PRAMANA
— journal of
physics

Vol. 52, No. 5
May 1999
pp. 453–457

Partial quantum recurrence in free and quasibound systems

A K SIKRI and M L NARCHAL
Department of Physics, Punjabi University, Patiala, India

MS received 19 May 1998; revised 23 December 1998

Abstract. The quantum motion of a periodically two-sided kicked free particle is studied under various boundary conditions. The quasienergies, quasistates and the energy of the system are determined exactly. It is found that the energy of the system recurs irrespective of boundary conditions whereas the wave function shows recurrence only for a completely bound particle.

Keywords. Quantum chaos; free system; quasibound system.

PACS No. **05.45**

1. Introduction

The study of quantal behaviour of periodically kicked systems has been a subject of much interest in recent years to understand the notion of quantum chaos [1–11]. Two types of behaviour have so far been discussed. One is recurrent behaviour in which both the energy and wave function of the system come infinitely close to their starting values infinitely often during the course of time evolution of the system. Second is the unstable behaviour in which the system energy grows with time in an unbounded manner whereas wave function shows no inclination to come close to its starting value [8]. In a recent paper by Eisenberg *et al* [12] an example of a two-sided kicked rotator has been discussed in which the system shows exact periodicity in time. This phenomenon has been called quantum resonance. However, the discussion is limited to systems whose quasienergies constitute a discrete set. The question as to what type of behaviour the two-sided kicked system will show whose quasienergy spectrum is continuous is still open and forms the subject matter of investigation in this report.

2. Wave function and energy of the system after N kicks

Consider a particle of mass m subject to a sequence of two sided periodic δ kicks. The Hamiltonian of the system is given by

$$H = H_0 + \epsilon x \sum_{n=-\infty}^{\infty} \left[\delta\left(\frac{t}{T} - 2n\right) - \delta\left(\frac{t}{T} - (2n+1)\right)\right] \epsilon > 0, \tag{1}$$

where $H_0 = p^2/2m$ is the free particle Hamiltonian, T is the half period of two-sided kick and ϵ determines the strength of the kick.

Since the Hamiltonian is periodic in time, the dynamics of the system is determined by one-step Floquet operator

$$U = \exp(-i\alpha x)\exp((-i/\hbar)H_0 T)\exp(i\alpha x)\exp((-i/\hbar)H_0 T), \tag{2}$$

where $\alpha = \epsilon T/\hbar$.

Using the result

$$\mathrm{e}^{-iv}\mathrm{e}^{\omega}\mathrm{e}^{iv} = \exp(\mathrm{e}^{-iv}\omega\mathrm{e}^{iv}). \tag{3}$$

Equation (2) takes the form [13]

$$U = \mathrm{e}^{-iF}, \tag{4}$$

where the operator F is given by

$$F = \frac{p^2 T}{m\hbar} + \frac{p\alpha T}{m} + \frac{\alpha^2 \hbar T}{2m}. \tag{5}$$

The quasistates of the system, which are eigenstates of U, corresponding to quasienergy E_β are given by

$$|\beta\rangle = A\exp(i(k - \alpha/2)x) + B\exp(-i(k + \alpha/2)x), \tag{6}$$

where A and B are arbitrary constants and $k^2 = E_\beta m/\hbar T$.

Let $|\psi_0\rangle$ be the wave function of the free particle at time $t = 0$. Then its wave function after N kicks is

$$|\psi_N\rangle = U^N|\psi_0\rangle. \tag{7}$$

Representing the free particle by the wave packet of width σ at $t = 0$ and using (4), the expression for $\psi_N(x)$ in position representation turns out to be

$$\psi_N = \frac{\mathrm{e}^{-iNT\alpha^2\hbar/2m}}{(2\pi\sigma^2)^{1/4}\left(1 + \frac{i\hbar NT}{m\sigma^2}\right)^{1/2}} \exp\left[-\left(x - \frac{2\hbar NTr}{m}\right)^2\left(1 - \frac{i\hbar NT}{m\sigma^2}\right)\right] \times \exp\left[\frac{i\hbar NTr}{m}\right]\mathrm{e}^{irx}. \tag{8}$$

Defining

$$||\psi_N||^2 = \int_0^\infty \psi_N^\star \psi_N \mathrm{d}x, \tag{9}$$

it can be shown easily that

$$||\psi_{N+p} - \psi_N||^2 = 2 - \frac{2\sqrt{2}}{\left(4+\frac{a^2}{\sigma^4}\right)^{1/4}} \exp\left[-\frac{2a^2r^2}{(b+3)}\right] \times \cos\left(\frac{\alpha a}{2} + \frac{r^2 a}{b} + \frac{\theta}{2} + \frac{3a^2r^2}{b(b+3)}\right), \tag{10}$$

where

$$a = pT\hbar/m, \tag{11}$$

$$b = 1 + a^2/\sigma^4, \tag{12}$$

$$\theta = \tan^{-1}\left(\frac{a}{2\sigma^2}\right). \tag{13}$$

The factor $2\sqrt{2}(4 + a^2/(\sigma^4))^{-1/4}$ in the second term on right hand side of (10) decreases monotonically with p. The exponential factor in this term is also monotonically decreasing function even though slowly. The third factor in this term oscillates between +1 and −1. The product of these three factors is monotonically decreasing and its value is going away from 2. Thus the wave function of the system shows no signs of periodicity or quasiperiodicity.

The energy of the particle after N kicks is given by

$$E_N = \langle\psi_N|H_0|\psi_N\rangle. \tag{14}$$

Using (7) it reduces to

$$E_N = E_0 = \langle\psi_0|H_0|\psi_0\rangle, \tag{15}$$

where E_0 is the initial energy of the particle. Thus we observe that the energy of the system remains constant with time. We conclude that the energy of the free particle subject to two sided δ impulses recurs whereas its wave function does not.

3. Wave functions of quasibound and bound systems

Suppose the free particle is now confined to one dimensional half space $x > 0$ and subject to two-sided δ kicks. The quasistates can be obtained from (6) by applying the boundary condition

$$|\beta\rangle = 0 \quad \text{at} \quad x = 0. \tag{16}$$

The Floquet states are given by

$$|\beta\rangle = \subseteq \exp\left(-\frac{i\alpha x}{2}\right) \sin kx, \tag{17}$$

where k can take any arbitrary value. Expanding initial wave function $|\psi_0\rangle$ in terms of $|\beta\rangle$ and using (7), the wave function of the system after N kicks turns out to be

$$|\psi_N\rangle = \int_{-\infty}^{\infty} a_k \exp(-i\alpha x/2) \exp(-iE_\beta^k N) \sin kx \, \mathrm{d}k. \tag{18}$$

It can be easily shown that

$$||\psi_{N+p} - \psi_N||^2 = 2 - 2\pi \int_0^\infty |a_k|^2 \cos\left(\frac{\hbar T k^2 p}{m}\right) \mathrm{d}k. \tag{19}$$

Since each $|a_k| \leq 1$, the above result can be approximately written as

$$||\psi_{N+p} - \psi_N||^2 \leq 2 - \pi^{3/2}\sqrt{\frac{m}{2\hbar T p}}. \tag{20}$$

This equation shows that the wave function does not show any recurrence whereas the energy of this system given by (15) does recur with zero period. This result is in sharp contrast with our earlier result where the wave function as well as energy of the particle confined to half space and subject to one-sided δ kicks has been found to recur [7].

Suppose the free particle is now enclosed in a box of length l and is subjected to two-sided δ kicks. Then Floquet states must satisfy the boundary conditions.

$$|\beta\rangle \begin{array}{l} = 0, \text{ at } x = 0 \\ = 0, \text{ at } x = l \end{array} \tag{21}$$

Applying these boundary conditions to (6), we obtain

$$E_\beta = p^2 \frac{\pi^2 \hbar T}{m l^2} \qquad p = 1, 2, \ldots. \tag{22}$$

The quasienergy spectrum becomes discrete. Thus both the wave function and energy of a particle enclosed in a box will recur with the passage of time in accordance with theorem of Hogg and Hubermann [3].

4. Conclusion

We have investigated the effect of two sided periodic kicks on a particle which may be (a) completely free ($-\infty < x < \infty$) (b) quasibound ($0 < x < \infty$) and (c) completely bound ($0 < x < l$). It turns out that

(a) the energy of the system remains constant irrespective of boundary conditions and may be considered periodic with zero period,

(b) the wave function shows recurrent behaviour only for a completely bound system whereas it is nonrecurrent for a completely free or for a quasibound system.

This seems to be a third type of quantal behaviour which periodically kicked system may show. This behaviour may be called partial recurrence in which only the energy recurs whereas the wave function does not. It appears that the quantum recurrence theorem of Hogg and Huberman [3], originally formulated for completely bound systems with discrete quasienergy spectrum may as well partially hold for systems with continuous quasienergy spectrum.

References

[1] G Casati, B V Chirikov, F M Izrailev and J Ford, *Stochastic behaviour in classical and quantum Hamiltonian systems*, edited by G Casati and J Ford, Lecture Notes in Physics (Springer-Berlin, 1979) Vol. 93
[2] S Fishman, D R Grempel and R E Prange, *Phys. Rev. Lett.* **49**, 509 (1982)
[3] T Hogg and B A Huberman, *Phys. Rev.* **A28**, 22 (1983)
[4] J V Jose and R Cordery, *Phys. Rev. Lett.* **56**, 290 (1986)
[5] G Casati, J Ford, I Guarneri and F Vivaldi, *Phys. Rev.* **A34**, 1413 (1986)
[6] B Milek and P Seba, *Phys. Rev.* **A42**, 3213 (1990)
[7] A K Sikri and M L Narchal, *Phys. Rev.* **A47**, 4605 (1993)
[8] J Bellissard, in *Trends and developments in the eighties*, edited by G Albeverio and Ph. Blanchard (World Scientific, Singapore, 1985)
[9] M V Berry, N L Balazs, M Tabor and A Voros, *Ann. Phys.* (NY) **122**, 26 (1979)
[10] R Blumel, R Meir and U Smilansky, *Phys. Lett.* **A103**, 353 (1984)
[11] A Cohen and S Fishman, *Int. J. Mod. Phys.* **B2**, 103 (1988)
[12] E Eisenberg and N Shnerb, *Phys. Rev.* **E49**, R941 (1994)
[13] W H Louisell, *Quantum statistical properties of radiation* (John Wiley and Sons, 1973)