PRAMANA
— journal of
physics

© Indian Academy of Sciences

Vol. 52, No. 4
April 1999
pp. 375–387

Suppression of Smale horseshoe structure via secondary perturbations in pendulum systems

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MS received 11 September 1998; revised 18 January 1999

Abstract. We analyse the use of parametric and quasiperiodic modulations in suppressing horseshoe structure in the phase plane of perturbed pendulum systems. Taking the Froude pendulum as a typical system, four different modulation mechanisms are studied by deriving analytic expressions for the window of the strength of modulation giving suppression in each case. A comparison of the four cases from the point of view of flexibility and efficiency is also given.

Keywords. Smale horseshoe; Melnikov criterion; Froude pendulum; parametric modulation; quasiperiodic forcing; control of chaos.

PACS Nos 05.45; 05.40

1. Introduction

It has been well established during the last decade that the separatrix orbit of integrable Hamiltonian systems develops into stochastic layers under dissipative or Hamiltonian perturbations [1]. These layers arise due to the transversal intersections of stable and unstable manifolds of the perturbed separatrix. The consequent stretchings and foldings result in the occurrence of Smale horseshoes in the underlying dynamics [2,3]. These structures need not, in general imply asymptotic chaos in the system; still in Hamiltonian cases the horseshoe chaos or stochasticity in the vicinity of the separatrix is often a precursor to global deterministic chaos in the system [4,5]. In the dissipative case, with multiple attractors, long-lived chaotic transients originate near the separatrix and can lead to a periodic or chaotic attractor asymptotically. Nevertheless these transients bear relevance in studies related to the escape scenario and stability properties of such systems [6,7].

The Melnikov analysis is till now the only analytic search procedure that can pick up the threshold for the first intersection between the stable (W^s) and the unstable (W^u) manifolds in terms of the relevant parameters of the system. The basic aim of the present work is to see how this threshold shifts when additional secondary perturbations of parametric or quasiperiodic type are introduced into the system. Interestingly, isolated regions are found to exist in the parameter space where horseshoe generation can be suppressed by the above procedure.

A few isolated attempts along these directions are seen to be reported, especially in systems of anharmonic oscillators [8–15]. In perturbations of Hamiltonian systems, additional forcing leading to inhibition of chaos was reported earlier [14]. For a typical anharmonic oscillator, suppression of horseshoe by the addition of a weak periodic forcing has been studied in certain specific parameter planes [8,14]. Suppression of chaos in a Duffing–Holmes oscillator by resonant parametric perturbations employing Melnikov method has been analysed by Lima and Pettini [15]. Regularisation by means of weak parametric modulations in the forced pendulum was recently reported by Chacon [16].

In the present work we envisage the problem in a broader sense by taking up the pendulum system and try to capture in detail the behaviour of the threshold under parametric modulation (PM) of the damping, driving and restoring terms and 'additive' quasiperiodic modulation. In each case analytic expression for the window of amplitude of modulation (denoted by $\Delta\eta$), where suppression is possible, is worked out and its dependence on the the relevant parameters of the problem studied graphically. Our analysis includes higher order resonances also. The results are supported by a detailed numerical analysis with parameters chosen from this window $\Delta\eta$. We have been able to establish with detailed bifurcation diagrams, phase portraits and computation of Lyapunov characteristic exponents (LCE) that the regions which are chaotic under damping and forcing show periodic behaviour after modulations are applied.

The paper is organized as follows. In §2 we formulate the procedure with the nonlinear pendulum as the basic system. The window of values for the amplitude of modulation or secondary perturbation is worked out in a general sense. The following section deals with a particular pendulum called Froude pendulum [17], so that detailed equations can be worked out. We try four different cases of modulations and present the results in detail. Section 4 contains our remarks regarding the efficiency of performance for the four different types of modulations.

2. General formalism

In order to set the analysis in a clear perspective we work with the nonlinear pendulum as our basic system. This system is chosen particularly because it serves as a model unperturbed system for many nonlinear systems like Josephson junctions, synchronous electric motors, phase locked loops, charge density waves etc. [18]. The equations of motion of the conservative system in $\mathbf{R}^2$ space can be written as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\sin x_1. \end{aligned} \tag{2.1}$$

with a Hamiltonian given by

$$H_0 = {x_2}^2/2 - \cos x_1 \; .$$

The solutions of the separatrix (heteroclinic orbit) are [1] given by

$$W_0^{\pm} = [x_{1_0}, x_{2_0}] = [\pm\,\tan^{-1}(\sinh t),\ \pm 2\,\mathrm{sech}\, t] \tag{2.2}$$

for $x_{1_0}\, \epsilon\, (-\pi, \pi)$ and +(–) sign refers to the upper(lower) part of the orbit.

The primary perturbations in the system can be damping and periodic driving that separates $W_0^\pm$ into $W_s^\pm$ and $W_u^\pm$; and the Melnikov function $M^\pm(t_0)$ furnishes a measure of their separation in a chosen Poincaré section Σ^{t_0} fixed by the arbitrary time t_0. Equation (2.1) then gets modified as

$$\begin{aligned} \dot{x}_1 &= x_2 \ , \\ \dot{x}_2 &= -\sin x_1 - g(x_1, x_2)x_2 + f \ \cos\omega t \ . \end{aligned} \tag{2.3}$$

where g is the damping function and f and ω are the amplitude and frequency respectively of the forcing term.

The Melnikov function M for (2.3) is defined by

$$M = \int_{-\infty}^{+\infty} \mathbf{h_0} \wedge \mathbf{h_1} \, \mathrm{d}t \ , \tag{2.4}$$

where

$$\mathbf{h_0} = \begin{bmatrix} h_{0_1} \\ h_{0_2} \end{bmatrix} = \begin{bmatrix} x_2 \\ -\sin x_1 \end{bmatrix}$$

and

$$\mathbf{h_1} = \begin{bmatrix} h_{1_1} \\ h_{1_2} \end{bmatrix} = \begin{bmatrix} 0 \\ -g(x_1, x_2) + f \cos\omega t \end{bmatrix} . \tag{2.5}$$

Here $\wedge$ is the antisymmetric wedge operator defined by

$$\mathbf{h_0} \wedge \mathbf{h_1} = h_{0_1} h_{1_2} - h_{0_2} h_{1_1} .$$

It can easily be verified that the Melnikov function now takes the form

$$M_0^\pm(t_0) = \delta \pm \alpha \ \cos\omega t_0, \tag{2.6}$$

where δ is decided by the actual dependence of the parameters of the damping function g on x_1 and x_2. It has the general form

$$\delta = \int_{-\infty}^{+\infty} g(x_{1_0}, x_{2_0}) \, x_{2_0}(t - t_0) \, \mathrm{d}t \, . \tag{2.7}$$

The term α can readily be found as

$$\alpha = 2\pi f \, \mathrm{sech}(\pi\omega/2) \, . \tag{2.8}$$

The development of Smale horseshoe due to intersections of $W_s^\pm$ and $W_u^\pm$ is indicated by the sign reversals of the function $M^\pm(t_0)$. This is possible only if

$$|\alpha| > \delta \, . \tag{2.9}$$

This sets the threshold for stochasticity in terms of the parameters of the problem as

$$f_0 = \delta / [2\pi \mathrm{sech}(\pi\omega/2)] \, . \tag{2.10}$$

Now secondary perturbations are introduced via additive or parametrically modulating functions of the type $\eta \sin(p\omega t + \phi)$ where η is the strength of the modulation, p is the frequency ratio and ϕ the phase difference between primary and secondary terms. This results in additional terms in $M^{\pm}(t_0)$, as

$$M^{\pm}(t_0) = M_0^{\pm} \pm \beta \sin(p\omega t_0 + \phi) \tag{2.11}$$

with

$$\beta = \eta \Gamma. \tag{2.12}$$

Here Γ depends on the term being modulated and also on the way in which it is modulated. If $\mathcal{J}$ represents the modulated term (for additive mode $\mathcal{J}$ takes the value 1) Γ can be written as

$$\Gamma = \int_{-\infty}^{+\infty} \cos\left[p\omega(t - t_0)\right] x_{2_0}(t - t_0) \mathcal{J} \mathrm{d}t\,. \tag{2.12a}$$

Then the necessary condition for suppression of horseshoe is

$$|\alpha - \beta| < \delta \tag{2.13}$$

and in terms of the modulation strength this can be rewritten as

$$\left(1 - \frac{\delta}{\alpha}\right)\zeta < \eta \leq \left(1 + \frac{\delta}{\alpha}\right)\zeta, \tag{2.14}$$

where ζ is a function involving p, ω and $\mathcal{J}$. It is given by

$$\zeta = \frac{2\pi f}{\Gamma} \mathrm{sech}\,(\pi\omega/2). \tag{2.15}$$

This introduces restrictions on the value of ϕ. With p as a positive integer the allowed values of ϕ are given by

$$p = \frac{2m + (3/2) - (\phi/\pi)}{2n} \quad : \; m\,, n \text{ integers.} \tag{2.16}$$

While the above expression determines ϕ values for M^-, a similar expression with denominator changed as $(2n + 1)$ can be obtained for M^+.

The sufficient condition for (2.13) to be always true can be expressed as

$$\eta \leq \zeta/p^2\,. \tag{2.17}$$

The maximum of η leading to horseshoe suppression is the lower one of the upper-bounds of (2.14) and (2.17). This gives us a range of η in the interval $[\eta_{\min}, \eta_{\max}]$ where suppression is possible. Further details can be worked out by taking up a specific case.

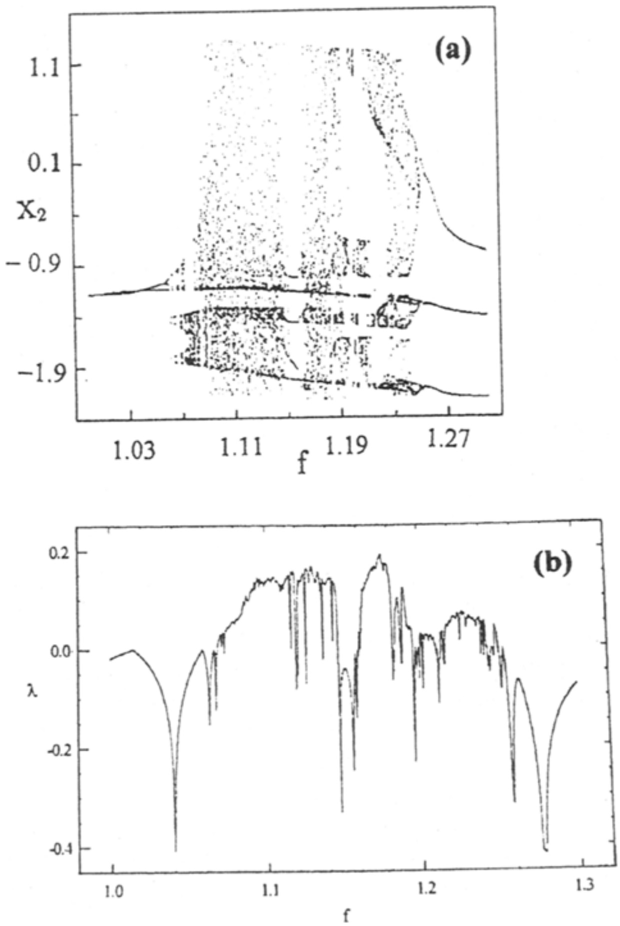
3. The Froude pendulum

The Froude pendulum is a nonlinear mechanical system involving a pendulum mounted on a rotating shaft [17,18]. The nonlinear coupling between the shaft and the pendulum keeps it oscillating. Mathematically, this coupling introduces a nonlinear damping term in the equation of motion that leads to self generated limit cycle in the system. With primary driving alone we have

$$\begin{aligned} \dot{x}_1 &= x_2 \,, \\ \dot{x}_2 &= -\sin x_1 - q_1 x_2 (q_2 {x_2}^2 - 1) + f \cos \omega t \,; \end{aligned} \tag{3.1}$$

where q_1 and q_2 are damping parameters.

Our earlier studies indicate that the self excited oscillations in the system go chaotic for certain values of f and ω [20]. Figure 1a is a bifurcation diagram showing x_2 sampled at ω against f, revealing chaotic behaviour for f in the interval (1.06,1.25); here $q_1 = 0.3$, $q_2 = 0.5$ and $\omega = 0.7$. For the same parameter values, the LCE is computed using the Wolf algorithm [21], where 500 initial cycles are discarded for settling of transients; calculations done over the next 5000 cycles yield the LCE. The variation of these exponents against f is shown in figure 1b. The attractor in the phase space of the system for $f = 1.1$, exhibiting chaotic nature is shown in figure 1c.



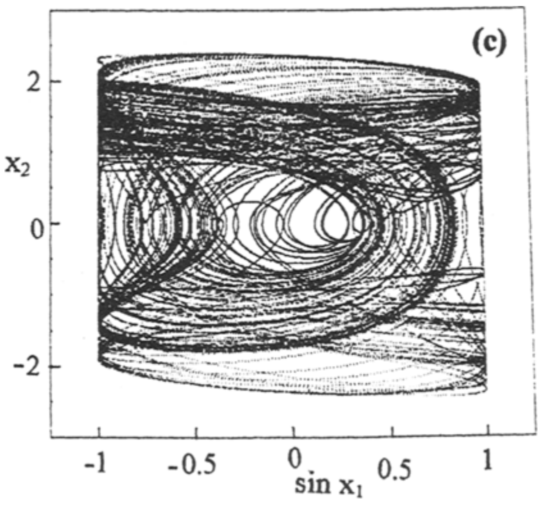


Figure 1. Bifurcation diagram for the Froude pendulum with x_2 along the vertical axis and the drive amplitude f as the control parameter is shown in (**a**) the other parameter values are fixed as $q_1 = 0.3$, $q_2 = 0.5$ and $\omega = 0.7$. (**b**). The LCE (λ) plotted as a function of f. (**c**). The chaotic trajectories in the phase space - x_2 against $\sin x_1$-.

The Melnikov function for the system in (3.1) is given by (2.6) with

$$\delta = 8q_1\ [(16q_2/3) - 1]$$

and

$$\alpha = 2\pi f\ \mathrm{sech}(\pi\omega/2)\,. \tag{3.2}$$

The threshold amplitude in accordance with (2.10) is given by

$$f_0 = \frac{4q_1}{\pi}\left(\frac{16q_2}{3} - 1\right)\cosh\left(\frac{\pi\omega}{2}\right)\,. \tag{3.3}$$

So for drive amplitudes f above f_0, horseshoe structure develops near the separatrix. For the parameter values chosen in figure 1, f_0 works out to be 1.0618. Above f_0, asymptotic chaos is exhibited as can be seen from the figures. However since the system supports multiple attractors, the asymptotic states for certain initial values may still be periodic. This aspect, though it falls outside the purview of the present paper is made obvious in the studies of the basin boundary patterns in similar systems by one of the authors [6].

Case 1. Parametric modulation of the drive term

We first consider a situation where the amplitude of the primary driving term is modulated by a weak periodic forcing of strength η, frequency $p\omega$, and phase difference ϕ. Then (3.1) becomes

$$\ddot{x}_1 = -\sin x_1 - q_1 x_2 (q_2 {x_2}^2 - 1) + f\,[1 + \eta \sin(p\omega t + \phi)] \cos \omega t. \quad (3.4)$$

Correspondingly, the Melnikov function gets modified as

$$M^{\pm}(t_0) = M_0^{\pm}(t_0) \mp \beta \sin[(p-1)\omega t_0 + \phi] \mp \gamma \sin[(p+1)\omega t_0 + \phi] \quad (3.5)$$

with

$$\beta = \pi f \eta \cosh[\pi(p-1)\omega/2)]$$

and

$$\gamma = \pi f \eta \cosh[(\pi(p+1)\omega/2)].$$

Now suppose the unmodulated system ($\eta = 0$) is set in a chaotic state with an $f > f_0$ given by (3.3). When the PM is switched on, the necessary condition for suppression of horseshoe becomes

$$|\alpha - (\beta + \gamma)| < \delta \quad (3.6)$$

or in terms of η,

$$\left(1 - \frac{\delta}{\alpha}\right) \psi < \eta \leq \left(1 + \frac{\delta}{\alpha}\right) \psi, \quad (3.7)$$

where α and δ are given by (3.2); and ψ has the form

$$\psi = 2\,\mathrm{sech}\,(\pi\omega/2) / \,\{\mathrm{sech}\,[\pi(p-1)\omega/2] + \mathrm{sech}\,[\pi(p+1)\omega/2]\,\}\,. \quad (3.8)$$

The sufficient condition for $M(t_0)$ not to change sign for all t_0 is

$$\eta < \psi/p^2\,, \quad (3.9)$$

where the frequency ratio p is a positive integer satisfying

$$p = \frac{2m + (3/2) - (\phi/\pi)}{2n} \quad : m,\ n \text{ integers}\,. \quad (3.10)$$

Similar results have been reported for PM of $\sin x_1$ term in a linearly damped pendulum [16]. It is obvious that then ϕ is limited to $(2l - 1/2)\pi$, with l an integer. Under these circumstances we can define a window of drive amplitudes, satisfying the necessary and sufficient conditions for horseshoe suppression as

$$\eta_{\min} < \eta \leq \eta_{\max},$$

with

$$\eta_{\min} = (1 - \delta/\alpha)\psi\,,$$
$$\eta_{\max} = \min\left[\left(1 + \frac{\delta}{\alpha}\right)\psi\,,\, \frac{\psi}{p^2}\right]. \quad (3.11)$$

This window $\Delta\eta = (\eta_{max} - \eta_{min})$ depends on ω, f, q_1, and q_2. In figure 2a we take $q_1 = 0.3$, $q_2 = 0.5$ and $\omega = 0.7$: a set of parameter values that give chaos, for $f > 1.07$ when $\eta = 0$ and then study how $\Delta\eta$ varies with the drive amplitude f as PM is switched on. The figure shows that the magnitude of modulation amplitude $\Delta\eta$ varies inversely as f, for all resonances. As the resonance order increases $\Delta\eta$ generally falls steeply. Figure 2b shows the bifurcation diagram similar to figure 1a with the PM (fundamental mode only) whose amplitude lies between η_{max} and η_{min}. It is evident that the PM has been successful in rendering the chaotic window periodic. Figure 2c is the LCE for the modulated system plotted against f. The negative value of the exponent throughout the f interval (1,1.3) proves the suppression effect of modulating the drive amplitude . The trajectory of the modulated Froude pendulum shown in figure 2d also indicates the stabilization effect of the particular type of modulation, on the system.

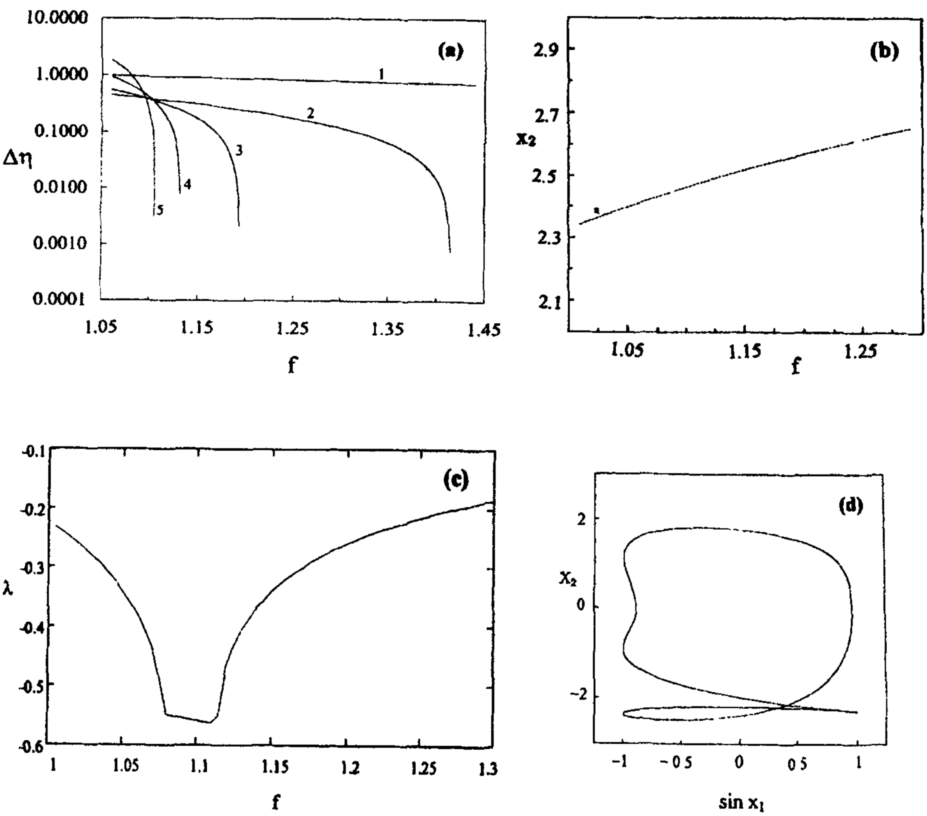


Figure 2. (**a**) Window $\Delta\eta$ (on logarithmic scale) of the amplitude of modulation capable of producing suppression of chaos indicated in case (1) as a function of f. Parameters are the same as in figure 1. The numbers refer to the order of resonance. In (**b**) bifurcation diagram for the chaotic region of figure 1 is shown with a value of η chosen from the window $\Delta\eta$; (**c**) is the LCE plot for $1.01 < f < 1.3$ with modulation; (**d**) is the trajectory of the modulated pendulum.

Case 2. Parametric modulation of the damping term

We study the effect of subjecting the damping term also to a modulation, giving

$$\dot{x}_2 = -\sin x_1 - q_1\,[1 + \eta\sin(p\omega t + \phi)]\,x_2(q_2{x_2}^2 - 1) + f\cos\omega t\,. \quad (3.12)$$

The Melnikov function in this case has the form

$$M^{\pm}(t_0) = M_0^{\pm} + \beta\sin(p\omega t_0 + \phi)\,, \quad (3.13)$$

with β given by

$$\beta = \frac{4q_1 p\omega\eta}{\sinh(\pi p w/2)}[2q_2\frac{(4 + p^2\omega^2)}{3} - 1]\,. \quad (3.14)$$

The window of η values restricting the region of suppression is

$$\left[1 - \frac{\delta}{\alpha}\right]\lambda < \eta \leq \min\left[\lambda(1 + \frac{\delta}{\alpha}),\, \frac{\lambda}{p^2}\right]\,, \quad (3.15)$$

where

$$\lambda = \frac{f\sinh(\pi p\omega/2)}{2q_1 p\omega\{\frac{2}{3}q_2(4 + p^2\omega^2) - 1\}\cosh(\pi p\omega/2)}\,; \quad (3.16)$$

here the admissible values of p are as in (3.10).

The window $\Delta\eta$ for suppression of chaos is plotted as a function of f in figure 3 choosing the other parameter values the same as in case(1), giving chaos in the absence of modulation. It can be seen that $\Delta\eta$ is more or less a constant for different drive amplitudes for the fundamental resonance mode; while for higher modes $\Delta\eta$ generally decreases with f. The bifurcation diagram for $1.00 < f < 1.3$ which was chaotic is found to be periodic with the introduction of the modulation. The LCE plot and trajectory of the modulated system also support the fact that the modulation has been efficient in suppressing chaos.

Case 3. Parametric modulation of the restoring term

Here we investigate the possibility of modulating the $\sin x_1$ term. A similar case has been studied by Chacon [16] for a linearly damped pendulum. The system in (3.1) is modulated with a sinusoidal term of amplitude η and frequency $p\omega$ with initial phase ϕ so that

$$\dot{x}_2 = -\sin x_1[\,1 + \eta\cos(p\omega t + \phi)\,] - q_1 x_2({x_2}^2 - 1) + f\cos\omega t\,. \quad (3.17)$$

By repeating the steps in the same sequence as given for case(2) we arrive at

$$M^{\pm}(t_0) = M_0^{\pm}(t_0) + \beta\sin(p\omega t_0 + \phi)\,,$$

with

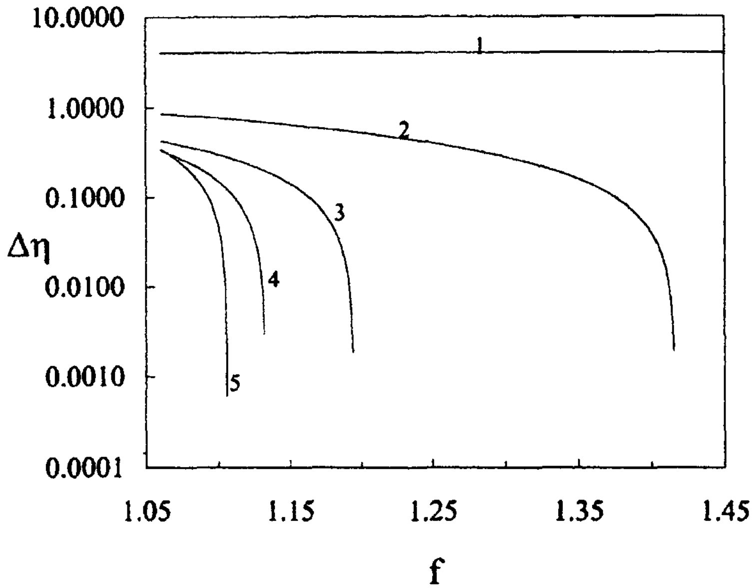


Figure 3. Modulation amplitude window $\Delta\eta$ for the case (2) plotted against f; parameters and numbers as in figure 2a.

$$\beta = 2\pi\eta p^2\omega^2 \text{cosech}\,(\pi p\omega/2)\,. \tag{3.18}$$

As before here also p is given by (3.10). The window of forcing amplitude for suppression becomes

$$\left(1-\frac{\delta}{\alpha}\right)\chi < \eta \leq \min\left[\left(1+\frac{\delta}{\alpha}\right)\chi\,,\,\chi/p^2\right]\,,$$

where

$$\chi = \frac{f\sinh(\pi p\omega/2)}{\cosh(\pi\omega/2)}\,. \tag{3.19}$$

The window $\Delta\eta$ is plotted as a function of f in figure 4 in the same way as in case(1). It is observed that the modulation strength $\Delta\eta$ giving suppression of horseshoe remains almost a constant for different drive amplitudes in the fundamental resonance mode; and for higher resonances $\Delta\eta$ generally decreases with f. With the help of the bifurcation diagram and the LCE plot of the modulated system it was established that the chaotic window for f $(1.0, 1.3)$ of the original system has been transformed to a periodic one.

Case 4. Addition of secondary forcing term

Addition of a second forcing term as a quasiperiodic driving, has been successfully tried in many systems earlier as a control mechanism [8–11]. We look into the mathematical reasoning behind such a possibility using the technique of Melnikov analysis along the

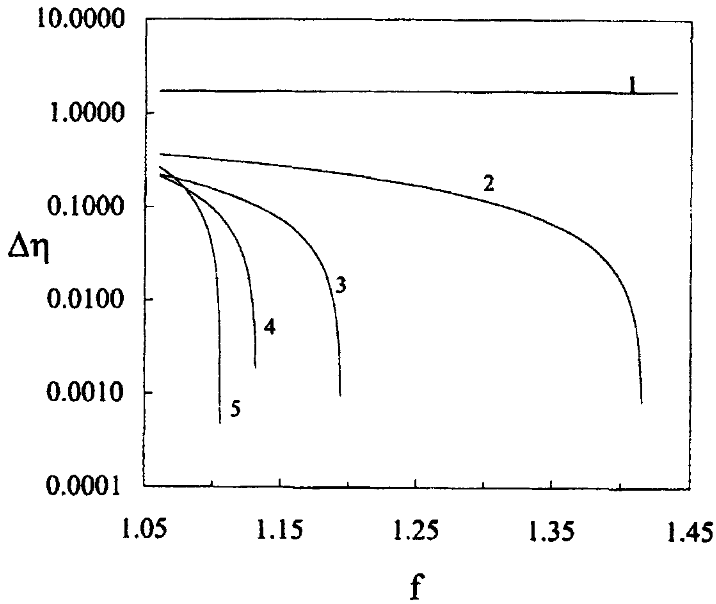


Figure 4. Window $\Delta\eta$ for the case (3) as a function of original drive amplitude f; details as in figure 2a.

lines of Lima and Pettini [15]. The system in (3.1) is driven with an additional forcing of amplitude η, frequency $p\omega$ and initial phase ϕ, so that

$$\dot{x}_2 = -\sin x_1 - q_1\, x_2\,({x_2}^2 - 1) + f\cos\omega t + \eta\sin(p\omega t + \phi)\,. \tag{3.20}$$

In this case, the modified Melnikov function works out to be

$$M^{\pm}(t_0) = M_0^{\pm}(t_0)\ \mp \beta\sin(p\omega t_0 + \phi)\,,$$

with

$$\beta = 2\pi\eta\,\mathrm{sech}\,(\pi p\omega/2)\,, \tag{3.21}$$

p being as in the previous cases. The window of forcing amplitude for suppression now becomes

$$[1 - (\delta/\alpha)]\xi < \eta \le \min\{[1 + (\delta/\alpha)]\psi, \xi/p^2]\}\,,$$

where

$$\xi = f\,\mathrm{sech}\,(\pi\omega/2)/\mathrm{sech}\,(\pi p\omega/2)\,. \tag{3.22}$$

The strength of the modulation amplitude $(\Delta\eta)$ giving synchronization of chaotic orbits is plotted as a function of f in figure 5 with values of q_1, q_2 and ω the same as used in the previous cases, giving rise to chaos in the unmodulated system (2.3). It is observed that while the modulation amplitude required for suppression decreases with f for higher resonances, for the fundamental term it is almost a constant. The bifurcation diagram,

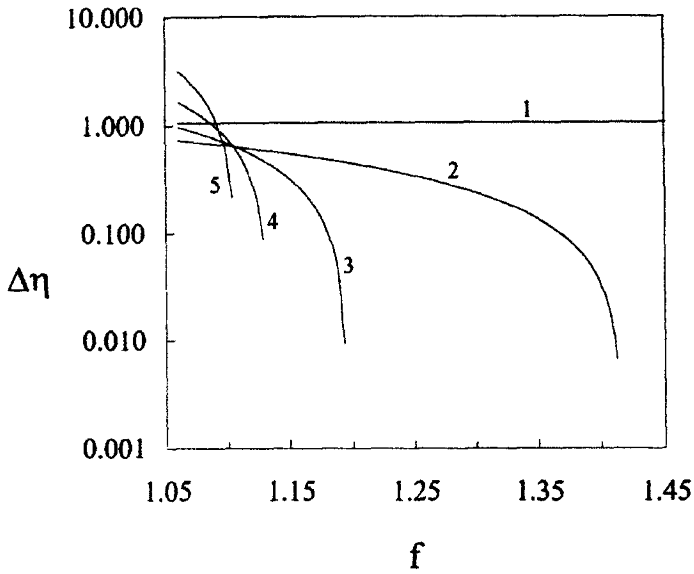


Figure 5. $\Delta\eta$ on adding a secondary forcing, plotted against original amplitude f; details as in figure 2a.

and LCE plot of this modulated system were obtained for the same parameter values as in case(1). It was found that the chaotic window for $1.00 < f < 1.30$ has been rendered periodic by the influence of the secondary term. The phase trajectory also is in support of the stabilization property of the modulation.

4. Conclusions

We have studied in this paper the effect of parametrically modulating the drive term, the damping term and the sine term on the chaotic state of the Froude pendulum. It has been found that suppression of horseshoe chaos is possible in all types of modulations applied. The stabilization effect of the addition of a secondary quasiperiodic forcing term on the chaotic regime of the pendulum is also included. In all cases, ranges of parameters for suppression of chaos have been analytically estimated using Melnikov's method. Though predictions by this method are for the transient chaos, the asymptotic behaviour also is found to be well within the predictions.

The window of modulation amplitude $\Delta\eta$ for all the four types of modulations, is in general a function of the frequency and amplitude of the primary forcing term and the order of resonance p. However by fixing f and ω, one can make a comparative study of the $\Delta\eta$ values for each resonance. Thus with $\omega = 0.7$, $f = 1.1$ and $p = 1$, we get maximum flexibility or width in $\Delta\eta$ values giving suppression for modulation of the damping term while it is least for the PM of the driving term. As another index in evaluating the performance of the different modulation mechanisms, we computed the response time τ for stabilization into the periodic mode. Then modulation of the $\sin x$ term gives the fastest response for a given ϕ and parametric and quasiperiodic modulations of the driving term give almost equal response time τ. The case of the PM applied to the damping term is slightly different from the other three cases because of the additional constraint on the q_1 and q_2 values required to keep the system bounded. This is a consequence of the particular nature of the nonlinear dissipation in the Froude pendulum. Provided this constraint is taken care of, case (2) gives a large window $\Delta\eta$ and comparatively small response time τ.

Although we have worked out explicit expressions only for one particular pendulum, it should work for all nonlinear systems with the pendulum as the unperturbed system. Hence the analysis is helpful in analytically estimating the parameter values for suppression of chaos, once PM is chosen as the technique for control, thus narrowing down the numerical search procedure to a smaller region of the parameter space.

Acknowledgement

One of the authors (KIT) is thankful to the University Grants Commission, New Delhi for financial assistance through a minor Research Project.

References

[1] A J Lichtenberg and M A Lieberman, *Regular and stochastic motion* (Springer-Verlag, 1983)
[2] S Wiggins, *Global bifurcations and chaos* (Springer, Berlin, 1988)
[3] P J Holmes and J E Marsden, *Comm. Math. Phys.* **82**, 523(1982)
[4] J Guckenheimer and P J Holmes, *Nonlinear oscillations, dynamical systems and bifurcations of vector fields* (Springer, New York, 1983)
[5] R Chacon and J Diaz Bejarano, *J. Sound and Vibrations* **186**, 269 (1995)
[6] G Ambika, *Pramana J. Phys.* **48**, 637 (1997)
[7] E Simiu and M Fray, *J. Engg. Mech.* **122**, 263 (1996)
[8] S Rajasekhar, S Parthasarathy and M Lakshmanan, *Chaos, solitons and fractals* **2**, 271 (1992)
[9] S Rajasekhar and M Lakshmanan, *International J. Chaos and Bifurcations* **2**, 201 (1992)
[10] Z Qu, G Hu, G Yang and G Qin, *Phys. Rev. Lett.* **74**, 1736 (1995)
[11] Y Braiman and I Goldhirsch, *Phys. Rev. Lett.* **66**, 2545(1991)
[12] J Heagy and W L Ditto, *J. Nonlinear Sci.* **1**, 423 (1991)
[13] S Wiggins, *Phys. Lett.* **A124**, 138 (1987)
[14] R Deb Shanker, *Phys. Rev.* **A41**, 726 (1990)
[15] R Lima and M Pettini, *Phys. Rev.* **A42**, 5975 (1990)
[16] R Chacon, *Phys. Rev.* **E52**, 2330 (1995)
[17] N Minorsky, *Nonlinear oscillations* (Van Nostrand, Princeton N J, 1962)
[18] F C Moon, *Chaotic and fractal dynamics* (John Wiley, N Y, 1992)
[19] H T Davis, *Introduction to nonlinear differential and integral equations* (Dover Publications Inc N Y, 1960)
[20] K I Thomas and G Ambika, in *Computational aspects in chaos and nonlinear dynamics*, edited by G Ambika and V M Nandakumaran (Wiley Eastern, New Delhi, 1994)
[21] A Wolf, J B Swift, H L Swinney and A Vastano, *Physica* **D16**, 285 (1985)