

Dirac current in expanding spacetime

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Abstract. We study the behaviour of Dirac current in expanding spacetime with Schrödinger and de Sitter form for the evolution of the scale-factor. The study is made to understand the particle-antiparticle rotation and the evolution of quantum vacuum leading to particle production in such spacetime.

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1. Introduction

The study of particle production is still an active area of research, especially in early universe inflationary cosmology. The non-linear dynamics of quantum fields in fixed cosmological backgrounds reveal new phenomena when compared to a Minkowski analysis. Such an analysis allows to obtain the metric dynamically from the quantum fields propagating in that metric. The dynamics of the scale factor $a(t)$ is driven by semiclassical Einstein equations

$$\frac{1}{8\pi G_R} G_{\mu\nu} = - \langle T_{\mu\nu} \rangle_R. \quad (1)$$

Recently [1] we solved equation (1) numerically taking in the right hand side of eq. (1) the contributions due to one loop quantum corrections plus the energy density due to particle production. The energy density of the produced particles was calculated using the method of the complex time WKB approximation (CWKB) and the energy density so calculated through particle-antiparticle rotation turns out to be ρ_r/a^4 , as if it acts like a radiation density term. The result is quite surprising. However, the numerical solution of (1) reveals that the universe evolves in a de Sitter phase, avoids singularity and at late time behaves like a radiation dominated or matter dominated universe depending upon the mode of particle production. It is now believed that the parametric and spinodal instabilities give rise to massive particle production and is a result of coherent oscillations of inflaton as it evolves

in time. The equilibrium dynamics or out of equilibrium dynamics, whichever one we prefer to use in (1), we need solutions of Dirac equation (considering only spin- $\frac{1}{2}$ fields) and solve numerically equation (1). Such kind of numerical solutions considering scalar field has been carried out by Anderson [2–5], evaluating $\langle T_{\mu\nu} \rangle$ for massless and massive fields with a chosen vacuum for quantum fields. For out of equilibrium situation, one evaluates $\langle T_{\mu\nu} \rangle$ for any state $|\rangle$ such that the quantum field operator $\Phi(x)$ is written as

$$\Phi(x) = \phi(x) + \psi(x), \tag{2}$$

where $\psi(x)$ is a new quantum operator. Here

$$\phi(x) = \langle \Phi(x) \rangle, \tag{3}$$

$$\langle \psi(x) \rangle = 0. \tag{4}$$

For a given Lagrangian using Hartree or large N approximation one evaluates $\phi(x)$ and $\psi(x)$ and calculates ϵ and P using renormalized equations of motion for dynamical evolution (for details please see [6–12]. We intend to apply this technique considering $\Phi(x)$ as spinor field. Leaving aside the fluctuation term $\psi(x)$ in (2), we study in this work the evolution of the fields on a fixed FRW background. Before going into the text of the paper we note that the numerical solutions of (1), with massless conformally coupled scalar fields contributing to $\langle T_{\mu\nu} \rangle$ through one loop quantum corrections (see [1–5]) supposed to be valid also in spinor case when curvature is much greater than the mass of the field, reveal that the universe evolves in a deSitter phase, bounces and then emerges into classical universe, depending upon the matter content to be fixed by the mechanism of particle production in the model. In the literature, though most works concentrate on scalar fields, the effect of spinor particle production contributing to $\langle T_{\mu\nu} \rangle$ are not well tackled. In our view at a very early stage, consideration of spinor particle production would be important if one has to investigate the preheating and reheating mechanism of early universe, where reheating (a process that starts when inflation ends) temperature is solely determined by the mechanism of particle production. The numerical investigation is no doubt a way to understand and study this aspect but it is a hard nut to crack. The analytical treatment allows one to channel the numerical work and this is what we intend to take for our discussion. For further details the reader is referred to [2–5], and for numerical solutions backed by analytical input (see [1]).

Our aim is now to study the evolution with scale factor $a(t) = (t/t_0)^n$. The cases $n = 1/2$, $n = 2/3$ and $n = 1$ correspond respectively to radiation, matter dominated and Schrödinger model backgrounds. The calculations have been carried out in all these backgrounds. We report here the calculations of $n = 1$ case and cite also the results of de Sitter background for comparison. This work is also motivated to find an answer to the emergence of ρ_r/a^4 term as mentioned earlier. We do not intend here to the solution of dynamical back reaction equation (1), rather concentrate on the role of fields on particle production scenario according to CWKB [13–18].

Let us now understand the basis of the CWKB as follows. We write down the temporal equation for Dirac particle in FRW spacetime in Klein–Gordon form. After separating out the spacetime part we determine the turning points as usual from the resulting one-dimensional Schrödinger-type equation in time variables. The turning points are found to be complex in the conformal time. The turning points are identified as points where

particle turns back, not in space but in time, which according to Feynmann–Stuckleberg prescription must be identified as an event of pair production. Now there is a mathematical analogy between the Dirac equation in a time varying electric field and the scattering off a potential which varies in space but not in time. The unitarity relation obtained from charge conservation relates the reflection and transmission coefficient R^c and T^c which refer to the reflection and transmission coefficient not in space but in time. Pair production is considered as the reflection of a positron from the vacuum and R^c is the pair production amplitude. It is also found that in the CWKB, the particle production is basically a resonance particle production and the essence of the phenomenon of resonance is the rotation of a free-particle solution to a free-antiparticle solution as if the Dirac current shows a smooth interpolation from $-J$ to $+J$. The problem in the interpolation is how to choose the initial conditions i.e., the particle and antiparticle states. To have a clear understanding on how to choose the initial conditions of quantum fields, we turn to the explicit solution of Dirac equation in fixed background and construct the current $J^\mu = \bar{\psi}\gamma^\mu\psi$ to look into the turnings of the current (if it exists) with the evolution of time. This is the main content of the present paper. We find marked differences for the cases $n = 1, 3/2, 1/2$ when compared to the de Sitter evolution $a(t) = \exp(Ht)$.

Though the method of CWKB was originally proposed by us in the context of the behaviour of quantum field in curved spacetime, recently many workers [19–21] are studying this aspect. Recently [22] we have established the rotation of currents that we are advocating during particle production and find that the CWKB results are gauge invariant result akin to Schwinger’s gauge invariant result related to particle production. The present work is an attempt to study this ‘rotation of current’ in the context of curved spacetime, in order to understand the interplay of Minkowski and Rindler vacuum in relation to particle production.

The plan of the paper is organized as follows. In §2 we discuss the Dirac equation in expanding spacetime. In §3 we describe the basics of CWKB. In §4 we discuss the solutions for (i) $a(t) = a_0 t$ and (ii) $a(t) = \exp(Ht)$ along with the corresponding currents. In §5, we discuss the numerical results. The de Sitter case is discussed with a view to understand the emergence of radiation dominated universe from the act of particle production in §6. We end up with a concluding section.

2. The Dirac equation in expanding non-flat spacetime

In curved spacetime the Dirac equation is taken to be

$$[i\gamma^\mu(x)\partial_\mu - i\gamma^\mu(x)\Gamma_\mu(x)]\Psi = m\Psi, \quad (5)$$

where $\gamma^\mu(x)$ are the curvature dependent Dirac matrices, $\Gamma_\mu(x)$ are the spin connections and m is the mass of the particle.

The metric, we shall consider here, the Robertson–Walker type

$$ds^2 = dt^2 - a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \right], \quad (6)$$

that describes a homogeneous isotropic non-flat universe, where $k = 0, +1, -1$, corresponds to Euclidean space, a spherical space and a pseudo-spherical space respectively. To obtain $\gamma^\mu(x)$ and $\Gamma_\mu(x)$, we introduce vierbeins L_μ^α as

$$ds^2 = L_\mu^\alpha L_{\alpha\nu} dx^\mu dx^\nu, \tag{7}$$

where

$$\gamma_\mu(x) = L_\mu^a \gamma_a, \tag{8}$$

having relations $\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2g_{\mu\nu}$ and $\gamma_a \gamma_b + \gamma_b \gamma_a = 2\eta_{ab}$. The spin connections are given by

$$\Gamma_\mu = -\frac{1}{8} \left[\gamma^\nu(x), \gamma_\nu(x); \mu \right]. \tag{9}$$

Evaluating (9), and using (8), the Dirac equation is finally reduced to the form

$$\left[\partial_t + \frac{3}{2} \frac{\dot{a}}{a} + im\gamma_0 - \frac{1}{a} \left\{ \vec{\alpha} \cdot \vec{\nabla} + (\rho - 1)\alpha_r \left(\partial_r + \frac{1}{r} \right) \right\} \right] \psi = 0, \tag{10}$$

where, $\rho^2 = 1 - kr^2$. For further details, the reader is referred to our earlier work [17].

3. Basic principles of CWKB

Let us consider a one dimensional Schrödinger equation, not in space but in time,

$$\frac{d^2}{dt^2} \psi + \omega^2(t) \psi = 0. \tag{11}$$

In CWKB we consider t to be a complex variable and assume $\omega(t)$ has complex turning points given by

$$\omega^2(t_{1,2}) = 0. \tag{12}$$

Defining

$$S(t_f, t_i) = \int_{t_i}^{t_f} \omega(t) dt, \tag{13}$$

the solution of (11) in CWKB is written as

$$\psi(t) \underset{t \rightarrow \infty}{\overset{t \rightarrow 0}{\longrightarrow}} \exp[iS(t, t_0)] + R \exp[-iS(t, t_0)]. \tag{14}$$

Here t_0 and t are real where t_0 is arbitrary and $t_0 > t$. In (14) R is given by

$$R = \frac{-i \exp[2iS(t_1, t_0)]}{1 + \exp[2iS(t_1, t_2)]}, \tag{15}$$

where $t_{1,2}$ are the complex turning points determined from (12). The interpretation of (14) and (15) is as follows. In (14) the first term is the direct ray. It starts from $t_0 > t_1$, moving leftward arrives at a real $t < t_0$. The second term in (14) corresponds to reflected part. A

wave starting from t_0 reaches the complex turning point t_1 and after bouncing back from t_1 reaches t . It is represented as

$$(-i) \exp [iS(t_1, t_0) - iS(t, t_1)]. \tag{16}$$

The contribution (16) is then multiplied by the repeated reflections between t_1 and t_2 and the multiple reflection is written as

$$\sum_{\mu=0}^{\infty} [-i \exp \{iS(t_1, t_2)\}]^{2\mu} = \frac{1}{1 + \exp [2iS(t_1, t_2)]}. \tag{17}$$

The combined contributions (16) and (17) comprise the second term of (14). For convenience we have neglected the WKB pre-exponential factor throughout.

Let us now obtain a dynamical picture of particle production from the foregoing discussion of reflection in time. Consider a potential $V \sim \exp(-i\omega t)$ to supply energy for the pair (e^+e^-) creation. In Feynmann's space-time diagram, we represent it as in figure 1(a)

$$V \rightarrow (e^+, E_{e^+}) \uparrow + (e^-, E_{e^-}) \uparrow. \tag{18}$$

Here the positron of energy E_{e^+} and the electron of energy E_{e^-} both are moving forward in time out of the potential site and is represented by the arrow beside them. According to Feynmann–Stuckleberg (F–S) prescription, the negative energy particle solution propagating backward in time as being equivalent to positive energy anti-particle solution moves forward in time. Using this prescription, (see figure 1(b))

$$(e^-, E_{e^-}) \uparrow \xrightarrow[\text{prescription}]{\text{F.S}} (e^+, -E_{e^-}) \downarrow$$

i.e., a negative energy positron ($E_{e^+} = -E_{e^-}$) moving backward in time (represented by down arrow) meets the potential site V (that acts as turning points) and moves forward in time. Thus F–S prescription applying to (18) now reads

$$(e^+, E_{e^+}) \uparrow + (e^-, E_{e^-}) \uparrow \xrightarrow[\text{prescription}]{\text{F.S}} (e^+, E_{e^+}) \uparrow + (e^+, -E_{e^-}) \downarrow.$$

Thus pair production can be viewed as a process of reflection in time. For further details the reader is referred to [18], [13] and [14] to find the equivalence,

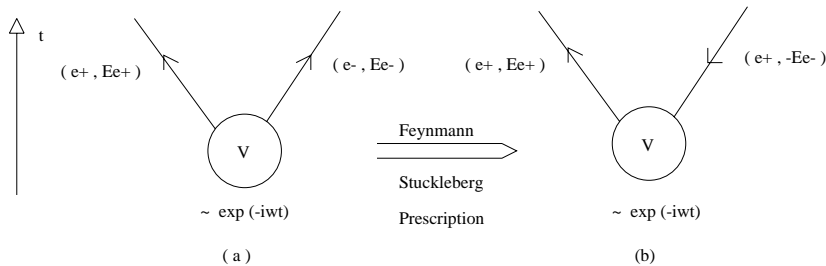


Figure 1. Feynmann Stuckleberg prescription.

$$|\text{pair production amplitude}| = |\text{reflection amplitude}|.$$

If we now consider $\exp [+iS(t, t_0)]$ as antiparticle solution, the reflected component is interpreted as a particle moving forward in time. This is the Klein paradox-like situation not in space but in time and R is interpreted as pair-production amplitude. The essence of (14) is that there is no particle at $t \rightarrow -\infty$, i.e.,

$$\psi_{in} \xrightarrow{t \rightarrow -\infty} \exp [iS(t, t_0)], \quad t < 0 \tag{19}$$

but at $t \rightarrow +\infty$, (19) evolves into

$$\psi_{in} \xrightarrow{t \rightarrow +\infty} \exp [iS(t, t_0)] + R \exp [-iS(t, t_0)].$$

We applied the results (14) and (15) in various expanding spacetime [12–14] with remarkable results.

4. Solutions of Dirac equation

In case of flat spacetime i.e. $k = 0$ (or $\rho = 1$), we will, here, find out the solutions of the Dirac eq. (10) and it will take its form as

$$\left[\partial_t + \frac{3}{2} \frac{\dot{a}}{a} + im\gamma_0 - \frac{1}{a} \vec{\alpha} \cdot \vec{\nabla} \right] \psi = 0. \tag{20}$$

Let

$$\psi(x, t) = \frac{e^{i\mathbf{k} \cdot \mathbf{x}}}{(2\pi)^{3/2}} \begin{pmatrix} f(\mathbf{k}, t) \\ g(\mathbf{k}, t) \end{pmatrix}, \tag{21}$$

and thus we get for two-component spinors $f(\mathbf{k}, t)$ and $g(\mathbf{k}, t)$ the coupled equations:

$$\left(\partial_t + \frac{3}{2} \frac{\dot{a}}{a} + im \right) f - \frac{i}{a} \vec{k} \cdot \vec{\sigma} g = 0, \tag{22}$$

and

$$\left(\partial_t + \frac{3}{2} \frac{\dot{a}}{a} - im \right) g - \frac{i}{a} \vec{k} \cdot \vec{\sigma} f = 0. \tag{23}$$

We get in turn the uncoupled equation for $f(\mathbf{k}, t)$ as

$$\left(\partial_t^2 - \frac{1}{2} \frac{\ddot{a}}{a} + \frac{1}{4} \frac{\dot{a}^2}{a^2} + im \frac{\dot{a}}{a} + m^2 + \frac{k^2}{a^2} \right) h = 0, \tag{24}$$

where we have used

$$f = \frac{1}{a^2} h. \tag{25}$$

4.1 Case I: Schrödinger model

In Schrödinger model, we find

$$a(t) = a_0 t. \quad (26)$$

Then eq. (24) becomes

$$\left[\partial_t^2 + \frac{k^2/a_0^2 + \frac{1}{4}}{t^2} + \frac{im}{t} + m^2 \right] h = 0. \quad (27)$$

After changing the variable from t to $z^2 = -4m^2 t^2$ or $z = \pm 2imt$, we get, from eq. (27),

$$\left[\frac{d^2}{dz^2} - \frac{1}{4} + \frac{\pm \frac{1}{2}}{z} + \frac{\frac{1}{4} - (\pm ik/a_0)^2}{z^2} \right] h = 0, \quad (28)$$

which is Whittaker differential equation having two independent solutions:

$$h_{\pm}(z) = W_{\pm \frac{1}{2}, \frac{ik}{a_0}}(\pm 2imt). \quad (29)$$

Thus the upper components of the wave function in (21) become

$$f_{\pm} = \frac{1}{a^2} h_{\pm}(z) = \frac{1}{a_0^2 t^2} W_{\pm \frac{1}{2}, \frac{ik}{a_0}}(\pm 2imt) \quad (30)$$

and the lower components will be

$$g_{\pm} = y_{\pm} \frac{1}{k^2} \begin{pmatrix} k_3 & k_- \\ k_+ & -k_3 \end{pmatrix}, \quad (31)$$

where

$$\begin{aligned} y_{\pm} &= -\frac{it}{a_0} \left(\partial_t + \frac{3}{2t} + im \right) \frac{1}{t^2} W_{\pm 1/2, ik/a_0}(\pm 2imt) \\ &= -\frac{it}{a_0} \left(\frac{\pm 2im}{t^2} \dot{W}_{\pm} - \frac{1}{2t^3} W_{\pm} + \frac{im}{t^2} W_{\pm} \right) \end{aligned} \quad (32)$$

and $k_{\pm} = k_1 \pm ik_2$. But Whittaker functions W_{\pm} satisfy the following identities:

$$\dot{W}_{\kappa, \mu}(z) = \left(\frac{1}{2} + \mu - \kappa \right) \left(\frac{1}{2} - \mu - \kappa \right) \frac{1}{z} W_{\kappa-1, \mu} + \left(\frac{\kappa}{z} - \frac{1}{2} \right) W_{\kappa, \mu} \quad (33)$$

and

$$\dot{W}_{\kappa, \mu}(z) = -\frac{1}{z} W_{\kappa+1, \mu} - \left(\frac{\kappa}{z} - \frac{1}{2} \right) W_{\kappa, \mu}. \quad (34)$$

We use the first identity for $\kappa = +\frac{1}{2}$, $z = 2imt$ and for second identity $\kappa = -\frac{1}{2}$, $z = -2imt$. Thus (32) gives, using (33) and (34),

$$y_+ = -\frac{ik^2}{a_0^3} \frac{1}{t^2} W_{-\frac{1}{2}, \frac{ik}{a_0}}(2imt) \tag{35}$$

and

$$y_- = \frac{i}{a_0} \frac{1}{t^2} W_{\frac{1}{2}, \frac{ik}{a_0}}(-2imt). \tag{36}$$

Denoting the constant spinor components of f by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, the four independent solutions of the form (21) will be

$$\psi_1 = N_1 \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{3/2}} \frac{1}{a_0^2 t^2} \begin{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} W_{\frac{1}{2}, \frac{ik}{a_0}}(2imt) \\ -\frac{i}{a_0} \begin{pmatrix} k_3 \\ k_+ \end{pmatrix} W_{-\frac{1}{2}, \frac{ik}{a_0}}(2imt) \end{pmatrix}, \tag{37}$$

$$\psi_2 = N_2 \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{3/2}} \frac{1}{a_0^2 t^2} \begin{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} W_{\frac{1}{2}, \frac{ik}{a_0}}(2imt) \\ -\frac{i}{a_0} \begin{pmatrix} k_- \\ -k_3 \end{pmatrix} W_{-\frac{1}{2}, \frac{ik}{a_0}}(2imt) \end{pmatrix}, \tag{38}$$

$$\psi_3 = N_3 \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{3/2}} \frac{1}{a_0^2 t^2} \begin{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} W_{-\frac{1}{2}, \frac{ik}{a_0}}(-2imt) \\ \frac{ia_0}{k^2} \begin{pmatrix} k_3 \\ k_+ \end{pmatrix} W_{\frac{1}{2}, \frac{ik}{a_0}}(-2imt) \end{pmatrix}, \tag{39}$$

$$\psi_4 = N_4 \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{3/2}} \frac{1}{a_0^2 t^2} \begin{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} W_{-\frac{1}{2}, \frac{ik}{a_0}}(-2imt) \\ \frac{ia_0}{k^2} \begin{pmatrix} k_- \\ -k_3 \end{pmatrix} W_{\frac{1}{2}, \frac{ik}{a_0}}(-2imt) \end{pmatrix}. \tag{40}$$

It is interesting to see the asymptotic form of these solutions for large times. Since $W_{\kappa, \mu}(z) \rightarrow z^\kappa e^{-z/2}$, for $-\frac{3\pi}{2} < \arg z < \frac{3\pi}{2}$, we obtain the normalized solutions as follows:

$$\psi_1 \simeq \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{3/2}} \frac{\sqrt{i}}{a_0^{3/2}} \frac{1}{t^{3/2}} \begin{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ -\frac{1}{2ma_0} \begin{pmatrix} k_3 \\ k_+ \end{pmatrix} \frac{1}{t} \end{pmatrix} e^{-imt}, \tag{41}$$

$$\psi_2 \simeq \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{3/2}} \frac{\sqrt{i}}{a_0^{3/2}} \frac{1}{t^{3/2}} \begin{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ -\frac{1}{2ma_0} \begin{pmatrix} k_- \\ -k_3 \end{pmatrix} \frac{1}{t} \end{pmatrix} e^{-imt}, \tag{42}$$

$$\psi_3 \simeq \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{3/2}} \frac{\sqrt{-i}}{a_0^{5/2}} \frac{k}{t^{3/2}} \begin{pmatrix} -\frac{1}{2im} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{1}{t} \\ \frac{ia_0}{k^2} \begin{pmatrix} k_3 \\ k_+ \end{pmatrix} \end{pmatrix} e^{imt}, \quad (43)$$

$$\psi_4 \simeq \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{3/2}} \frac{\sqrt{-i}}{a_0^{5/2}} \frac{k}{t^{3/2}} \begin{pmatrix} -\frac{1}{2im} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \frac{1}{t} \\ \frac{ia_0}{k^2} \begin{pmatrix} k_- \\ -k_3 \end{pmatrix} \end{pmatrix} e^{imt}, \quad (44)$$

where we have determined the normalization constant N_i in such a way that asymptotically, i.e., in the flat-space limit, we have the usual $\delta(\mathbf{k} - \mathbf{k}')$ normalization of the electron's wave function. The norm of ψ is defined by

$$\begin{aligned} (\psi^k, \psi^{k'}) &= \int_t d^3x a^3(t) \bar{\psi}^k \gamma^0 \psi^{k'} \\ &= \int_t d^3x a_0^3 t^3 \psi_k^\dagger \psi_{k'} \\ &\rightarrow \delta(\mathbf{k} - \mathbf{k}') a_0^3 t^3 \frac{\sqrt{2im}}{a_0^2} \frac{\sqrt{-2im}}{a_0^2} \frac{N^2}{t^3}. \end{aligned} \quad (45)$$

This procedure gives

$$N_1 = N_2 = \sqrt{\frac{a_0}{2m}}, \quad N_3 = N_4 = \frac{k}{\sqrt{2ma_0}}. \quad (46)$$

4.2 Case II: de Sitter spacetime

In case of de Sitter spacetime, the function $a(t)$ is given by

$$a(t) = e^{Ht}. \quad (47)$$

In a similar way, as in § 4.1, we get four asymptotic normalized solutions as

$$\psi_1 = \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{3/2}} \left[\frac{\frac{\pi k}{2H}}{\cos\left(\frac{i\pi m}{H}\right)} \right]^{\frac{1}{2}} e^{-2Ht} \begin{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} J_\nu(z) \\ \frac{i}{k} \begin{pmatrix} k_3 \\ k_+ \end{pmatrix} J_{\nu-1}(z) \end{pmatrix}, \quad (48)$$

$$\psi_2 = \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{3/2}} \left[\frac{\frac{\pi k}{2H}}{\cos\left(\frac{i\pi m}{H}\right)} \right]^{\frac{1}{2}} e^{-2Ht} \begin{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} J_\nu(z) \\ \frac{i}{k} \begin{pmatrix} k_- \\ -k_3 \end{pmatrix} J_{\nu-1}(z) \end{pmatrix}, \quad (49)$$

$$\psi_3 = \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{3/2}} \left[\frac{\frac{\pi k}{2H}}{\cos\left(\frac{i\pi m}{H}\right)} \right]^{\frac{1}{2}} e^{-2Ht} \begin{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} J_{-\nu}(z) \\ -\frac{i}{k} \begin{pmatrix} k_3 \\ k_+ \end{pmatrix} J_{-\nu+1}(z) \end{pmatrix} \quad (50)$$

and

$$\psi_4 = \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{3/2}} \left[\frac{\frac{\pi k}{2H}}{\cos\left(\frac{i\pi m}{H}\right)} \right]^{\frac{1}{2}} e^{-2Ht} \begin{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} J_{-\nu}(z) \\ -\frac{i}{k} \begin{pmatrix} k_- \\ -k_3 \end{pmatrix} J_{-\nu+1}(z) \end{pmatrix}. \quad (51)$$

We now evaluate the Dirac currents for these cases I and II.

Dirac current in Case I

Particle production in Schrödinger model may easily be interpreted from the nature of the curves of Dirac current in a given spacetime. Using eq. (5)

$$\left[i\gamma^\mu(x)\partial_\mu - i\gamma^\mu(x)\Gamma_\mu(x) \right] \psi = m\psi, \quad (52)$$

in an external field A_μ and its conjugate we can express the Gordon decomposition of current as follows [23]:

$$j^\mu = \bar{\psi}\gamma^\mu\psi = \frac{1}{2m} \left[i\bar{\psi}\gamma^\mu\gamma^\lambda\partial_\lambda\psi - i\partial_\lambda\bar{\psi}\gamma^\lambda\gamma^\mu\psi - \bar{\psi} \{ i(\gamma^\mu\gamma^\lambda\Gamma_\lambda + \Gamma_\lambda\gamma^\lambda\gamma^\mu) + eA_\lambda g^{\lambda\mu} \} \psi \right]. \quad (53)$$

For $k = 0$ i.e. in the flat spacetime the metric (6) reduces to

$$ds^2 = dt^2 - a^2(t) (dx_1^2 + dx_2^2 + dx_3^2), \quad (54)$$

and also the space-dependent Dirac matrices get the following forms

$$\gamma^0(x) = \gamma_0, \quad \gamma^1(x) = -\frac{1}{a}\gamma_1, \quad \gamma^2(x) = -\frac{1}{a}\gamma_2, \quad \gamma^3(x) = -\frac{1}{a}\gamma_3. \quad (55)$$

Using this equation (55) we have the current densities in components as

$$j_0 = \frac{i}{2ma} \nabla_j (\bar{\psi}\gamma_j\gamma_0\psi) + \frac{1}{2m} \bar{\psi} \left[i \overset{\leftrightarrow}{\partial}_0 + eA_0 \right] \psi \quad (56)$$

and

$$j_k = \frac{i}{2ma} \partial_i (\bar{\psi}\gamma_k\gamma_0\psi) + \frac{i}{4ma^2} \partial_j (\bar{\psi} [\gamma_j, \gamma_k] \psi) + \frac{1}{2ma^2} \bar{\psi} \left[i \overset{\leftrightarrow}{\partial}_k + eA_k \right] \psi + \frac{3i}{2m} \frac{\dot{a}}{a^2} \bar{\psi}\gamma_k\gamma_0\psi. \quad (57)$$

Now using equations (56) and (57), taking $A_\mu = 0$, we get the same results for current densities j_0, j_1, j_2 , and j_3 for the solutions ψ_1 and ψ_2 given in (41) and (42). These are as follows:

$$j_0 = C_0 T^3 + C'_0 T^5, \quad j_1 = -C_1 T^5, \quad j_2 = -C_2 T^5, \quad j_3 = -C_3 T^5 \quad (58)$$

and for the solutions ψ_3 and ψ_4 of eqs (43) and (44), the components of the current densities get their forms as

$$j_0 = C_0 T^3 + C'_0 T^5, \quad j_1 = C_1 T^5, \quad j_2 = C_2 T^5, \quad j_3 = C_3 T^5, \quad (59)$$

where $C_0 = \frac{1}{8\pi^3 a_0^3}$, $C'_0 = \frac{k^2}{32\pi^3 m^2 a_0^5}$, $C_1 = \frac{k_1}{8\pi^3 m a_0^5}$, $C_2 = \frac{k_2}{8\pi^3 m a_0^5}$ and $C_3 = \frac{k_3}{8\pi^3 m a_0^5}$ with $T = \frac{1}{t}$.

Dirac current in Case II

Similarly the four components of the current densities, in de Sitter spacetime, will get their forms as

$$\begin{aligned} j_0 &= C_0 T^3, \\ j_1 &= C \left(k_1 \sinh \frac{\pi m}{H} - k_2 \sin T \right) T^4, \\ j_2 &= C \left(k_2 \sinh \frac{\pi m}{H} + k_1 \sin T \right) T^4, \\ j_3 &= C k_3 T^4 \sinh \frac{\pi m}{H}, \end{aligned} \quad (60)$$

or in brief,

$$\begin{aligned} j_0 &= C_0 T^3, \\ j_1 &= C_1 (1 - D \sin T) T^4, \\ j_2 &= C_2 (1 + D_1 \sin T) T^4, \\ j_3 &= C_3 T^4 \end{aligned} \quad (61)$$

with the solution ψ_1 and where, $T = (2k/H) e^{+Ht}$, $C_0 = H^3/64\pi^3 k^3$, $C = H^4/(128\pi^3 k^5 \cos(i\pi m/H))$, $C_1 = C k_1 \sinh(\pi m/H)$, $C_2 = C k_2 \sinh(\pi m/H)$, $C_3 = C k_3 \sinh(\pi m/H)$, $D = k_2/(k_1 \sinh(\pi m/H))$ and $D_1 = (k_1^2/k_2^2)D$.

Let us mention some arguments for citing the Gordon decomposition, though it is not required to obtain the expression of currents in the present work. We also calculated $\bar{\psi}\gamma^\mu\psi$ using the exact solutions of Dirac equation and find that this result obtained by us is coincident with the Gordon decomposition (57). In Barut's work [23] this Gordon decomposition

is wrongly calculated and has forced us to check this result. The Gordon decomposition is carried out to investigate the effect of spin towards the rotation of currents which is extremely important in understanding the particle production leaving aside the complicity of vacuum state definition in curved spacetime.

5. Interpretation of currents

In order to interpret the currents qualitatively, we carry out the numerical analysis. For Schrödinger model as well as de Sitter spacetime we observe that $j_0 = \bar{\psi}\gamma_0\psi$ varies as $\frac{1}{a^3}$. For the case I, the nature of the curves for j_1, j_2, j_3 are of the same form for the set of solutions ψ_1 and ψ_2 . Similar results are also obtained for ψ_3 and ψ_4 . We observe that $j_i(\psi_{1,2}) = -j_i(\psi_{3,4})$, though j_0 remains the same for both the sets. For convenience we take $C_0 = C'_0 = C_1 = C_2 = C_3 = 1$ and the curves are reproduced in figure 2a for j_0 , figure 2b for the first set of solutions and figure 2c for the second set of solutions. The curves in figures 2a and 2c are of the same nature.

The current in Schrödinger model has interesting properties. From the figures 2b and 2c it is clear that the current of ψ_1 and ψ_2 is negative whereas for ψ_3 and ψ_4 it is positive. So from equations (41) to (44) it is evident that the turning (also shown in figure 2d) $(\psi_1, \psi_2) \rightarrow (\psi_3, \psi_4)$ would lead to particle production. Though the situation here is less attractive and interpretative compared to de Sitter spacetime, the plot of current density versus time provides an interesting way to understand particle production through particle-antiparticle rotation. Though we do not get a definite conclusion, we believe that the particle production should somehow be dependent on initial states and the initial conditions would greatly effect particle production as well as the evolution of the universe. It is worthwhile to mention that the behaviour of current in radiation dominated universe is very much similar to Schrödinger model. It is evident from the curves that if there are some fluctuations that take the system out of equilibrium there will be oscillations between curves I and II in figure 2d.

In case of de Sitter spacetime the forms of j_0, j_1, j_2 , and j_3 are of the same form as that of eq. (61) when we consider any one of the solutions ψ_2, ψ_3 or ψ_4 separately. Though there is some differences in constants, only we look into the behaviour of currents with time. The nature of the curves are of the same form when the current density j_1, j_2 , or j_3 are plotted against time t . For convenience we take $C_0 = C_1 = C_2 = C_3 = D = D_1 = 1, 2k = H = 1$ and the curve is reproduced in the figure 2e. The oscillating time-dependent current densities j_1 and j_2 always remain symmetrical about the exponentially decaying one j_3 . The nature of the curve of j_0 is of the same form as that of j_3 , not shown in the figure. A single oscillation indicates particle production (though the nature of the vacuum is unspecified). Repeated oscillations indicates particle production (at 2a, 2b, 2c, ...) and annihilation (at 2d, 2e, 2f, ...) in course of time. Without going into the details of quantum vacuum, it can be said that repeated oscillation is an indication of generation of photon along with particle production. The occurrence of oscillations at t large negative indicate the existence of repeated reflected trajectories as is demanded in our CWKB approach [13,14,17,18]. The emergence of radiation dominated universe may thus be explained considering production and annihilation (trough and crest in figure 2e) as a mode for generation of photon at late time. In view of our work on de Sitter spacetime [15] the oscillating behaviour of de Sitter current is very interesting and worth-pointing in the

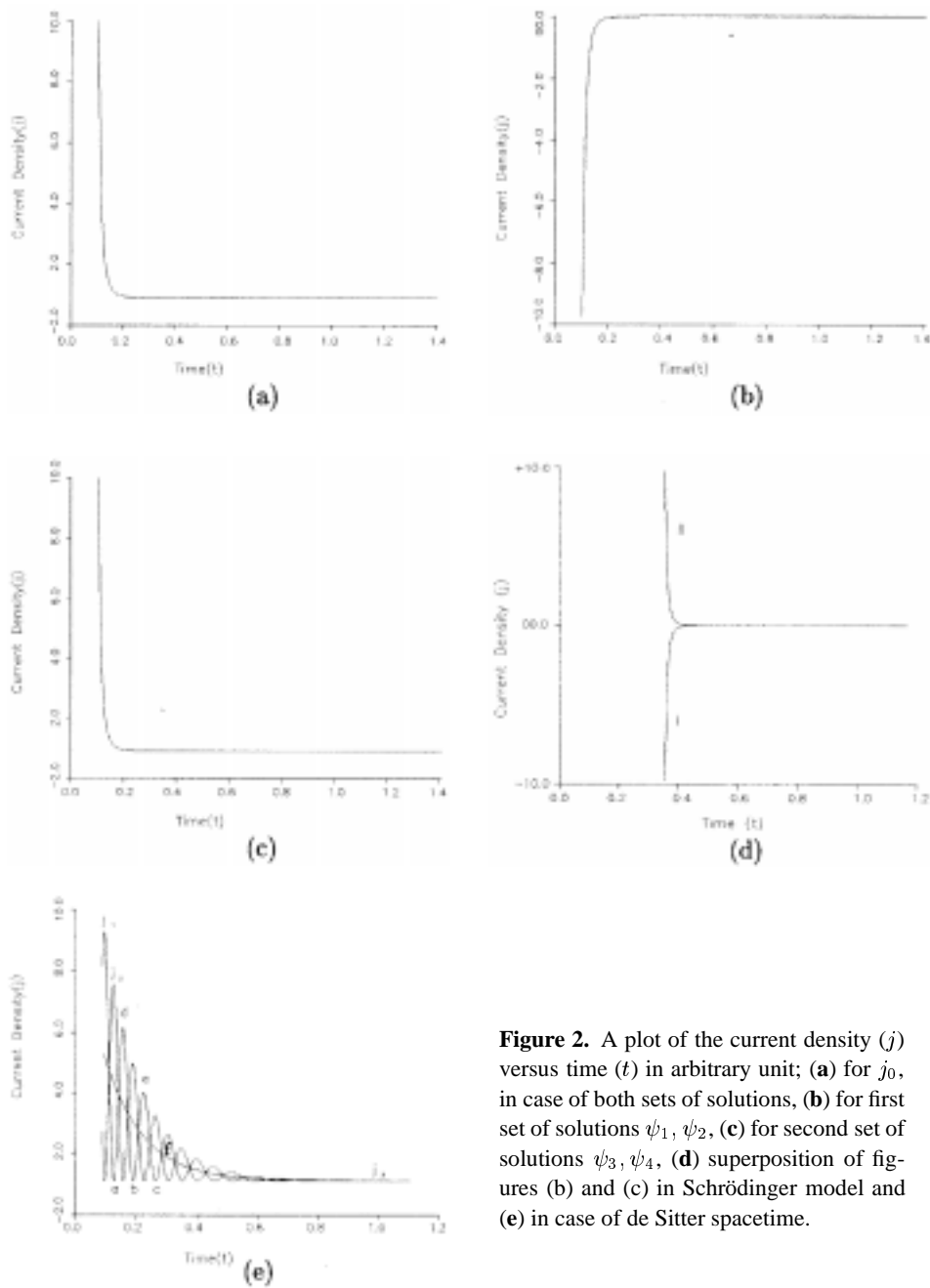


Figure 2. A plot of the current density (j) versus time (t) in arbitrary unit; (a) for j_0 , in case of both sets of solutions, (b) for first set of solutions ψ_1, ψ_2 , (c) for second set of solutions ψ_3, ψ_4 , (d) superposition of figures (b) and (c) in Schrödinger model and (e) in case of de Sitter spacetime.

sense that the emergence of radiation dominated universe might also help to understand the baryon-asymmetry problem in early universe.

The numerical parameters adopted in this work are used only to have a quantitative understanding of photon production in the early universe. We will publish shortly the results taking parameters that corresponds to early universe situation.

6. Photon dominance in de Sitter spacetime

Before we discuss the production of huge number of photons due to particle production in de Sitter spacetime, let us consider the arguments of Bernido [24] in this respect. The Green function for a relativistic scalar particle is expressed as

$$G(x'', x; m) = (i\hbar)^{-1} \int_0^\infty \exp\left(-\frac{im^2\lambda}{2}\right) K(x'', x'; \lambda), \quad (62)$$

where $x = (\mathbf{r}, t)$ and λ is a time-like variable. Antiparticles in the path integral approach are identified by particles which move backwards in ordinary time such that $\partial t/\partial \lambda < 0$, with λ always increasing. Pair production is understood as follows. Let $A(\mathbf{r}, t, \lambda = 0)$, $B(\mathbf{r}', t, \lambda = \lambda_B)$ and $C(\mathbf{r}_s, t_s, \lambda)$ be three spacetime points. Let a particle leave the point A and move backwards in time, scatter at C , turn back and move forward in time to arrive at B . Here $\mathbf{r} < \mathbf{r}_s < \mathbf{r}'$ and $t_s < t$. We note that here the path from A to B represents a particle moving backward in ordinary time t . In Feynmann's language we interpret it by saying that an observer at time t sees both a particle and antiparticle coming from C .

In de Sitter spacetime, eq. (62) has been evaluated in [24]. It is found that the Green function yields no bound state spectrum and it exists provided $t'' > t'$. It is to be noted that the spacetime points $x' (= \mathbf{r}', t')$ and $x'' (= \mathbf{r}'', t'')$ can be viewed as the points where a particle and antiparticle can be detected. The consequence of the constraint $t'' > t'$ is interpreted as follows.

- (i) For a given t , such that $t' < t < t''$, an observer can only detect a particle and not its antiparticle. This gives rise to an observed matter universe.
- (ii) To obtain huge number of photons compared with the baryons as observed today, consider the particle starting at point $x' = (\mathbf{r}', t')$. It may follow a path that jogs up and down (with respect to t coordinate) and a huge number of photons is thus generated from pair annihilation (i.e., creation at the trough and annihilation at the crest).

The arguments placed in (i) and (ii) though seem encouraging but have some shortfalls. What makes the particle jog up and down? A plausible answer comes from the CWKB approach where the mechanism described in (i) and (ii) fit very nicely. The details will be placed elsewhere and here we mention the essential aspects. The second order Dirac equation or the equation (24) shows turning points at

$$\frac{k}{a} = \pm i \left(m + \frac{iH}{2} \right), \quad (63)$$

when we put $a(t) = \exp(Ht)$.

According to CWKB, the particle starts from t_i and arrives at t with $t < t_i$ without any reflection. We call it direct rays. There is also a reflected part. A particle starts from t_i and

moving backwards suffers a reflection at one of the turning points, say $t_1 = +i(m + \frac{iH}{2})$, and moving forward arrives at t . This is identified as pair production. The reflected part is enhanced by the repeated reflections between the turning points, in complex t -plane, given by equation (63). As like the arguments of Bernido [24], the particle jogs up and down resulting in a huge number of photons. The remarkable equivalence of CWKB with the standard techniques of particle production has already been established [15,16,18]. In CWKB, the repeated reflections gives rise to thermal spectrum and a reflection in time is considered as a rotation of currents i.e., $-J$ to $+J$. In CWKB, we also find an answer to the Bernido's plausible assumptions. The matter-antimatter asymmetry can also be explained from our approach. However, we do not consider it here. Before ending, it is justified to ask whether the turning points given by (63) have anyway been reflected in the expression of Dirac currents in the curved spacetime.

Consider the situation just after Big-Bang. We take $k \ll Ha$ and $m \ll H$, the plausible conditions in inflationary cosmology. Under these conditions the expressions of the currents in eq. (60) give

$$\frac{k}{a} \simeq \pm \left(\frac{k_2}{k_1} \right) m,$$

quite consistent with eq. (63), if we consider the case $m^2 < 0$ (broken symmetry). The condition $k \ll Ha$ correspond to long wavelength fluctuations greater than the Hubble radius, indicating the emergence of spinodal instabilities, a crude picture of which is reflected in the expression of Dirac current. However, the situation is not so simple and straightforward as mentioned. We need here to understand the interplay of vacuum energy and inflationary universe. The reader is referred to [25] in this respect.

It should be pointed out that in a realistic approach, huge photon production should follow from high temperature nonperturbative QED, but in curved spacetime a complete theory for which is still lacking. The work of Lotze [26] may be consulted for simultaneous creation of e^+e^- pairs and photons in RW universe.

7. Conclusion

The present work reveals that

- (i) the quantum vacuum at an early epoch starts from a state to be characterized at a given time,
- (ii) during the evolution of quantum vacuum there is rapid particle production and annihilation, of course in expanding spacetime.
- (iii) amplitude of oscillation decreases with time indicating a gradual attainment of stable ground state (stable vacuum),
- (iv) at late time particle production ends.

In view of the above results it would thus be interesting to study the behaviour of Dirac currents in other expanding spacetimes. Rapid oscillation in the current is a characteristic feature of inflationary spacetime and may be introduced through an inflaton field or through an oscillating scale-factor (as is found in Starobinsky-type model without phase transition). We hope to discuss, in future, particle production, back reaction, avoidance of

singularity (see [1]), catastrophic particle-production through parametric resonance within our framework.

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