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# Group of contact transformations: Symmetry classification of Fokker–Planck type equations

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**Abstract.** Fokker–Planck type equations have been classified according to the groups of contact transformations to which they belong. It has been found that there are only five classes as in the case of groups of point transformations. We have also obtained the algebraic structures of the corresponding Lie algebras. However, there are isomorphies in their group properties. The corresponding basis sets of functionally independent invariants formed by the generators of these groups have also been obtained.

**Keywords.** Fokker–Planck type equation; group of contact transformations; symmetry classification.

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## 1. Introduction

Fokker–Planck equation [1–4] and its generalization now occupy a permanent position in many fields of physical, biological and sociological problems. Basically this equation describes Gaussian stochastic systems with a white spectrum. The symmetry properties of point transformations of these types of equations have also been extensively investigated [5–10]. In a previous publication [11] we have shown that Fokker–Planck type equations belong to five basic groups of point transformations.

In this work we extend our work to groups of contact transformations. Groups of point transformations for any set of differential equations involve transformations of the dependent and independent variables. The groups of contact transformations [12], on the other hand, contain these variables as well as the gradients of the dependent variables. Contact transformations thus lie in-between the point transformations and the Lie–Bäcklund transformations. In studying the dynamics of a system, the group of contact transformations has an important role.

Our study of this system shows that the generalized Fokker–Planck equations belong to *five* classes:

1. Double harmonic symmetry type with 6 generators,
2. Genetic equation type with 4 generators,

3. Heat equation type with 6 generators,
4. Plasma equation type with 4 generators, and
5. General type with 2 generators.

We shall show that there is isomorphy in the symmetries of these equations. It is to be remembered that as in the case of point transformations [11] these generators correspond to the factor group of physical interest formed from the full group modulo the infinite-parameter invariant subgroup of linear combinations of solutions.

The number of invariants forming the functional basis is 2 in all cases except for one subclass of case 3.

## 2. Group of contact transformations

The method for obtaining [12] the group of contact transformations is a generalization of the method of extended group used in obtaining the group of point transformations. The set of differential equations is augmented by the contact conditions defining the gradient variables. The generators of the Lie group now contain also the gradient variables.

In the case of generalized Fokker–Planck equation in one space and one time variable, we have

$$\Delta_1 \equiv \frac{\partial P}{\partial t} - \beta_0(q)P - \beta_1(q)P_q - \beta_2(q)\frac{\partial P_q}{\partial q} = 0, \tag{1}$$

and the contact condition

$$\Delta_2 \equiv \frac{\partial P}{\partial q} - P_q = 0. \tag{2}$$

Here $P(q,t)$ is the probability function and $P_q$ is its derivative $\partial P/\partial q$, $\beta$s are the 'transition probabilities'. Equation (2) is actually the contact condition for $\partial P/\partial q$.

We have taken a more generalized form than the normal form of the Fokker–Planck equation and are calling it Fokker–Planck type equations. In this form there are *three* independent 'transition probabilities', $\beta_i(q)$, instead of the *two* in the normal form. The Fokker–Planck equation in the normal form is essentially the continuity equation for the probability function without any source term. In this case the rate of decay of the probability of remaining in a particular state is equal to the sum of transition probability rates from this state to all the other states [2]. However, in the presence of source term in the system this assumption fails, and we need the three independent 'transition probabilities' in the generalized Fokker–Planck type equation.

In our analysis we have not only found the complete contact symmetry classification of this generalized Fokker–Planck type equation, but also the relations that the three 'transition probabilities' must satisfy. These relations are obtained simultaneously with the group symmetry of the equation.

The generators of the Lie group of contact transformations

$$X = \xi^0(t,q,P,P_q)\frac{\partial}{\partial t} + \xi^1(t,q,P,P_q)\frac{\partial}{\partial q} + \phi_0(t,q,P,P_q)\frac{\partial}{\partial P} + \phi_1(t,q,P,P_q)\frac{\partial}{\partial P_q} \tag{3}$$

have the vectors $\xi^i$ and $\phi_i$. First extension of $X$ and the original equations (1), (2) will give us the following determining equations for the vectors $\xi^i$s and $\phi_i$s.

$$\beta_2 \xi^0_{P_q} = 0, \tag{4}$$

$$\beta_2 \xi^0_q + \beta_2 \xi^0_P P_q + P_q \xi^1_{P_q} - \phi_{0;P_q} + \xi^0_{P_q} \left[\beta_0 P + \beta_1 P_q\right] = 0, \tag{5}$$

$$\begin{aligned} -\phi_1 + \phi_{0;q} - P_q \left[\xi^1_q - \phi_{0;P}\right] - \xi^0_q \left[\beta_0 P + \beta_1 P_q\right] \\ -P_q \xi^0_P \left[\beta_0 P + \beta_1 P_q\right] - (P_q)^2 \xi^1_P = 0, \end{aligned} \tag{6}$$

$$P_q \xi^1_{P_q} - \phi_{0;P_q} - \beta_2 \xi^0_q - \beta_2 P_q \xi^0_P + \xi^0_{P_q} \left[\beta_0 P + \beta_1 P_q\right] = 0, \tag{7}$$

$$\beta_2 \xi^1_{P_q} - (\beta_1)^2 \xi^0_P = 0, \tag{8}$$

$$\begin{aligned} &-\beta_2 \left[\xi^1_q - \phi_{1;P_q}\right] + \xi^1 \beta_2' + \beta_2 \left[\xi^0_t - \phi_{0;P}\right] \\ &+2\xi^0_P \beta_1 \left[\beta_0 P + \beta_1 P_q\right] + \left[\beta_1 + \beta_2'\right] \xi^0_{P_q} \left[\beta_0 P + \beta_1 P_q\right] \\ &- \left[\beta_1 + \beta_2'\right] \left[\beta_2 \xi^0_q + \beta_2 P_q \xi^0_P + \phi_{0;P_q} - P_q \xi^1_{P_q}\right] = 0, \end{aligned} \tag{9}$$

$$\begin{aligned} &\xi^1 \beta_0' P + \xi^1 \beta_1' P_q + \phi_0 \beta_0 + \phi_1 \beta_1 - \phi_{0;t} + \xi^1_t P_q + \beta_2 \phi_{1;P_q} \\ &+\beta_2 P_q \phi_{1;P} + \left[\beta_0 P + \beta_1 P_q\right] \left[\xi^0_t - \phi_{0;P}\right] + P_q \xi^1_P \left[\beta_0 P + \beta_1 P_q\right] \\ &+\xi^0_{P_q} \left[\beta_0 P + \beta_1 P_q\right] \left[\beta_0' P + \beta_1' P_q + \beta_0 P_q\right] + \xi^0_P \left[\beta_0 P + \beta_1 P_q\right]^2 \\ &- \left[\beta_0' P + \beta_1' P_q + \beta_0 P_q\right] \left[\beta_2 \xi^0_q + \beta_2 P_q \xi^0_P + \phi_{0;P_q} - P_q \xi^1_{P_q}\right] = 0. \end{aligned} \tag{10}$$

Here a prime on a function of a single variable means derivative of the function with respect to that variable and a variable subscript of any $\xi$ or $\phi$ means derivative with respect to that variable.

Equations (4), (5) and (7) give

$$\xi^0 \equiv \xi^0(t), \tag{11}$$

and

$$\xi^1 = -\frac{\partial F}{\partial P_q}, \tag{12}$$

where

$$F = \phi_0 - P_q \xi^1. \tag{13}$$

Hence equations (6) and (8) give

$$\xi^1 \equiv \xi^1(t,q,P), \tag{14}$$
$$\phi_0 \equiv \phi_0(t,q,P), \tag{15}$$
$$\phi_1 = \frac{\partial F}{\partial q} + P_q \frac{\partial F}{\partial P}, \tag{16}$$

and $F$ is linear in $P_q$. Terms independent of and linear in $P$ in equation (9) now give

$$\xi^1 \equiv \xi^1(t,q), \tag{17}$$

$$2\xi^1_q - \xi^1 \frac{\beta_2'}{\beta_2} - \xi^0_t = 0, \tag{18}$$

and $\phi_0$ is linear in $P$.

From (18) we get

$$\xi^0 = b(t), \tag{19}$$

$$\xi^1(t,q) = \frac{1}{B(q)'}\left[A(t) + \frac{1}{2}B(q)b(t)'\right], \tag{20}$$

where

$$B(q) \equiv \int \frac{\mathrm{d}q}{\sqrt{\beta_2(q)}} \neq \mathrm{constant}, \tag{21}$$

and $A(t)$ is as yet an arbitrary function of $t$. From the different powers of $P$ and $P_q$ in equation (10) we get

$$\xi^0 = b(t), \tag{22}$$
$$\xi^1 = \frac{1}{B(q)'}\left[A(t) + \frac{1}{2}B(q)b(t)'\right], \tag{23}$$
$$\phi_0 = f^0(t,q) + Pf^1(t,q), \tag{24}$$
$$\phi_1 = \frac{\partial f^0}{\partial q} + P\frac{\partial f^1}{\partial q} - P_q\left[\frac{\partial \xi^1}{\partial q} - f^1\right]. \tag{25}$$

Here $f^0(t,q)$ and $f^1(t,q)$ satisfy

$$\frac{\partial f^0}{\partial t} - \beta_0(q)f^0 - \beta_1(q)\frac{\partial f^0}{\partial q} - \beta_2(q)\frac{\partial^2 f^0}{(\partial q)^2} = 0, \tag{26}$$

$$\frac{\partial \xi^1}{\partial t} - \beta_1(q)\frac{\partial \xi^1}{\partial q} - \beta_2(q)\frac{\partial^2 \xi^1}{(\partial q)^2} + \xi^1\beta_1(q)' + \beta_1(q)b(t)' + 2\beta_2(q)\frac{\partial f^1}{\partial q} = 0, \tag{27}$$

$$\xi^1\beta_0(q)' + \beta_0(q)b(t)' + \beta_1(q)\frac{\partial f^1}{\partial q} + \beta_2(q)\frac{\partial^2 f^1}{(\partial q)^2} - \frac{\partial f^1}{\partial t} = 0. \tag{28}$$

Solving for $f^1$ from equation (28) we get

$$f^1(t,q) = C(t) + \frac{1}{2}A(t)\left[\frac{1}{B(q)'}\right]' + \frac{1}{4}b(t)'B(q)\left[\frac{1}{B(q)'}\right]' - \frac{1}{2}A(t)\beta_1(q)B(q)' - \frac{1}{4}b(t)'\beta_1(q)B(q)B(q)' - \frac{1}{2}A(t)'B(q) - \frac{1}{8}b(t)''\left[B(q)\right]^2. \tag{29}$$

From eqs (28) and (29) it follows that

$$C(t)' + \frac{1}{4}b(t)'' = A(t)f_1(q) + A(t)''f_3(q) + b(t)'f_2(q) + b(t)'''f_4(q), \tag{30}$$

with

$$f_1(q) = \frac{1}{B(q)'}\left[\beta_0(q) - F_0(q)\right]', \tag{31}$$

$$f_2(q) = \left[\beta_0(q) - F_0(q)\right] + \frac{B(q)}{2B(q)'}\left[\beta_0(q) - F_0(q)\right]', \tag{32}$$

$$f_3(q) = \frac{1}{2}B(q), \tag{33}$$

$$f_4(q) = \frac{1}{8}\left[B(q)\right]^2. \tag{34}$$

Here

$$F_0(q) = \frac{1}{4}\left[\Psi(q)\right]^2 + \frac{1}{2}\frac{\Psi(q)'}{B(q)'} \tag{35}$$

with

$$\Psi(q) = \beta_1(q)B(q)' - \left[\frac{1}{B(q)'}\right]'. \tag{36}$$

## 3. Symmetry classification

The vectors of the generators for the symmetry group of contact transformations of the generalized Fokker–Planck equation are given in eqs (22)–(25). The different functions appearing there are given in eqs (26), (29)–(36). Since $f^0(t,q)$ satisfies the original equation (1) the generators obtained from $f^0(t,q)$,

$$X_\infty = f^0(t,q)\frac{\partial}{\partial P} + \frac{\partial f^0(t,q)}{\partial q}\frac{\partial}{\partial P_q} \tag{37}$$

correspond to the invariant subgroup of the linear combinations of solutions arising of the original equation.

The factor group modulo of this invariant subgroup is designated as the symmetry group of contact transformations. Different classes of Fokker–Planck equation will arise depending on different relations between $f_1(q)$, $f_2(q)$, $f_3(q)$ and $f_4(q)$ appearing in eqs (30)–(34). We find the following five classes given in table 1. Their group algebras are described in terms of the generators given in table 2. In actual calculations with the group algebras, the structure constants (i.e. the commutation relations) are unavoidable tools. These are given in the Appendix.

Any Lie algebra $L$ is a semi-direct product $L \equiv R \bigwedge L/R$ of its radical $R$ and a semisimple part $L/R$. The semisimple part $L/R$ is again a direct sum of ideals which, as subalgebras, are simple [13,14]. These characteristics of the Lie algebras for the different types are given in table 3.

We note that the radical, being solvable, has only 1-dimensional irreducible representations (irreps) [13]. Thus the irreps of $L$ are obtained if the irreps of $L/R$ are known. To this end we give in table 4 the algebraic characteristics of the semisimple parts $L/R$ for the four different non-trivial classes of equations, which are simple for all the four classes.

## 4. Functional bases of invariants

In order to investigate integrability of a dynamical system we require the functional bases of invariants in terms of which all invariants of the dynamical system can be expressed functionally. These base invariants of a Lie algebra $L$ generate the Centre of the Universal

**Table 1.** Specification of the five types of Fokker–Planck type equations.

| Type of equation | Relation between different $f_i\,(q)$ | $\beta_0\,(q) - F_0\,(q)$ |
|---|---|---|
| Double harmonic | $f_1(q) = -\frac{1}{3}b_3 + \frac{1}{4}\omega^2 f_3(q)$, $f_2(q) = b_1 - b_3 f_3(q) + \omega^2 f_4(q)$, with constant $b_1,\ b_3$, and constant $\omega \neq 0$ | $b_1 - \frac{1}{3}b_3 B(q) + \frac{1}{16}\omega^2\,[B(q)]^2$ |
| Genetic equation | $f_2(q) = b_1 - b_2 f_1(q)$ $+\omega^2 b_2 f_3(q) + \omega^2 f_4(q)$ with constant $b_1,\ b_2$, and constant $\omega \neq 0$ | $\left[b_1 - \frac{1}{2}\,(b_2)^2\right] + \frac{\omega^2}{16}\,[2b_2 + B(q)]^2$ $+\frac{b_0}{[2b_2+B(q)]^2}$ with constant $b_0 \neq 0$ |
| Heat equation | $f_2(q) = b_1 - b_3 f_3(q)$, $f_1(q) = -\frac{1}{3}b_3$ with constant $b_1,\ b_3$ | $b_1 - \frac{1}{3}b_3 B(q)$ |
| Plasma equation | $f_2(q) = b_1 - b_2 f_1(q)$ with constant $b_1,\ b_2$ | $b_1 + \frac{b_0}{[2b_2+B(q)]^2}$ with constant $b_0 \neq 0$ |
| General | No relationship | Arbitrary |

**Table 2.** Generators that describe the different Lie algebras of the five classes of equations.

| Symbol of the generator | Form of the generator |
|---|---|
| $X_S$ | $P\frac{\partial}{\partial P} + \frac{\partial}{\partial P_q}$, |
| $X_{(\pm)}$ | $\frac{i}{\omega}e^{\pm i\omega t}\left\{ i\frac{\partial}{\partial t} \pm \frac{i}{B(q)'}\left[\frac{4b_3}{3\omega} - \frac{\omega}{2}B(q)\right]\left(-i\frac{\partial}{\partial q}\right) + \left[\left(ib_1 + \frac{4i(b_3)^2}{9\omega^2} \pm \frac{\omega}{4}\right) \mp \frac{2b_3}{3\omega}\Psi(q) - \frac{2ib_3}{3}B(q) \pm \frac{\omega}{4}B(q)\Psi(q) + \frac{i\omega^2}{8}(B(q))^2\right]X_S + \left[\mp \frac{2b_3}{3\omega}\Psi(q)' - \frac{2ib_3}{3}B(q)' \pm \frac{\omega}{4}B(q)'\Psi(q) \pm \frac{\omega}{4}B(q)\Psi(q)' + \frac{i\omega^2}{4}B(q)B(q)'\right]P\frac{\partial}{\partial P_q} \pm \left[\frac{\omega}{2} - \frac{4b_3}{3\omega}\left(\frac{1}{B(q)'}\right)' + \frac{\omega}{2}B(q)\left(\frac{1}{B(q)'}\right)'\right]P_q\frac{\partial}{\partial P_q}\right\}$ |
| $Y_{(\pm)}$ | $\sqrt{\frac{2}{\omega}}e^{\pi i/4}e^{\pm i\omega t/2}\left\{\frac{1}{B(q)'}\left(-i\frac{\partial}{\partial q}\right) + \left[\pm\frac{2b_3}{3\omega} \mp \frac{\omega}{4}B(q) + \frac{i}{2}\Psi(q)\right]X_S + \left[\frac{i}{2}\Psi(q)' \mp \frac{\omega}{4}B(q)'\right]P\frac{\partial}{\partial P_q} + i\left(\frac{1}{B(q)'}\right)'P_q\frac{\partial}{\partial P_q}\right\}$ |
| $Z_{(\pm)}$ | $\frac{i}{\omega}e^{\pm i\omega t}\left\{i\frac{\partial}{\partial t} - \frac{i\omega}{2B(q)'}[2b_2 + B(q)]\left(-i\frac{\partial}{\partial q}\right) + \left[\left(ib_1 \pm \frac{\omega}{4}\right) + \frac{\omega}{4}(2b_2 + B(q))\Psi(q) + \frac{ib_2\omega^2}{4}B(q) + \frac{i\omega^2}{8}(B(q))^2\right]X_S + \left[\frac{i\omega^2}{4}B(q)'(2b_2 + B(q)) \pm \frac{\omega}{4}\Psi(q)'(2b_2 + B(q)) \pm \frac{\omega}{4}B(q)'\Psi(q)\right]P\frac{\partial}{\partial P_q} + \frac{\omega}{2}\left[1 + (2b_2 + B(q))\left(\frac{1}{B(q)'}\right)'\right]P_q\frac{\partial}{\partial P_q}\right\}$ |
| $X_1$ | $i\left\{it\frac{\partial}{\partial t} - \frac{1}{2B(q)'}\left[b_3t^2 + B(q)\right]\left(-i\frac{\partial}{\partial q}\right) + \frac{i}{4}\left[-1 + 4\left(b_1t - \frac{1}{18}(b_3)^2t^3\right) - 2b_3tB(q) - b_3t^2\Psi(q) - B(q)\Psi(q)\right]X_S - \frac{i}{4}\left[2b_3tB(q)' + b_3t^2\Psi(q)' + B(q)'\Psi(q) + B(q)\Psi(q)'\right]P\frac{\partial}{\partial P_q} - \frac{i}{2}\left(1 + \left[b_3t^2 + B(q)\right]\left(\frac{1}{B(q)'}\right)'\right)P_q\frac{\partial}{\partial P_q}\right\}$ |

contd....

**Table 2**. Contd...

| Symbol of the generator | Form of the generator |
|---|---|
| $X_2$ | $2\sqrt{2}\Big\{ \frac{it^2}{2}\frac{\partial}{\partial t} - \frac{1}{2B(q)'}\left[\frac{1}{3}b_3t^3 + tB(q)\right]\left(-i\frac{\partial}{\partial q}\right) - \frac{i}{4}\Big[\left(t - 2b_1t^2 + \frac{1}{18}(b_3)^2t^4\right) + b_3t^2B(q) + \frac{1}{2}(B(q))^2 + \left(\frac{1}{3}b_3t^3 + tB(q)\right)\Psi(q)\Big]X_S - \frac{i}{4}\left[\left(b_3t^2 + B(q)\right)B(q)' + tB(q)'\Psi(q) + \left(\frac{1}{3}b_3t^3 + tB(q)\right)\Psi(q)\right]P\frac{\partial}{\partial P_q} - \frac{i}{2}\left[t + \left(\frac{1}{3}b_3t^3 + tB(q)\right)\left(\frac{1}{B(q)'}\right)'\right]P_q\frac{\partial}{\partial P_q}\Big\}$ |
| $X_3$ | $2^{1/4}e^{-\pi i/4}\Big\{\frac{1}{B(q)'}\left(-i\frac{\partial}{\partial q}\right) + i\left[\frac{1}{3}b_3t + \frac{1}{2}\Psi(q)\right]X_S + \frac{i}{2}\Psi(q)'P\frac{\partial}{\partial P_q} + i\left(\frac{1}{B(q)'}\right)'P_q\frac{\partial}{\partial P_q}\Big\}$ |
| $X_4$ | $8^{1/4}e^{-\pi i/4}\Big\{\frac{t}{B(q)'}\left(-i\frac{\partial}{\partial q}\right) + \frac{i}{2}\left[\frac{1}{3}b_3t^2 + B(q) + t\Psi(q)\right]X_S + \frac{i}{2}\left[B(q)' + t\Psi(q)'\right]P\frac{\partial}{\partial P_q} + it\left(\frac{1}{B(q)'}\right)'P_q\frac{\partial}{\partial P_q}\Big\}$ |
| $Y_1$ | $i\Big\{it\frac{\partial}{\partial t} - \frac{1}{2B(q)'}\left[2b_2 + B(q)\right]\left(-i\frac{\partial}{\partial q}\right) + i\Big[-\frac{1}{4} + b_1t - \frac{1}{4}(2b_2 + B(q))\Psi(q)\Big]X_S - \frac{i}{4}\Big[(2b_2 + B(q))\Psi(q)' + \Psi(q)B(q)'\Big]P\frac{\partial}{\partial P_q} - \frac{i}{2}\left[1 + (2b_2 + B(q))\left(\frac{1}{B(q)'}\right)'\right]P_q\frac{\partial}{\partial P_q}\Big\}$ |
| $Y_2$ | $2\sqrt{2}\Big\{\frac{it^2}{2}\frac{\partial}{\partial t} - \frac{t}{2B(q)'}\left[2b_2 + B(q)\right]\left(-i\frac{\partial}{\partial q}\right) - \frac{i}{2}\Big[2(b_2)^2 + \left(\frac{t}{2} - b_1t^2\right) + \frac{t}{2}(2b_2 + B(q))\Psi(q)b_2B(q) + \frac{1}{4}(B(q))^2\Big]X_S - \frac{i}{2}\Big[\frac{t}{2}(2b_2 + B(q))\Psi(q)' + \frac{t}{2}\Psi(q)B(q)' + \frac{1}{2}(2b_2 + B(q))B(q)'\Big]P\frac{\partial}{\partial P_q} - \frac{it}{2}\Big[1 + (2b_2 + B(q))\left(\frac{1}{B(q)'}\right)'\Big]P_q\frac{\partial}{\partial P_q}\Big\}$ |
| $X^t$ | $-\frac{i}{\omega}\frac{\partial}{\partial t} - \frac{i}{\omega}\left[b_1 - \frac{4(b_3)^2}{9\omega^2}\right]X_S$ |
| $Z^t$ | $-\frac{i}{\omega}\frac{\partial}{\partial t} - \frac{i}{\omega}\left[b_1 - \frac{1}{2}(b_2)^2\omega^2\right]X_S$ |
| $X_t$ | $\frac{i}{\sqrt{2}}\frac{\partial}{\partial t} + \frac{ib_1}{\sqrt{2}}X_S$ |
| $Y^t$ | $\frac{i}{\sqrt{2}}\frac{\partial}{\partial t} + \frac{ib_1}{\sqrt{2}}X_S$ |

**Table 3.** Characteristics of the Lie algebras of the symmetry groups of the different classes of equations

| Type of equation | Double harmonic | Genetic equation | Heat equation | Plasma equation | General equation |
|---|---|---|---|---|---|
| Generators of Lie algebra | $\{X_S, X^t, X_{(\pm)}, Y_{(\pm)}\}$ | $\{X_S, Z^t, Z_{(\pm)}\}$ | $\{X_S, X_t, X_1, X_2, X_3, X_4\}$ | $\{X_S, Y^t, Y_1, Y_2\}$ | $\{X_S, i\frac{\partial}{\partial t}\}$ |
| Dimension | 6 | 4 | 6 | 4 | 2 |
| Solvability, nilpotency, simplicity, semisimplicity | None | None | None | None | Solvable nilpotent |
| Centre, $Z(L)$ | $\{X_S\}$ | $\{X_S\}$ | $\{X_S\}$ | $\{X_S\}$ | $\{X_S, i\frac{\partial}{\partial t}\}$ |
| Radical, $R$ | $\{X_S, Y_{(\pm)}\}$ | $\{X_S\}$ | $\{X_S, X_3, X_4\}$ | $\{X_S\}$ | $\{X_S, i\frac{\partial}{\partial t}\}$ |
| Semisimple part | $\{X^t, X_{(\pm)}\}$ | $\{Z^t, Z_{(\pm)}\}$ | $\{X_t, X_1, X_2\}$ | $\{Y^t, Y_1, Y_2\}$ | Void |
| Cartan subalgebra | $\{X_S, X^t\}$ | $\{X_S, Z^t\}$ | $\{X_S, X_1\}$ | $\{X_S, Y_1\}$ | $\{X_S, i\frac{\partial}{\partial t}\}$ |

**Table 4.** Characteristics of the semisimple parts in table 3.

| Type of equation | Double harmonic | Genetic equation | Heat equation | Plasma equation |
|---|---|---|---|---|
| Generators of $L/R$ | $\{X^t, X_{(\pm)}\}$ | $\{Z^t, Z_{(\pm)}\}$ | $\{X_t, X_1, X_2\}$ | $\{Y^t, Y_1, Y_2\}$ |
| Dimension | 3 | 3 | 3 | 3 |
| Cartan subalgebra $H$ | $\{X^t\}$ | $\{Z^t\}$ | $\{X_1\}$ | $\{Y_1\}$ |
| Rank | 1 | 1 | 1 | 1 |
| Base of simple roots : $\Delta$ | $\alpha = 1$ | $\alpha = 1$ | $\alpha = 1$ | $\alpha = 1$ |
| Root system | $\pm\alpha$ | $\pm\alpha$ | $\pm\alpha$ | $\pm\alpha$ |
| Isomorphy to | $O(3)$ | $O(3)$ | $O(3)$ | $O(3)$ |
| Standard set of generators | $E_{+\alpha} = X_{(+)}$ $E_{-\alpha} = X_{(-)}$ | $E_{+\alpha} = Z_{(+)}$ $E_{-\alpha} = Z_{(-)}$ | $E_{+\alpha} = X_t$ $E_{-\alpha} = X_2$ | $E_{+\alpha} = Y^t$ $E_{-\alpha} = Y_2$ |
| Structure constants | $[H, E_{\pm\alpha}] = \pm E_{\pm\alpha}$, $[E_{+\alpha}, E_{-\alpha}] = 2H$ | $[H, E_{\pm\alpha}] = \pm E_{\pm\alpha}$, $[E_{+\alpha}, E_{-\alpha}] = 2H$ | $[H, E_{+\alpha}] = +E_{+\alpha} - \frac{b_3}{\sqrt{2}} X_4$, $[H, E_{-\alpha}] = -E_{-\alpha}$, $[E_{+\alpha}, E_{-\alpha}] = 2H$ | $[H, E_{+\alpha}] = \pm E_{\pm\alpha}$, $[E_{+\alpha}, E_{-\alpha}] = 2H$ |

**Table 5.** Functional base of invariants for the different types of equations

| Type of equation | Base invariants |
|---|---|
| Double harmonic | $I_S = X_S,$<br>$I_t = X_S \left(X^t\right)^2 + \frac{1}{2} X_S \left[X_{(+)}X_{(-)} + X_{(-)}X_{(+)}\right]$<br>$+\frac{1}{2}\mathcal{S}X^t\left[Y_{(+)}Y_{(-)} + Y_{(-)}Y_{(+)}\right]$<br>$-\frac{i}{2}\mathcal{S}\left[X_{(+)}\left(Y_{(-)}\right)^2 + X_{(-)}\left(Y_{(+)}\right)^2\right]$ |
| Genetic equation | $I_S = X_S,$<br>$I_t = \left(Z^t\right)^2 + \frac{1}{2}\left[Z_{(+)}Z_{(-)} + Z_{(-)}Z_{(+)}\right]$ |
| Heat equation | $I_S = X_S,$ for all $b_3$, and<br>$I_t = X_S\left(X_1\right)^2 + \frac{1}{2}X_S\left[X_tX_2 + X_2X_t\right]$<br>$+\frac{1}{2}\mathcal{S}X_1\left[X_3X_4 + X_4X_3\right]$<br>$-\frac{i}{2}\mathcal{S}\left[X_2\left(X_3\right)^2 + X_t\left(X_4\right)^2\right]$ only if $b_3 = 0.$ |
| Plasma equation | $I_S = X_S,$<br>$I_t = \left(Y_1\right)^2 + \frac{1}{2}\left[Y^tY_2 + Y_2Y^t\right]$ |
| General type | $I_S = X_S,$<br>$I_t = i\frac{\partial}{\partial t}$ |

Enveloping algebra of $L$. Since they commute with all the generators of $L$, they appear as conserved quantities of the system. Lie's method [15–18] has been utilized to obtain the base invariants for the five symmetry groups of §3. If the functional base has $s$ invariants $I_1, I_2, \ldots, I_s$, each a function of the $r$ generators $X_a$ of the symmetry group, then

$$[X_a, I_b] = 0, \;\; a = 1, \ldots, r, \;\; b = 1, \ldots, s.$$

If $I$ is any other invariant so that

$$[X_a, I] = 0, \;\; a = 1, \ldots, r,$$

then $I$ can be functionally expressed as

$$I \equiv I(I_1, \ldots, I_s).$$

If the base has the constant as its only member, then the system is completely chaotic. If on the other hand $s = r$ and all the generators give mutually commuting invariants, then the system is fully integrable. The first four special symmetry classes obtained in §3, are in between these two extreme cases, while the general type describes a fully integrable system. The functional bases of invariants for the five distinct types are given in table 5.

The $\mathcal{S}$ in some of the expressions for base invariants indicate that the symmetrized form has to be taken.

## 5. Discussion

These cases of symmetry classification for the groups of contact transformations are worth comparison with those for point transformations [11]. The forms of the generators are different for the two cases but the commutation relations are the same. Hence the types and forms of the base invariants are the same.

The semisimple part $L/R$ of all the four non-trivial algebras $L$ are isomorphic to O$(3)$ having the same Dynkin diagram as SL$(2,R)$. The occurrence of O$(3)$ symmetry in a 1-dimensional coordinate space is surprising, since we generally ascribe O$(3)$ to an isotropic 3-dimensional coordinate space.

In the case of the double harmonic case, the genetic equation, the heat equation with $b_3 = 0$ and the plasma equation, the semi-direct product is actually a direct product. But in the case of the heat equation with $b_3 \neq 0$ it is a true semi-direct product. This can be readily seen from the structure constants given in the Appendix.

In spite of the differences in the forms of the equations and the generators, the genetic equation and the plasma equation have the same group structures and are isomorphic to each other. The forms of their base invariants are also equivalent. The same holds for the double harmonic case and the heat equation with $b_3 = 0$. When $b_3 \neq 0$, the symmetry of the heat equation is *not* isomorphic to that of the double harmonic case. Thus there are essentially *four* symmetry classes:

1. Double harmonic symmetry type and the heat equation with $b_3 = 0$,
2. Heat equation type with $b_3 \neq 0$,
3. Genetic equation and plasma equation type, and
4. General type.

The groups of contact transformations for equations of motion for dynamical systems has a direct physical meaning. The space then consists of the space-time variables as well as of the canonical momenta. For the Fokker–Planck equation the derivative of the probability function does not have such a physical interpretation. However, even in this case the group of contact transformation helps us in obtaining both invariant and partially invariant solutions [8].

## Appendix

In this Appendix we give the non-zero structure constants of the generators given in table 2.

$$\begin{aligned}
&\left[X^t, X_{(\pm)}\right] = \pm X_{(\pm)}, \left[X_{(+)}, X_{(-)}\right] = 2X^t,\\
&\left[X_{(+)}, Y_{(-)}\right] = iY_{(+)}, \left[X_{(-)}, Y_{(+)}\right] = -iY_{(-)},\\
&\left[X^t, Y_{(\pm)}\right] = \pm\frac{1}{2}Y_{(\pm)}, \left[Y_{(+)}, Y_{(-)}\right] = X_S,
\end{aligned} \tag{A1}$$

$$\left[Z^t, Z_{(\pm)}\right] = \pm Z_{(\pm)}, \left[Z_{(+)}, Z_{(-)}\right] = 2Z^t, \tag{A2}$$

$$\begin{aligned}
&[X_1, X_t] = X_t - \frac{b_3}{\sqrt{2}}X_4, [X_1, X_2] = -X_2,\\
&[X_t, X_2] = 2X_1,\\
&[X_t, X_4] = iX_3, [X_2, X_3] = -iX_4,\\
&[X_1, X_3] = \frac{1}{2}X_3, [X_1, X_4] = -\frac{1}{2}X_4,\\
&[X_3, X_4] = X_S, [X_t, X_3] = -\mathrm{e}^{-\pi i/4}\frac{b_3}{3\cdot 2^{1/4}}X_S,
\end{aligned} \tag{A3}$$

$$\left[Y_1, Y^t\right] = Y^t,\ [Y_1, Y_2] = -Y_2.$$
$$\left[Y^t, Y_2\right] = 2Y_1. \tag{A4}$$

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