional of the gravitational field and M^1- order gives the source free Hamilton–Jacobi equation

$$\left(\frac{\partial A_0}{\partial Q}\right)^2 + Q^2 = 0. \tag{64}$$

We introduce a time operator as

$$\frac{\mathrm{d}}{\mathrm{d}t} = -\frac{1}{Q}\,\frac{\partial A_0}{\partial Q}\,\frac{\partial}{\partial Q} \tag{65}$$

and substitute (63) in (60) to get

$$i\frac{\partial\psi(Q,x)}{\partial t} = \left[-\frac{1}{2Q^3}\,\frac{\partial^2}{\partial x^2} + \frac{Q^3}{2}U(x) + \frac{1}{2Q}\right]\psi(Q,x). \tag{66}$$

In obtaining (66) we have neglected terms $\sim (\partial^2\psi/\partial Q^2)$. After the WKB reduction we have set $M = 1$. Thus we have

$$\Psi = \psi_0(Q)\psi(Q,x), \tag{67}$$

where

$$\psi_0 = \mathrm{e}^{iA_0(Q)}. \tag{68}$$

From (64) and (68) one finds

$$\psi_0(Q) = \mathrm{e}^{\pm(Q^2/2)}. \tag{69}$$

In terms of the q variable

$$\psi_0(q) = \mathrm{e}^{\pm(q^2/2\lambda)}. \tag{70}$$

The solution with the negative sign in the exponent was obtained by Vilenkin with the assumption that $q \ll 1$ and

$$\frac{\partial\Psi}{\partial x}(0,x) = 0 \tag{71}$$

and the tunneling solution exponentially decreases in the classically forbidden region. From (55) the classically forbidden region lies between $q = 0$ and $q \simeq 1 + (1/2)\mu^2(x)$ and acts as turning points.

Now to solve (66) we make a Gaussian ansatz

$$\psi = N(t)\mathrm{e}^{-(\Omega(t)/2)x^2}. \tag{72}$$

Substituting this in (66) leads to coupled equations for Ω and N

$$i\frac{\mathrm{d}}{\mathrm{d}t}\ln N = \frac{\Omega}{Q^3} + Q^3\lambda + \frac{1}{Q} \tag{73}$$

and

$$i\dot{\Omega} = \frac{\Omega^2 + Q^6\lambda m^2}{Q^3}. \tag{74}$$

In (73) and (74), t is the WKB time defined by (65) and parameterizes the classical trajectories in the minisuperspace which is now spanned by Q. With the ansatz

$$\Omega = -iQ^3 \frac{\dot{y}}{y}, \tag{75}$$

one finds from (74)

$$\ddot{y} + 3\frac{\dot{Q}}{Q}\dot{y} - \lambda m^2 y = 0. \tag{76}$$

Introducing the conformal time η by the relation $\mathrm{d}t = Q\mathrm{d}\eta$, we get from (76)

$$y'' + 2\frac{Q'}{Q}y' - \lambda m^2 Q^2 y = 0, \tag{77}$$

where prime denotes a derivative with respect to η. We now specify the model by specifying $Q = -(1/\sqrt{\lambda}\eta)$ such that the background undergoes an exponential expansion. Equation (76) now reduces to

$$y'' - \frac{2}{\eta}y' - \frac{m^2}{\eta^2}y = 0 \tag{78}$$

and is solved by

$$y = \eta^{3/2 \pm \sqrt{9/4 + m^2}}. \tag{79}$$

Approximating $\sqrt{9/4 + m^2} = (3/2 + m^2/3)$ (since $m^2 = M_0^2/H_0^2 \ll 1$), and using (75) in conformal time coordinate we get

$$\Omega = -im^2\sqrt{\lambda}\frac{Q^3}{3}. \tag{80}$$

As Ω is imaginary the state (72) will not be normalizable. One of the ways to obtain the real part in Ω is to consider various mode solutions of the scalar field x as in the work of Kiefer [3]. However we will follow a different line based on our previous work [29] having interpretational significance along with settling the boundary conditions. Remembering that $m^2\sqrt{\lambda}$ is a very small quantity, we approximate (73) for large Q,

$$i\frac{\mathrm{d}}{\mathrm{d}t}\ln N = \Omega^3\lambda. \tag{81}$$

With $\mathrm{d}/\mathrm{d}t = \sqrt{\lambda}Q(\mathrm{d}/\mathrm{d}Q)$, one finds from (81)

$$N = N_0 \exp\left[-\frac{iQ^3\sqrt{\lambda}}{3}\right] \tag{82}$$

so that

$$\psi = N_0 \exp\left[-\frac{iQ^3\sqrt{\lambda}}{3} + \frac{im^2\sqrt{\lambda}Q^3}{6}x^2\right], \tag{83}$$

where N_0 is a constant to be determined. The bracketed term in the exponent (83) is approximated as

$$
\begin{aligned}
S_{\text{eff}} &= -\frac{iQ^3\sqrt{\lambda}}{3} + \frac{im^2\sqrt{\lambda}Q^3}{6}x^2 \\
&= -\frac{i}{3}\left[Q^2\lambda(1-\mu^2(x))\right]^{3/2}\ \frac{1+\mu^2}{\lambda} \\
&\simeq -\frac{i}{3}\left[Q^2\lambda(1-\mu^2)-1\right]^{3/2}\ \frac{1+\mu^2(x)}{\lambda}.
\end{aligned} \tag{84}
$$

In our previous work [29] we have shown that the multiple reflections between the turning points will contribute to the normalization factor and are supposed to arise from the wormhole contributions. Using (84) and the result of [29] the normalization constant N_0, according to wormhole dominance proposal, turns out to be

$$
N_0 = \frac{\exp S_{\text{eff}}(Q_x, 0)}{1-\exp[2S_{\text{eff}}(Q,0)]}, \tag{85}
$$

where

$$
S_{\text{eff}}(Q_x, 0) = S_{\text{eff}}(Q)|_0^{Q_x} \tag{86}
$$

and $Q=0$ and $Q=Q_x=1/(\lambda(1-\mu^2(x)))^{1/2}$ are the turning points as can be seen from (55) and (57). We thus get from (85) and (83)

$$
\psi \sim \exp\left\{+\frac{1}{3\lambda}(1+\mu^2(x))\left[1-i(q-q^2\mu^2-1)^{3/2}\right]\right\}. \tag{87}
$$

For simplicity we have dropped the denominator of (85) but is nonetheless important while discussing the normalization. Equation (87) continued in classically forbidded region gives

$$
\psi \sim \frac{\exp\left\{\frac{1}{3\lambda}(1+\mu^2(x))\left[1-(1-q^2+q^2\mu^2)^{3/2}\right]\right\}}{(1-\exp\{2/3\lambda(1+\mu^2(x))\})}. \tag{88}
$$

Equations (87) and (88) are the wave functions that one gets from the Hawking's proposal.

We can now discuss the curvature fluctuations in the newly-born universe. From (87), the probability of nucleation with a certain value of x is proportional to

$$
|\psi^2| \propto \exp\left[-\frac{2\mu^2(x)}{3\lambda}\right]. \tag{89}
$$

In obtaining (89), a factor $(\mathrm{e}^{-(2/3\lambda)(1+\mu^2)}-1)^2$ in the denominator is kept as multiplying (89). For $x \ll 1, \mu^2(x)=m^2x^2=(M_0^2/H_0^2)x^2$ and we get

$$
|\psi|^2 \propto \exp\left[-\frac{4\pi x^2}{GH_0^2}\right]. \tag{90}
$$

Now it follows from (56) that $x = 1/2(\delta R/R_0)$, where $\delta R = R - R_0$ is the curvature fluctuations. Hence we can write

$$
\left\langle \left(\frac{\delta R}{R_0}\right)^2 \right\rangle = 4\langle x^2\rangle = \frac{GH_0^2}{2\pi}. \tag{91}
$$

Exactly this result was obtained by Vilenkin from the tunneling proposal, but our wave function corresponds to the Hawking's proposal.

Though the result (91) is identical with the Vilenkin, there are some marked interesting differences. In Vilenkin [7] $1/3\lambda(1+\mu^2(x))$ term is added in (87) to get the behaviour $S \to q^2/2\lambda$ for $q \to 0$. In our approach when the denominator in (87) is taken into account, $S \to -(q^2/2\lambda)$ as $q \to 0$. If we set aside the denominator in (87), the wave function looks apparently as the Vilenkin wave function. By analytic continuation, in the classically allowed region, Vilenkin gets back the term $-(1/3\lambda)(1+\mu^2)$ in the exponent to find

$$\psi \sim \exp\left\{-\frac{1}{3\lambda}(1+\mu^2(x))\left[1-i(q^2-q^2\mu^2-1)^{3/2}\right]\right\}. \tag{92}$$

This analytic continuation seems questionable. Thus we see that the Gaussian ansatz and the normalization prescription serve as the boundary conditions to settle between the two current proposals in favour of the Hawking's prescription. We also stressed this in our previous work [29]. The present work works out explicitly the equivalence between the timeless Wheeler–DeWitt equation (55) and the time contained Wheeler–DeWitt equation (66) and hence the time parameter prescription as mentioned in this work suits also in the frame work of the quantum cosmology.

5. Discussion

We find that the normalization factor arises from the repeated reflections between the turning points and hence corresponds to the higher order corrections related to the quantum fluctuations in the WKB description. In our previous work [29] we have shown that the factor N_0 can be interpreted as contributions from the wormholes using Klebanov and Coleman's arguments [30,31]. Apart from the wormhole picture, it can be said that the quantum force has its origin in the curvature fluctuations with 'to and fro' motion within $0 < q < 1 + (1/2)\mu^2(x)$ and it necessitates to uphold the probabilistic interpretation. Leaving aside the interpretational hindrance, 'to and fro motion' is assigned to terms like e^{iS} and e^{-iS} but the Gaussian ansatz required only the form (72) in the classically allowed region as if there is a suppression of interference terms. It has been argued that some boundary conditions at small scale [32] would lead to the quantum effects in the vicinity of the turning points. Thus the only way to interpret e^{iS} and e^{-iS} superposition (i.e., the quantum effects) is to turn back towards Klebanov and Coleman's arguments of wormholes connections around $q \approx 0$. There is also an objection about the 'to and fro' motion with respect to time because there is no external time parameter with respect to which the universe can turn. But our calculation in the time variable description correctly obtains the wave function both in the classically allowed and in the forbidden region. The classical turning points serve as a clue to obtain the peculiar behaviour of wave packets at a late time $(q \gg 1)$ whereas the quantum turning point (we identify it to be $q = 0$) leads to 'to and fro' quantum fluctuations. The Euclidean and the Lorentzian time both are manifestation of the spacetime structure latent in the description of the Wheeler–DeWitt equation. There is also a question about the influence of excited matter states and whether they lead to decoherence. In the present work we prove the decoherence in reverse way i.e., leading to the form (88) from the Gaussian ansatz (72); however we have been able to show [33] that the presence of effective decoherence even in the Starobinsky description, using the technique of ref. [16]. Perhaps the arrow of time starts functioning from the classical turning point when the universe emerges from the Euclidean to the Lorentzian regime with a signature change. It should be pointed out that the denominator in (88) arises from the multiple reflections from the turning points and indicates decisively in favour of the Hawking wave functions. If we accept the Klebanov and Coleman's [30,31] arguments, it would not be unjustified to comment that wormholes contribution generates the initial randomness, and the quantum behaviour thus generated even persists in the classical spacetime. The emergence of the classical universe is then explained through the decoherence to suppress the interference of exp iS and exp $-iS$ like terms. The code of the decoherence is seeded in the normalization constant rather than in the initial conditions.

It is worthwhile to point out that, at the semiclassical level, the Starobinsky inflationary scenario has been criticized [34] on the ground that the inflationary solutions are not perturbatively expandable in the parameter of 'quantum corrections' terms in the equation of motion or the Lagrangian. There are also some questions of treating the higher order terms in the Lagrangian on an equal footing with the Einstein terms. Even then, studying the quantum cosmology of the Starobinsky description from the standpoint of quantum to classical transition would help us rethink about the criticism labelled against the model. Leaving aside this fact, it is instructive to look at the model as a toy example, to understand the current boundary condition proposals, decoherence mechanism and also the origin of the quantum force in the early universe.

Further there are also some drawbacks in expanding the Wheeler–DeWitt wave function as a power of Planck mass as in (43) and in obtaining the Schrödinger–Wheeler–DeWitt equation (SWD) from it. Recently [26,35] we have been able to obtain the SWD equation without using the WD equation and also the expansion (43) and it has been found that the SWD wave function can be defined on the standard Hilbert space of quantum mechanics. Though the Starobinsky description itself has some drawbacks as discussed in [30], our attempt is to understand the quantum to classical transition in the framework of the model and to investigate which of the boundary condition proposal suits the inflationary description. We find that (i) the classical Starobinsky model is consistent with the quantum description, (ii) the wormhole dominance proposal [29] correctly connects the Wheeler–DeWitt wave function and the Schrödinger–Wheeler–DeWitt wave function in the respective regime, (iii) the curvature fluctuation is correctly reproduced from the wormhole dominance wave function, (iv) the wormholes initiate the quantum randomness in the initial stage (i.e., $0 < a < H^{-1}$) and (v) the decoherence is effectively [33] reproduced in the Starobinsky description, provided we accept the wormhole dominance proposal keeping intact the fruits of the inflationary scenario.

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References

[1] A H Guth, *Phys. Rev.* **D23**, 347 (1981)
[2] A A Starobinsky, *Phys. Lett.* **B91**, 99 (1980)
[3] W Boucher, G W Gibbons and G T Horewitz, *Phys. Rev.* **D30**, 2447 (1984)
[4] M B Mijic, M S Morris and W M Suen, Suen, *Phys. Rev.* **D34**, 2934 (1986)
[5] K Maeda, *Phys. Rev.* **D37**, 858 (1988)
[6] A Vilenkin, *Phys. Rev.* **D27**, 2848 (1983)
[7] A Vilenkin, *Phys. Rev.* **D32**, 2511 (1985)
[8] A Vilenkin, *Phys. Rev.* **D37**, 888 (1987)
[9] S W Hawking, *Nucl. Phys.* **B239**, 257 (1984)
S W Hawking and D N Page, *Nucl. Phys.* **B264**, 184 (1986)
[10] A A Starobinsky and H J Schmidt, *Class. Quantum. Gravit.* **4**, 694 (1987)
[11] R M Wald, *Phys. Rev.* **D28**, 2118 (1983)
[12] E Mottola, *Phys. Rev.* **D31**, 754 (1985)
[13] D E Nevelli, *Phys. Rev.* **D8**, 1695 (1984)
[14] J J Halliwell and S W Hawking, *Phys. Rev.* **D31**, 1777 (1985)
[15] C Kiefer, *Class. Quantum. Gravit.* **4**, 1369 (1987)
[16] C Kiefer, *Phys. Rev.* **D46**, 1658 (1992); **D45**, 2044 (1992)
[17] C Kiefer, in *Time, temporality, Now* edited by H Atmanspacher and R Buhnau (Springer, Berlin, 1997) pp 227–240
[18] C Kiefer, D Polarski and A A Starobinsky, gr-qc/9802003
[19] S Wada, *Nucl. Phys.* **B276**, 729 (1986)
[20] K V Kuchar, in *Proceedings of the fourth Canadian conference on general relativity and relativity astrophysics* edited by G Kunstatter, D Vincent and J Williams (World Scientific, Singapore, 1992) pp. 211–314
[21] E Santamato, *Phys. Rev.* **D29**, 216 (1984)
[22] E Madelung, *Z. Phys.* **40**, 332 (1926)
[23] D Bohm, *Phys. Rev.* **85**, 166 (1952)
[24] I Feynes, *Z. Phys.* **132**, 81 (1952)
D Kershaw, *Phys. Rev.* **B136**, 1850 (1962)
[25] E Nelson, *Phys. Rev.* **150**, 1079 (1966)
[26] S Biswas, A Shaw, B Modak and D Biswas, *Quantum gravity equation in Schrödinger form in minisuperspace description*, gr-qc/9906011
[27] J Butterfield and C J Isham, gr-qc/9901024
[28] C J Isham, gr-qc/9210011
[29] S Biswas, B Modak and D Biswas, *Phys. Rev.* **D55**, 4673 (1996)
[30] I Klebanov, L Susskind and T Banks, *Nucl. Phys.* **B317**, 665 (1989)
[31] S Coleman, *Nucl. Phys.* **B310**, 643 (1988)
[32] H D Conradi and H D Zeh, *Phys. Lett.* **A154**, 321 (1991)
[33] S Biswas, A Shaw and B Modak, *General Relativity and Gravitation* **31**, 1015 (1999)
[34] J Z Simon, *Phys. Rev.* **D45**, 1953 (1992)
L Parker and J Z Simon, *Phys. Rev.* **D47**, 1339 (1993)
[35] S Biswas, A Shaw and B Modak, *Time in Quantum Gravity*, gr-qc/9906010