PRAMANA — journal of physics
© Indian Academy of Sciences
Vol. 52, No. 6
June 1999
pp. 579–591

A combination of Yang's equations for $SU(2)$ gauge fields and Charap's equations for pion dynamics with exact solutions

SUSANTO CHAKRABORTY and PRANAB KRISHNA CHANDA*
Central Drugs Labdratory, 3 Kyd Street, Calcutta 700 016, India
*Govt. Teacher's Training College, Malda 732 101, India

MS received 3 March 1998; revised 13 May 1999

Abstract. Two sets of nonlinear partial differential equations originating from two different physical situations have been combined and a new set of nonlinear partial differential equations has been formed wherefrom the previous two sets can be obtained as particular cases. One of the two sets of equations was obtained by Yang [1] while discussing the condition of self-duality of $SU(2)$ gauge fields on Euclidean four-dimensional space. The second one was reported by Charap [2] for the chiral invariant model of pion dynamics under tangential parametrization. Using the same type of ansatz in each case De and Ray [16] and Ray [7] obtained physical solutions of the two sets of equations. Here exact solutions of the combined set of equations with particular values of the coupling constants have been obtained for a similar ansatz. These solutions too are physical in nature.

Keywords. Exact solutions; combined field equations; $SU(2)$ gauge field; self duality; pion dynamics; chiral invariance.

PACS Nos 11.30; 11.10

1. Introduction

In this paper we have combined the equations originating from two physical situations. We also present exact solutions to the combined equations for some of the particular values of the coupling constants. The solutions are physical in nature. In the following the two sets of equations leading to the combination are given. Then the motivation for such a combination is presented.

Equations leading to the combination

(i) *The equations due to Yang* [1]: These were obtained by Yang [1] while discussing the condition of self-duality of $SU(2)$ gauge fields on Euclidean four-dimensional space. The equations are given by,

$$\phi(\phi_{y\bar{y}} + \phi_{z\bar{z}}) - \phi_y\phi_{\bar{y}} - \phi_z\phi_{\bar{z}} + \lceil_y\lceil_{\bar{y}} + \lceil_z\lceil_{\bar{z}} = 0, \tag{1.1a}$$

$$\phi(\lceil_{y\bar{y}} + \lceil_{z\bar{z}}) - 2\lceil_y\phi_{\bar{y}} - 2\lceil_z\phi_{\bar{z}} = 0, \tag{1.1b}$$

where an overbar denotes the complex conjugate, ϕ and $\lceil$ are functions of $y, \bar{y}, z$ and $\bar{z}, \phi$ is real, $\lceil$ is complex and

$$\sqrt{2}y = x^1 + ix^2, \sqrt{2}z = x^3 - ix^4, \tag{1.1c,d}$$

x^1, x^2, x^3, x^4 are real.

Once one has found $\lceil$ and ϕ, the corresponding R-gauge potentials are given by Yang [1]

$$\phi\mathbf{b}_y = (i\lceil_y, \lceil_y, -i\phi_y), \phi\mathbf{b}_{\bar{y}} = (-i\lceil_{\bar{y}}, \bar{\lceil}_{\bar{y}}, i\phi_{\bar{y}}), \tag{1.2a,b}$$

$$\phi\mathbf{b}_z = (i\lceil_z, \lceil_z, -i\phi_z), \phi\mathbf{b}_{\bar{z}} = (-i\lceil_{\bar{z}}, \bar{\lceil}_{\bar{z}}, i\phi_{\bar{z}}), \tag{1.2c,d}$$

and R-gauge field strengths $F_{\mu\nu}$ are given by

$$F_{\mu\nu} = B_{\mu,\nu} - B_{\nu,\mu} - B_\mu B_\nu + B_\nu B_\mu, \tag{1.3a}$$

$$B_\mu = b^i_\mu X_i, \tag{1.3b}$$

and

$$X_i = -\frac{1}{2}i\sigma_i, \tag{1.3c}$$

where σ_i are 2×2 Pauli matrices.

All such solutions represent the condition of self-duality except when ϕ is zero. Because when ϕ is zero $F_{\mu\nu}$ become singular and the solutions obtained can only be treated as solutions of Yang's R-gauge equations and not self-dual solutions unless a transformation like, $F_{\mu\nu} \to U^{-1}F_{\mu\nu}U$ removes the singularities.

(ii) *The equations due to Charap* [2]: These were reported by Charap [2] for the chiral invariant model of pion dynamics under tangential parametrization [3]. The equations are given by

$$\Box\phi = \eta^{\mu\nu}\frac{\partial\phi}{\partial x^\mu}\cdot\frac{\partial\beta}{\partial x^\nu}, \tag{1.4a}$$

$$\Box\psi = \eta^{\mu\nu}\frac{\partial\psi}{\partial x^\mu}\cdot\frac{\partial\beta}{\partial x^\nu}, \tag{1.4b}$$

$$\Box\chi = \eta^{\mu\nu}\frac{\partial\chi}{\partial x^\mu}\cdot\frac{\partial\beta}{\partial x^\nu}, \tag{1.4c}$$

where,

$$\begin{aligned} \eta^{\mu\nu} &= 0 \quad \text{for} \quad \mu \neq \nu, \\ &= 1 \quad \text{for} \quad \mu = \nu \neq 4, \\ &= -1 \quad \text{for} \quad \mu = \nu = 4, \end{aligned} \tag{1.4d}$$

$$\beta = \ln(f_\pi^2 + \phi^2 + \psi^2 + \chi^2), \tag{1.4e}$$

$$f_\pi = \text{constant}. \tag{1.4f}$$

The Lagrangian is given by

$$L = \frac{1}{2}(g_{11}\partial_\mu\phi\partial^\mu\phi + g_{22}\partial_\mu\psi\partial^\mu\psi + g_{33}\partial_\mu\chi\partial^\mu\chi + 2g_{12}\partial_\mu\phi\partial^\mu\psi$$
$$+2g_{13}\partial_\mu\phi\partial^\mu\chi + 2g_{23}\partial_\mu\psi\partial^\mu\chi), \tag{1.5}$$

where the g_{ij} are such that Γ^l_{ij}, the Christroffel symbols, take the form,

$$\Gamma^l_{ij} = -(f_\pi^2 + \phi^2 + \psi^2 + \chi^2)^{-1}(\delta^l_i\phi_i + \delta^l_j\phi_j). \tag{1.6}$$

In (1.6), ϕ_1, ϕ_2 and ϕ_3 represent ϕ, ψ and χ respectively.

Motivation

In addition to the physical significance described above the equations have some common characteristics which are mathematically interesting. It has been observed that there is considerable similarity between the two sets of equations (1.1) and (1.4). First, when the equations (1.1) are written in terms of the variables x^1, x^2, x^3 and x^4, it can be shown that they are similar in form to equations (1.4). Second, both of them allow (i) reduction to equations in two independent variables which are conformally invariant equations permitting one to obtain infinitely many other solutions from any solution of these conformally invariant equations (4)–(6), and (ii) those reduced equations closely resemble the generalized Lund–Regge equations [4, 7–9].

The generalized Lund–Regge equations are

$$\theta_{11} + \theta_{22} - 4g(\theta) + h(\theta)(\lambda_1^2 + \lambda_2^2) = 0, \tag{1.7a}$$

$$\left[\lambda_1 \exp\left(-\int p(\theta)\mathrm{d}\theta\right)\right]_1 + \left[\lambda_2 \exp\left(-\int p(\theta)\mathrm{d}\theta\right)\right]_2 = 0, \tag{1.7b}$$

where, $\theta = \theta(x^1, x^2)$, $\lambda = \lambda(x^1, x^2), \theta = \partial\theta/\partial x^1$ and so on.

With $g = 0$, the equations (1.7) reduce to a conformally invariant set of equations, a particular example of which is the physically interesting equations of two dimensional Heisenberg ferromagnets [10, 11]. The reduced form of equations mentioned above (eq. (1.7)) which originate from Yang's equations (1.1) closely resembles this situation. There is, however, at least a difference that there are two equations for the Heisenberg ferromagnets, whereas the equations obtained from those due to Yang (1.1) [5] have three equations. On the otherhand, all of the reduced equations mentioned above [4, 6] which originate from Charap's equations (1.4) are of the same form as one of the two equations in (1.7), namely (1.7b). However, the similarity between the solutions of the nonlinear sigma model and self-dual gauge fields is well-known and, for example, forms the basis for the Atiyah–Manton [12] approach to constructing approximate solutions to the Skyrme model [12]. This has been quite widely applied to the study of nucleon–nucleon interactions in that model. For a recent example, see Leese *et al* [13].

These observations have inspired us to study the general set of equations having Yang's equations (1.1) which were obtained at the time of discussing the condition of self-duality for $SU(2)$ gauge fields on Euclidean space and Charap's equations (1.6) for chiral invariant pion dynamics under tangential parametrizations as the particular cases.

Since the solutions to some of the particular forms of the generalized equations have been shown to be physical in nature they may be useful in a wide area of physical research such as field theories and particle physics particularly with chiral models and Skyrme models in relation to solition solutions and spatio-temporal chaos [14, 15].

2. Formulation of the combined equations

All $k_j, j = 0, 1, 2, 3, \ldots$ appearing in the following discussion are constants. When written in terms of real variables the equations in (1.1) read

$$\begin{aligned}\phi_{11} + \phi_{22} + \phi_{33} + \phi_{44} = {}& (1/\phi)(\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2) - (1/\phi)(\psi_1^2 + \psi_2^2 + \psi_3^2 + \psi_4^2) \\ & -(1/\phi)(\chi_1^2 + \chi_2^2 + \chi_3^2 + \chi_4^2) - (2/\phi)(\psi_1\chi_2 - \psi_2\chi_1 + \psi_4\chi_3 - \psi_3\chi_4), \end{aligned} \quad (2.1a)$$

$$\begin{aligned}\psi_{11} + \psi_{22} + \psi_{33} + \psi_{44} = {}& (2/\phi)(\phi_1\psi_1 + \phi_2\psi_2 + \phi_3\psi_3 + \phi_4\psi_4) \\ & +(2/\phi)(\phi_1\chi_2 - \phi_2\chi_1 + \phi_4\chi_3 - \phi_3\chi_4), \end{aligned} \quad (2.1b)$$

$$\begin{aligned}\chi_{11} + \chi_{22} + \chi_{33} + \chi_{44} = {}& (2/\phi)(\phi_1\chi_1 + \phi_2\chi_2 + \phi_3\chi_3 + \phi_4\chi_4) \\ & +(2/\phi)(\phi_2\psi_1 - \phi_1\psi_2 + \phi_3\psi_4 - \phi_4\psi_3), \end{aligned} \quad (2.1c)$$

where $[= \psi + i\chi$, $\phi_1 = \partial\phi/\partial x^1$, $\phi_{11} \equiv \partial^2\phi/\partial x^2$ etc. On the other hand when written explicitly the equations in (1.4) read

$$\begin{aligned}\phi_{11} + \phi_{22} + \phi_{33} - \phi_{44} = {}& 2\phi[\exp(-\beta)](\phi_1^2 + \phi_2^2 + \phi_3^2 - \phi_4^2) \\ & +2\psi[\exp(-\beta)](\phi_1\psi_1 + \phi_2\psi_2 + \phi_3\psi_3 - \phi_4\psi_4) \\ & +2\chi[\exp(-\beta)](\phi_1\chi_1 + \phi_2\chi_2 + \phi_3\chi_3 - \phi_4\chi_4), \end{aligned} \quad (2.2a)$$

$$\begin{aligned}\psi_{11} + \psi_{22} + \psi_{33} - \psi_{44} = {}& 2\psi[\exp(-\beta)](\psi_1^2 + \psi_2^2 + \psi_3^2 - \psi_4^2) \\ & +2\phi[\exp(-\beta)](\phi_1\psi_1 + \phi_2\psi_2 + \phi_3\psi_3 - \phi_4\psi_4) \\ & +2\chi[\exp(-\beta)](\psi_1\chi_1 + \psi_2\chi_2 + \psi_3\chi_3 - \psi_4\chi_4), \end{aligned} \quad (2.2b)$$

$$\begin{aligned}\chi_{11} + \chi_{22} + \chi_{33} - \chi_{44} = {}& 2\chi[\exp(-\beta)](\chi_1^2 + \chi_2^2 + \chi_3^2 - \chi_4^2) \\ & +2\phi[\exp(-\beta)](\phi_1\chi_1 + \phi_2\chi_2 + \phi_3\chi_3 - \phi_4\chi_4) \\ & +2\psi[\exp(-\beta)](\psi_1\chi_1 + \psi_2\chi_2 + \psi_3\chi_3 - \psi_4\chi_4), \end{aligned} \quad (2.2c)$$

where $\beta = \ln(f_\pi^2 + \phi^2 + \psi^2 + \chi^2)$.

Combining the equations (2.1) and (2.2) one can write the equations in which all the terms of (2.1) and (2.2) are present. These equations are given by

$$\begin{aligned}\phi_{11} + \phi_{22} + \phi_{33} + \varepsilon\phi_{44} = k'[&(1/\phi)(\phi_1^2 + \phi_2^2 + \phi_3^2 + \epsilon\phi_4^2)\\ &-(1/\phi)(\psi_1^2 + \psi_2^2 + \psi_3^2 + \epsilon\psi_4^2) - (1/\phi)(\chi_1^2 + \chi_2^2 + \chi_3^2 + \epsilon\chi_4^2)\\ &-(2/\phi)(\psi_1\chi_2 - \psi_2\chi_1 + \psi_4\chi_3 - \psi_3\chi_4)]\\ &+k''\{2\phi[\exp(-\beta)](\phi_1^2 + \phi_2^2 + \phi_3^2 + \epsilon\phi_4^2)\\ &+2\psi[\exp(-\beta)](\phi_1\psi_1 + \phi_2\psi_2 + \phi_3\psi_3 + \epsilon\phi_4\psi_4)\\ &+2\chi[\exp(-\beta)](\phi_1\chi_1 + \phi_2\chi_2 + \phi_3\chi_3 + \epsilon\phi_4\chi_4)\},\end{aligned} \tag{2.3a}$$

$$\begin{aligned}\psi_{11} + \psi_{22} + \psi_{33} + \epsilon\phi_{44} = k'[&(2/\phi)(\phi_1\psi_1 + \phi_2\psi_2 + \phi_3\psi_3 + \epsilon\phi_4\psi_4)\\ &+(2/\phi)(\phi_1\chi_2 - \phi_2\chi_1 - \phi_3\chi_4 + \phi_4\chi_3)]\\ &+k''\{2\psi[\exp(-\beta)](\psi_1^2 + \psi_2^2 + \psi_3^2 + \epsilon\psi_4^2)\\ &+2\phi[\exp(-\beta)](\phi_1\psi_1 + \phi_2\psi_2 + \phi_3\psi_3 + \epsilon\phi_4\psi_4)\\ &+2\chi[\exp(-\beta)](\psi_1\chi_1 + \psi_2\chi_2 + \psi_3\chi_3 + \epsilon\psi_4\chi_4)\},\end{aligned} \tag{2.3b}$$

$$\begin{aligned}\chi_{11} + \chi_{22} + \chi_{33} + \epsilon\chi_{44} = k'[&(2/\phi)(\phi_1\chi_1 + \phi_2\chi_2 + \phi_3\chi_3 + \epsilon\phi_4\chi_4)\\ &+(2/\phi)(\phi_2\psi_1 - \phi_1\psi_2 + \phi_3\psi_4 - \phi_4\psi_3)]\\ &+k''\{2\chi[\exp(-\beta)](\chi_1^2 + \chi_2^2 + \chi_3^2 + \epsilon\chi_4^2)\\ &+2\phi[\exp(-\beta)](\phi_1\chi_1 + \phi_2\chi_2 + \phi_3\chi_3 + \epsilon\phi_4\chi_4)\\ &+2\psi[\exp(-\beta)](\psi_1\chi_1 + \psi_2\chi_2 + \psi_3\chi_3 + \epsilon\psi_4\chi_4)\},\end{aligned} \tag{2.3c}$$

where $\beta = \ln(f_\pi^2 + \phi^2 + \psi^2 + \chi^2)$ and k', k'' are arbitrary coupling constants.

Equations (2.3) reduce to (2.1) when $\epsilon = 1, k' = 1, k'' = 0$ and to (2.2) when $\epsilon = -1, k' = 0, k'' = 1$. For convenience we call the equations (2.1) as the 'Yang equations', the equations (2.2) as 'Charap equations' and the equations (2.3) as the 'combined Yang–Charap (Y–C) equations'. For $\epsilon = 1, k' \neq 0, k'' \neq 0$ the equations (2.3) are 'extended Yang equations'. And for $\epsilon = -1$, $k' \neq 0$, $k'' \neq 0$ the equations (2.3) are 'extended Charap equations'.

3. Solutions

The ansatz used here is given by

$$\phi = \phi(u), \quad \psi = \psi(u), \quad \chi = \chi(u), \tag{3.1}$$

where u is an unspecified function of x^1, x^2, x^3, x^4.

One of the motivations of analysing the ansatz (3.1) is that Ray [7] and De and Ray [16] obtained physical solutions of Charap equation (2.2) and Yang equations (2.1) respectively using the ansatz (3.1) in both the cases. (The ansatz used by De and Ray [16] was $\psi = \psi(\phi)$, $\chi = \chi(\phi)$. However, $\phi = \phi(u)$ can be rewritten as $u = u(\phi)$ when (3.1) reduces to $\psi = \psi(\phi), \chi = \chi(\phi)$.)

The solutions have been presented in two cases. Each case again has two parts. In the first part of Case 1, solutions of the Charap equations (2.2) obtained by Ray [7] for the ansatz (3.1) along with the procedure have been described. And in the second part of Case 1 the solutions of extended Charap equations (equations (2.3) with $\epsilon = -1$) with $k' = 1, k'' = 1$ for the ansatz (3.1) have been obtained. Similarly, in the first part of Case 2, the solutions of Yang equations (2.1) obtained by De and Ray [16] for the ansatz (3.1) have been rediscovered. And in the second part of Case 2 the solutions of extended Yang equations (equations (2.3) with $\epsilon = 1$) with $k' = 1, k'' = 1$ for the ansatz (3.1) have been obtained.

The procedure adopted by Ray [7] for obtaining the solutions of (2.2) for the ansatz (3.1) has been used for obtaining all the solutions mentioned in this paper. This has been done with the purpose of visualizing the effect of combining the two sets of equations.

Case 1

Part 1 (*Solutions of the Charap equations* (2.2)): The solutions presented here are due to Ray [7]. After the use of (3.1) the equations (2.2) reduce to

$$(u_{11} + u_{22} + u_{33} - u_{44}) + A(u_1^2 + u_2^2 + u_3^2 - u_4^2) = 0, \tag{3.2a}$$

$$(u_{11} + u_{22} + u_{33} - u_{44}) + D(u_1^2 + u_2^2 + u_3^2 - u_4^2) = 0, \tag{3.2b}$$

$$(u_{11} + u_{22} + u_{33} - u_{44}) + E(u_1^2 + u_2^2 + u_3^2 - u_4^2) = 0, \tag{3.2c}$$

$$A = (\phi_{uu}/\phi_u) - 2(\phi\phi_u + \psi\psi_u + \chi\chi_u)/(f_\pi^2 + \phi^2 + \psi^2 + \chi^2), \tag{3.2d}$$

$$D = (\psi_{uu}/\psi_u) - 2(\phi\phi_u + \psi\psi_u + \chi\chi_u)/(f_\pi^2 + \phi^2 + \psi^2 + \chi^2), \tag{3.2e}$$

$$E = (\chi_{uu}/\chi_u) - 2(\phi\phi_u + \psi\psi_u + \chi\chi_u)/(f_\pi^2 + \phi^2 + \psi^2 + \chi^2), \tag{3.2f}$$

so that either

$$A = D = E \quad \text{or} \tag{3.3}$$

$$u_1^2 + u_2^2 + u_3^2 - u_4^2 = 0 \quad \text{and} \tag{3.4a}$$

$$u_{11} + u_{22} + u_{33} - u_{44} = 0. \tag{3.4b}$$

Equations (3.4) have simple solutions and are given in the work of Ray [7] and Ghosh *et al* [17].

The general solutions of (3.3) are given by

$$\psi = k_1\phi + k_2, \tag{3.5a}$$

$$\chi = k_3\phi + k_4, \tag{3.5b}$$

where k_i are arbitrary constants of integration.

Let us now define consistent with (3.1) and without any loss of generality

$$\phi = k_5 v + k_6, \tag{3.6}$$

where k_5 and k_6 are arbitrary constants and v is some unspecified function of u.

Since u has been defined in (3.1) to be an unspecified function of x^1, x^2, x^3 and x^4 one can conclude till now that v is an unspecified function of x^1, x^2, x^3 and x^4.

Putting (3.6) in (3.5a, b) one immediately gets

$$\phi = k_5 v + k_6, \tag{3.7a}$$

$$\psi = k_7 v + k_8, \tag{3.7b}$$

$$\chi = k_9 v + k_{10}, \tag{3.7c}$$

where $k_1 k_5 = k_7, k_3 k_5 = k_9, k_1 k_6 + k_2 = k_8, k_3 k_6 + k_4 = k_{10}$. The use of (3.7) reduces the equations (2.2) to a single equation given by

$$(v_{11} + v_{22} + v_{33} - v_{44}) + A'(v_1^2 + v_2^2 + v_3^2 - v_4^2) = 0, \tag{3.8a}$$

where

$$A' = \frac{\mathrm{d}}{\mathrm{d}v}[\ln 1/\{(k_5^2+k_7^2+k_9^2)v^2+2(k_5k_6+k_7k_8+k_9k_{10})v+(f_\pi^2+k_6^2+k_8^2+k_{10}^2)\}]. \tag{3.8b}$$

The equation (3.8a) can be rewritten as

$$\mathrm{h}_{11} + \mathrm{h}_{22} + \mathrm{h}_{33} - \mathrm{h}_{44} = 0, \tag{3.9a}$$

where

$$\mathrm{h} = \int [\exp(A'(v)\mathrm{d}v)]\mathrm{d}v + k_0, \tag{3.9b}$$

where k_0 is an arbitrary constant of integration. Using (3.8b) in (3.9b) one can write

$$\mathrm{h} = \int \frac{\mathrm{d}v}{(k_5^2 + k_7^2 + k_9^2)v^2 + 2(k_5k_6 + k_7k_8 + k_9k_{10})v + (f_\pi^2 + k_6^2 + k_8^2 + k_{10}^2)} + k_0. \tag{3.10}$$

In order to obtain a compact form of the solutions without the loss of much of generality we choose $k_6 = 0$, $k_8 = 0$, $k_{10} = 0$ when (3.10) reduces to

$$v = \{f_\pi/\sqrt{(k_5^2 + k_7^2 + k_9^2)}\}\tan[f_\pi/\sqrt{(k_5^2 + k_7^2 + k_9^2)}](\mathrm{h} - k_0). \tag{3.11}$$

Finally ϕ, ψ and χ are given by (from (3.7a, b, c) with $k_6 = 0$, $k_8 = 0$, $k_{10} = 0$ and $k_0 = 0$)

$$\phi = \left\{(k_3 f_\pi)/\sqrt{(k_5^2 + k_7^2 + k_9^2)}\right\}\tan\left[f_\pi\sqrt{(k_5^2 + k_7^2 + k_9^2)}\right]\mathrm{h}, \tag{3.12a}$$

$$\psi = \left\{(k_4 f_\pi)/\sqrt{(k_5^2 + k_7^2 + k_9^2)}\right\}\tan\left[f_\pi\sqrt{(k_5^2 + k_7^2 + k_9^2)}\right]\mathrm{h}, \tag{3.12b}$$

$$\chi = \left\{(k_5 f_\pi)/\sqrt{(k_5^2 + k_7^2 + k_9^2)}\right\} \tan\left[f_\pi \sqrt{(k_5^2 + k_7^2 + k_9^2)}\right] \mathrm{h}, \tag{3.12c}$$

where h is given by (3.9a).

A particular solution of (3.9a) is given by [7]

$$\mathrm{h} = \{(\sin\tau)/\tau\}\cos t, \quad t = x^4 \tag{3.13a}$$

$$\tau^2 = (x^1)^2 + (x^2)^2 + (x^3)^2, \tag{3.13b}$$

$$\max(\sin\tau/\tau) < \left\{\pi/2f_\pi\sqrt{(k_5^2 + k_7^2 + k_9^2)}\right\}. \tag{3.13c}$$

Part 2 (*Solutions of the extended Charap equations* (*Equation* (2.3) *with* $\epsilon = -1$) *for* $k' = 1$, $k'' = 1$): After the use of (3.1) the equations (2.3) (with $\epsilon = -1$, $k' = 1$, $k'' = 1$) reduce to

$$(u_{11} + u_{22} + u_{33} - u_{44}) + A(u_1^2 + u_2^2 + u_3^2 - u_4^2) = 0, \tag{3.14a}$$

$$(u_{11} + u_{22} + u_{33} - u_{44}) + D(u_1^2 + u_2^2 + u_3^2 - u_4^2) = 0, \tag{3.14b}$$

$$(u_{11} + u_{22} + u_{33} - u_{44}) + E(u_1^2 + u_2^2 + u_3^2 - u_4^2) = 0, \tag{3.14c}$$

where

$$A(u) = (\phi_{uu}/\phi_u) - [\{(\phi_u^2 - \psi_u^2 - \chi_u^2)/\phi\phi_u\} + \{2(\phi\phi_u + \psi\psi_u + \chi\chi_u)/(f_\pi^2 + \phi^2 + \psi^2 + \chi^2)\}], \tag{3.14d}$$

$$D(u) = (\psi_{uu}/\psi_u) - [(2\phi_u/\phi) + \{2(\phi\phi_u + \psi\psi_u + \chi\chi_u)/(f_\pi^2 + \phi^2 + \psi^2 + \chi^2)\}], \tag{3.14e}$$

$$E(u) = (\chi_{uu}/\chi_u) - [(2\phi_u/\phi) + \{2(\phi\phi_u + \psi\psi_u + \chi\chi_u)/(f_\pi^2 + \phi^2 + \psi^2 + \chi^2)\}], \tag{3.14f}$$

so that either

$$A = D = E \quad \text{or} \tag{3.15}$$

$$u_1^2 + u_2^2 + u_3^2 - u_4^2 = 0 \quad \text{and} \tag{3.16a}$$

$$u_{11} + u_{22} + u_{33} - u_{44} = 0. \tag{3.16b}$$

The equations (3.16) are same as those numbered (3.4). In the following the situation given by (3.15) has been discussed. Considering $D = E$ in (3.15) one arrives at

$$\chi = k_{11}\psi + k_{12}, \tag{3.17}$$

where k_{11} and k_{12} are arbitrary constants.

Let us define

$$\psi = k_{13}v + k_{14}, \tag{3.18}$$

where
(i) v is some unspecified function of u,
(ii) k_{13}, k_{14} are arbitrary constants.

Since u has been defined in (3.1) and it is an unspecified function of x^1, x^2, x^3, x^4, one can conclude till now that v is an unspecified function of x^1, x^2, x^3, x^4.

Putting (3.18) in (3.17) one gets

$$\chi = k_{15}v + k_{16}, \tag{3.19}$$

where $k_{11}k_{13} = k_{15}, k_{11}k_{14} + k_{12} = k_{16}$.

Now, from (3.1) we have $\phi = \phi(u)$, where ϕ is an unspecified function of u. Here we set $v = v(u)$, where v is an unspecified function of u. Hence one can write without any loss of generality

$$\phi = \phi(v). \tag{3.20}$$

The use of (3.18), (3.19) and (3.20) reduce the equations (2.3) (with $\epsilon = -1, k' = 1, k'' = 1$) to two equations given by

$$(v_{11} + v_{22} + v_{33} - v_{44}) + A'(v_1^2 + v_2^2 + v_3^2 - v_4^2) = 0, \tag{3.21a}$$

$$(v_{11} + v_{22} + v_{33} - v_{44}) + D'(v_1^2 + v_2^2 + v_3^2 - v_4^2) = 0, \tag{3.21b}$$

where

$$A' = (\phi_{vv}/\phi_v) - (\phi_v^2 - k_{13}^2 - k_{15}^2)/(\phi\phi_v) - F', \tag{3.22a}$$

$$D' = -2(\phi_v/\phi) - F', \tag{3.22b}$$

$$F' = \frac{2\{\phi\phi_v + (k_{13}^2 + k_{15}^2)v + k_{13}k_{14} + k_{15}k_{16}\}}{f_\pi^2 + \phi^2 + (k_{13}^2 + k_{15}^2)v^2 + 2(k_{13}k_{14} + k_{15}k_{16})v + (k_{14}^2 + k_{16}^2)}. \tag{3.22c}$$

Just as in the above the possibility other than

$$v_{11} + v_{22} + v_{33} - v_{44} = 0, \tag{3.23a}$$

$$v_1^2 + v_2^2 + v_3^2 - v_4^2 = 0 \tag{3.23b}$$

requires that

$$A' = D'. \tag{3.24}$$

From (3.24) one gets

$$\phi\phi_{vv} + \phi_v^2 + (k_{13}^2 + k_{15}^2) = 0 \tag{3.25}$$

which on integration leads to

$$\phi = \sqrt{[k_{18} + 2k_{17}v - (k_{13}^2 + k_{15}^2)v^2]}, \tag{3.26}$$

where k_{17} and k_{18} are arbitrary constants of integration.

Thus when ϕ, ψ and χ are given by (3.26), (3.18) and (3.19) respectively the equations (2.3) (with $\epsilon = -1, k_1 = 1, k_2 = 1$) reduce to a single equation given by

$$(v_{11} + v_{22} + v_{33} - v_{44}) + A''(v_1^2 + v_2^2 + v_3^2 - v_4^2) = 0, \tag{3.27a}$$

where

$$A'' = \frac{\mathrm{d}}{\mathrm{d}v} \ln \left[\frac{1}{\phi^2\{f_\pi^2 + \phi^2 + (k_{13}^2 + k_{15}^2)v^2 + 2(k_{13}k_{14}k_{15}k_{16})v + (k_{14}^2 + k_{16}^2)\}} \right]. \tag{3.27b}$$

Equations (3.27) can again be rewritten as

$$\mathrm{h}_{11} + \mathrm{h}_{22} + \mathrm{h}_{33} - \mathrm{h}_{44} = 0, \tag{3.28a}$$

where

$$\mathrm{h} = \int [\exp(A''(v)\mathrm{d}v]\mathrm{d}v + k_{20} \tag{3.28b}$$

and where k_{20} is an arbitrary constant of integration and $A''(v)$ is given by (3.27b).

Using (3.27b) and (3.26) in (3.28b) one gets

$$\mathrm{h} = \left[\int \mathrm{d}v/(G.H) \right] + k_{20}, \tag{3.29}$$

$$G = k_{18} + 2k_{17}v - (k_{13}^2 + k_{15}^2)v^2$$

$$H = f_\pi^2 + k_{18} + 2k_{17}v + 2(k_{13}k_{14} + k_{15}k_{16})v + (k_{14}^2 + k_{16}^2).$$

k_{20} is an arbitrary constant of integration.

Then for having a compact form of the solutions without the loss of much generality we choose $k_{17} = 0, k_{14} = 0, k_{16} = 0$ when (3.29) reduces to

$$\mathrm{h} = [1/\{(f_\pi^2 + k_{18})(k_{13}^2 + k_{15}^2)\}] \cdot \int [\mathrm{d}v/(k_{19}^2 - v^2)] + k_{20}, \tag{3.30a}$$

where

$$k_{19}^2 = k_{18}/(k_{13}^2 + k_{15}^2). \tag{3.30b}$$

For $k_{17} = 0$ the equation (3.26) along with (3.30b) reduces to

$$\phi = \sqrt{(k_{18}/k_{19})}\sqrt{(k_{19}^2 - v^2)}. \tag{3.31}$$

Since ϕ is real one must have $k_{19} > v$.

Then integrating the right hand side of (3.30) one can write v in terms of h as follows:

$$v = k_{19} \tanh[k_{19}(f_\pi^2 + k_{18})(k_{13}^2 + k_{15}^2)(\mathrm{h} - k_{20})]. \tag{3.32}$$

Finally ϕ, ψ and χ are given by (from (3.32), (3.31), (3.18) and (3.19) with $k_{14} = 0$, $k_{16} = 0, k_{20} = 0$)

$$\phi = \sqrt{k_{18}}\, \mathrm{sech}[k_{19}(f_\pi^2 + k_{18})(k_{13}^2 + k_{15}^2)\mathrm{h}], \tag{3.33a}$$

$$\psi = k_{13}k_{19}\tanh[k_{19}(f_\pi^2 + k_{18})(k_{13}^2 + k_{15}^2)\mathrm{h}], \tag{3.33b}$$

$$\chi = k_{15}k_{19}\tanh[k_{19}(f_\pi^2 + k_{18})(k_{13}^2 + k_{15}^2)\mathrm{h}], \tag{3.33c}$$

where h satisfies (3.28a) and a particular example of such h is given by (3.13).

Comparing (3.12) and (3.33) one can easily observe that the solutions of the Charap equations (2.2) and the extended Charap equations ((2.3) with $\epsilon = -1$) $k' = 1, k'' = 1$ differ considerably. One can also notice that both solutions are physical in nature.

Case 2

Here the solutions have been obtained using the techniques identical to those for Case 1. Arbitrary constants of this case which are similar to those of the Case 1 have been chosen in the same fashion to be equal to zero.

Part 1 (*Solutions of the Yang equations* (2.1)): The solutions presented here are due to De and Ray [16] and are given by

$$\phi = \sqrt{k_{27}}\,\mathrm{sech}[k_{29}(k_{22}^2 + k_{24}^2)\mathrm{h}], \tag{3.34a}$$

$$\psi = k_{22}k_{29}\tanh[k_{29}(k_{22}^2 + k_{24}^2)\mathrm{h}], \tag{3.34b}$$

$$\chi = k_{24}k_{29}\tanh[k_{29}(k_{22}^2 + k_{24}^2)\mathrm{h}], \tag{3.34c}$$

where h satisfies

$$\mathrm{h}_{11} + \mathrm{h}_{22} + \mathrm{h}_{33} + \mathrm{h}_{44} = 0 \tag{3.35}$$

and a particular example of such h is

$$\mathrm{h} = (\sin\tau/\tau)\cosh t, \quad t = x^4 \tag{3.36a}$$

$$\tau^2 = (x^1)^2 + (x^2)^2 + (x^3)^2, \tag{3.36b}$$

$$k_{29}^2 = k_{27}/(k_{22}^2 + k_{24}^2). \tag{3.36c}$$

Part 2 (*Solutions of the extended Yang equations* (*equations* (2.3) *with* $\epsilon = 1$) *for* $k' = 1, k'' = 1$): Here the solutions are given by

$$\phi = \sqrt{k_{35}}\,\mathrm{sech}[k_{36}(f_\pi^2 + k_{35})(k_{30}^2 + k_{32}^2)\mathrm{h}], \tag{3.37a}$$

$$\psi = k_{30}k_{36}\tanh[k_{36}(f_\pi^2 + k_{35})(k_{30}^2 + k_{32}^2)\mathrm{h}], \tag{3.37b}$$

$$\chi = k_{32}k_{36}\tanh[k_{36}(f_\pi^2 + k_{35})(k_{30}^2 + k_{32}^2)\mathrm{h}], \tag{3.37c}$$

where h satisfies (3.35) and a particular example of such h is given by (3.36).

Comparing (3.34) and (3.37) one observes that the solutions of Yang equation (2.1) and the extended Yang equations ((2.3) with $\epsilon = 1$) for $k' = 1, k'' = 1$) for the ansatz (3.1) are same in form. Both the solutions are physical in nature.

Summary

(i) Equations (2.1) were obtained by Yang [1] while discussing the condition of self-duality of $SU(2)$ gauge fields on Euclidean four-dimensional space.

(ii) The equations (2.2) were obtained by Charap [2] for the chiral invariant model of tangential parametrization [3].

(iii) In addition to physical significance these two sets of equations described above have some common characteristics which are mathematically interesting. Details about this have been presented as 'motivation' in the Introduction.

(iv) With the help of (2.3) we have represented the combined form of nonlinear partial differential equations wherefrom the two sets of equations mentioned above can be obtained as particular cases. Equations (2.3) are termed as the 'combined Yang–Charap (Y–C) equations'. For $\epsilon = 1$, $k' = 1$, $k'' = 0$ the equations (2.3) reduce to 'Yang equations' (2.1). For $\epsilon = -1, k' = 0, k'' = 1$ the equations (2.3) reduce to 'Charap equations' (2.2). For $\epsilon = 1$, $k' \neq 0$, $k'' \neq 0$ the equations (2.3) are 'extended Yang equations'. For $\epsilon = -1, k' \neq 0, k'' \neq 0$ the equations (2.3) are 'extended Charap equations'. The same type of ansatz (3.1) generates physical solutions of all the four sets of equations: (i) the Yang equations (2.1), (ii) the Charap equations (2.2), (iii) the extended Yang equations (equations (2.3) with $\epsilon = 1$) for $k' = 1$, $k'' = 1$, (iv) the extended Charap equations (equations (2.3) with $\epsilon = -1$) for $k' = 1, k'' = 1$.

Such solutions in compact form for the Yang equations (2.1) and the extended Yang equations (equations (2.3) with $\epsilon = 1$) for $k' = 1, k'' = 1$ are given by (3.34) and (3.37) respectively. They are same in form.

Such solutions in compact form for the Charap equations (2.2) and the extended Charap equations (equations (2.3) with $\epsilon = -1$) for $k' = 1, k'' = 1$ are given by (3.12) and (3.33). They differ considerably.

Acknowledgement

One of the authors (PKC) thanks the University Grants Commission, India for financial assistance. They are also grateful to the referee for some constructive comments.

References

[1] C N Yang, *Phys. Rev. Lett.* **38**(24), 1377 (1977)
[2] J M Charap, *J. Phys.* **A9**, 1331 (1976)
[3] J M Charap, *J. Phys.* **A6**, 987 (1973)
[4] P K Chanda, D Ray and U K De, *J. Math. Phys.* **11**(5), 995 (1984)
[5] S Chakraborty, P K Chanda and D Ray, *Int. J. Theor. Phys.* **34**(11), 2223 (1995)
[6] S Chakraborty and P K Chanda, *Pramana – J. Phys.* **52**(3), 245 (1999)
[7] D Ray, *J. Phys.* **11**(5), 995 (1978)
[8] D Ray, *J. Math. Phys.* **23**, 2155 (1982)

[9] J Corones, *J. Math. Phys.* **19**, 2431 (1978)

[10] S Trimper, *Phys. Lett.* **A70**, 1144 (1979)

[11] D Ray, *Phys. Lett.* **A70**, 274 (1980)

[12] M F Atiyah and N S Manton, *Phys. Lett.* **B222**(3, 4), 438 (1987)

[13] R A Leese, N S Manton and B J Schroers, *Nucl. Phys.* **B442**, 228 (1995)

[14] S Lakshmibala, Bindu A Bambah, M S Sriram and C Mukku, *Pramana – J. Phys.* **48**(2), 617 (1997)

[15] M S Sriram and J Segar, *Pramana – J. Phys.* **48**(1), 205 (1997)

[16] U K De and D Ray, *Phys. Lett.* **B101**(5), 335 (1981)

[17] K K Ghosh, D Ray and P Chanda, *Int. J. Theor. Phys.* **28**(1), 111 (1989)