PRAMANA
— journal of
physics

Vol. 52, No. 4
April 1999
pp. 359–367

On spacetimes dual to spherically symmetric solutions

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MS received 5 February 1999

Abstract. By defining a duality transformation which implies interchange of active and passive electric parts of gravitational field, it is possible to construct spacetimes dual to solutions of the Einstein equation. Under the duality transformation a fluid spacetime maps into a fluid spacetime with density and pressure transforming as $\rho \rightarrow (\rho + 3p)/2$ and $p \rightarrow (\rho - p)/2$. On the other hand a vacuum solution will acquire a global monopole charge. The remarkable feature of spherically symmetric solutions is that it is possible to give a general prescription for writing dual solutions. We demonstrate its application by writing dual solutions to the McVittie solution for a Schwarzschild particle in an expanding universe and to Vaidya's radiating star solution.

Keywords. Duality; Schwarzschild black hole; global monopole; expanding universe.

PACS Nos 04.20; 04.50; 14.80

1. Introduction

It has recently been shown [1] that it is possible to construct a spacetime which is in a certain sense dual to the Schwarzschild solution. The duality is defined in terms of the electric components of the gravitational field. We first resolve relative to a unit timelike vector the Riemann curvature, in analogy with the electromagnetic field, into electric and magnetic parts. The electric part has further resolution into active (anchored onto non-gravitational matter-energy) and passive (anchored onto gravitational field energy) parts. By duality we mean interchange between the active and passive parts, which in general implies interchange between the Ricci and the Einstein curvatures. That is the two are dual of each-other. It then follows [2] that a fluid solution under the duality transformation goes over to another fluid solution with $\rho \rightarrow (\rho + 3p)/2, p \rightarrow (\rho - p)/2$, and heat flux and pressure anisotropy remaining unaltered. Obviously the vacuum equation will be invariant under the duality transformation. Then how do we construct a spacetime dual to the Schwarzschild solution?

It further turns out that it is possible to break the duality-symmetry of the vacuum equation by putting in an appropriate distribution, and yet the modified equation admits the Schwarzschild's as the general solution. This happens because in obtaining the

Schwarzschild solution (and also other black hole solutions), there remains one equation free which is implied by the others. Then throwing in some distribution on the right hand side of this equation will not disturb the black hole solution, because the other equations will imply it to be zero. Since the black hole solutions are unique, the modified equation can equally well characterize vacuum/electrovac. This will however make the equation duality non-invariant. The general solution of the dual equation will then give a black hole with global monopole charge [1,3]. What really happens is the modification required to make the equation duality non-invariant generates stresses precisely of the form, generated by a global monopole at large distance. This is how we can construct spacetimes dual to the black hole solutions [4–5], which will imbibe global monopole charge.

The stresses produced by the duality transformation in spherically symmetric vacuum, which correspond to global monopole, can be easily incorporated by a general prescription that amounts to creating a solid angle deficit similar to the angle deficit of cosmic string [6–7]. Note that the solid angle deficit produces non-zero curvature. That is spacetimes dual to spherically symmetric vacuum/electrovac solutions can be straightaway written, without solving any equation, by introducing a constant in front of the angular part of the metric. We demonstrate its application by obtaining solutions dual to, (i) the unique McVittie solution [8–9] representing a Schwarzschild particle embedded in an expanding universe and (ii) the Vaidya solution of radiating star [10–11]. In the former case, density and pressure of the cosmological background will transform as given above and the particle will acquire a global monopole charge while in the latter case, the radiating star will have global monopole charge [12].

Global monopoles may have been created by spontaneous breaking of global symmetry during phase transition in the early universe [6–7]. They are stable topological defects. The study of global monopole and spacetime associated with it is of cosmological interest and it has recently received considerable attention [13–15]. Barriola and Vilenkin [3] derived the metric associated with the spacetime of a static black hole with an internal global monopole. Yu [12] obtained an exact solution of Einstein's equation with the energy momentum tensor corresponding to a spherically symmetric global monopole plus a radial outgoing radiation, the Vaidya radiating star [12] with global monopole charge.

In §2 we formulate the duality transformation and the alternate duality non-invariant vacuum equation. In §3 we give the energy-momentum tensor for the general spherically symmetric metric, which is followed in §4 by discussion of global monopole and the stresses it generates. In §5, we give the general prescription for incorporating the duality-generated global monopole stresses into any spherically symmetric solution of the Einstein equation and demonstrate its application by writing solutions dual to the McVittie and the Vaidya solutions. The remarkable feature is that dual spacetimes automatically imbibe global monopole charge.

2. Duality transformation

Relative to a unit timelike vector, we resolve the Riemann curvature into electric and magnetic parts as follows [1]:

$$E_{ac} = R_{abcd}u^b u^d, \ \tilde{E}_{ac} = *R*_{abcd}\, u^b u^d \tag{1}$$

$$H_{ac} = *R_{abcd}u^b u^d = H_{(ac)} - H_{[ac]} \tag{2}$$

where

$$H_{(ac)} = *C_{abcd}u^b u^d \tag{3}$$

$$H_{[ac]} = \frac{1}{2}\eta_{abce}R^e_d u^b u^d. \tag{4}$$

C_{abcd} is the Weyl conformal curvature and η_{abcd} is the 4-dimensional volume element. E_{ab} and $\tilde{E}_{ab}$ are electric parts, and we shall term them respectively as active and passive electric parts, and H_{ab} is the magnetic part, which consists of the symmetric Weyl magnetic part and an antisymmetric part representing energy flux. Electric parts are symmetric while magnetic part is trace-free and they are all orthogonal to the resolving timelike vector. There are 12 electric components, 6 each of active and passive parts, and 8 magnetic components making the 20 independent components of the Riemann curvature. Electric field is produced by source-charge, which is matter-energy for gravitation, and magnetic field by motion of source. In GR there is an additional contribution from gravitational field energy itself, which also serves as a source. The active part refers to the usual Coulombic field and is represented by spacetime components, while passive part refers to the relativistic contribution of gravitational field energy represented by space-space components of the curvature [16].

Since the Riemann curvature is decomposed into electric and magnetic parts, we could also write the Ricci curvature in terms of the electromagnetic components. It would read as [1],

$$R^b_a = E^b_a + \tilde{E}^b_a + (E + \tilde{E})u_a u^b - \tilde{E}g^b_a + \frac{1}{2}(\eta_{amn}H^{mn}u^b + \eta^{bmn}H_{mn}u_a). \tag{5}$$

The traces of active and passive parts respectively denote the active gravitational charge density $\frac{1}{2}(T^4_4 - T^a_a)$ and the energy density T^4_4 relative to the timelike observer u^a.

The vacuum equation, $R_{ab} = 0$ will read as

$$E_{ab} + \tilde{E}_{ab} = 0 = H_{[ab]},\ E \text{ or } \tilde{E} = 0 \tag{6}$$

which is symmetric in active and passive parts. We define the duality transformation as

$$E_{ab} \leftrightarrow \tilde{E}_{ab},\ H_{ab} = H_{ab}. \tag{7}$$

From (1) it is obvious that the duality transformation would map Ricci into Einstein and vice-versa, because contraction of Riemann yields Ricci and that of its double dual Einstein. The vacuum equation is invariant under the duality transformation.

It turns out for obtaining the Schwarzschild solution we do not use the equation $E^1_1 + \tilde{E}^1_1 = 0$, it is implied by the others. That is even if we introduce some distribution in the 1-direction, it would not affect the Schwarzschild solution but it would render the equation duality non-invariant. This is what we are looking for finding solution dual to the Schwarzschild's. Thus instead of (6) we write

$$\tilde{E} = 0 = H_{[ab]}, E_{ab} + \tilde{E}_{ab} = KW_a W_b \tag{8}$$

where K is a scalar function and W_a is a spacelike unit vector, $W_a W^a = -1$ and $u_a\ W^a = 0$. This modified equation would also admit the Schwarzschild solution as the unique solution and hence could as well be taken as characterizing vacuum for spherical symmetry. Under the duality transformation (7), we shall now have

$$E = 0 = H_{[ab]}, E_{ab} + \tilde{E}_{ab} = KW_a W_b. \tag{9}$$

This dual equation also admits the general solution which represents the Schwarzschild particle with global monopole charge [3].

Equations (8) and (9) would respectively imply stress-energy of the form

$$T_{ab} = K(u_a u_b - W_a W_b - g_{ab}) \tag{10}$$

$$T_{ab} = K(u_a u_b - W_a W_b). \tag{11}$$

Note that the expression (11) represents the energy-momentum of the geometric string or string dust [17–18]. That is spacetime dual to the Schwarzschild solution will have stress-energy distribution of the geometric string or global monopole. We shall give a general prescription in §5 for incorporation of these stresses (11) in any spherically symmetric solution. On the other hand solution of the Einstein equation with (10) as source would always lead to $K = 0$, implying vacuum or flat spacetime.

3. Spherically symmetric spacetime

We consider the general spherically symmetric spacetime with the metric

$$ds^2 = e^{2\alpha} dt^2 - e^{2\lambda} dr^2 - r^2 e^{2\beta}(d\theta^2 + \sin^2\theta d\phi^2) \tag{12}$$

where α, λ and β are functions of r and t and $(x^1, x^2, x^3, x^4) = (r, \theta, \phi, t)$.

The surviving components of the Einstein tensor G^i_k for the metric (12) are given by

$$G^1_4 = -2e^{-2\lambda}\left[\dot{\beta}' + \dot{\beta}\left(\beta' - \alpha' + \frac{1}{r}\right) - \dot{\lambda}\left(\beta' + \frac{1}{r}\right)\right] \tag{13}$$

$$G^1_1 = e^{-2\lambda}\left[\beta'^2 + \frac{2\beta'}{r} + \frac{1}{r^2} + 2\alpha'\left(\beta' + \frac{1}{r}\right)\right] - \frac{e^{-2\beta}}{r^2} - e^{-2\alpha}[2\ddot{\beta} + 3\dot{\beta}^2 - 2\dot{\alpha}\dot{\beta}] \tag{14}$$

$$G^2_2 = G^3_3 = -e^{2\alpha}[\ddot{\lambda} + \dot{\lambda}^2 - \dot{\lambda}\dot{\alpha} + \ddot{\beta} + \dot{\beta}^2 - \dot{\alpha}\dot{\beta} + \dot{\lambda}\dot{\beta}] + e^{-2\lambda}\left[\alpha'' + \alpha'^2 - \lambda'\alpha' + \beta'' + \beta'^2 + \frac{2\beta'}{r} + (\alpha' - \lambda')\left(\beta' + \frac{1}{r}\right)\right] \tag{15}$$

$$G_4^4 = -e^{-2\alpha}(\dot{\beta}^2 + 2\dot{\lambda}\dot{\beta}) - \frac{e^{-2\beta}}{r^2} + e^{-2\lambda}\left[2\beta'' + 3\beta'^2 + \frac{6\beta'}{r} + \frac{1}{r^2} - 2\lambda'\left(\beta' + \frac{1}{r}\right)\right]. \tag{16}$$

Here, and in what follows an overhead dash and a dot denote differentiations with respect to r and t respectively.

Let us solve the modified vacuum equation (8) and its dual (9) for the metric (12) with $\beta = 0$. Here u^a is along the t-line orthogonal to 3-space and W_a is radial. Clearly $H_{[ab]} = 0$ will lead to time independence while $E_2^2 + \tilde{E}_2^2 = 0$ yields $\lambda + \mu = 0$ and $\tilde{E} = 0$ then determines $e^{\lambda} = (1 - 2M/r)^{-1/2}$, the Schwarzschild solution, yielding $K = 0$. For the solution of the dual set (9), we shall continue to have $\lambda + \mu = 0$ and then $E = 0$ will determine $e^{\lambda} = (1 - \eta^2 - 2M/r)^{-1/2}$. This is, like the Schwarzschild's, the general solution of the dual-vacuum equation (9) and represents a Schwarzschild particle with global monopole charge η [3]. It would give rise to the stresses

$$T_1^1 = T_4^4 = \frac{\eta^2}{r^2} = K \tag{17}$$

and rest of the stresses vanish. These are the stresses generated by the duality transformation which will be shown in the next section to be the same as for global monopole.

4. Global monopole

Global monopole is supposed to be created when global $0(3)$ symmetry is spontaneously broken into $U(1)$ in phase transition in the early universe. For a global monopole, the Lagrangian of the isoscalar triplet ψ^a with $a = 1, 2, 3$ is given by [3],

$$L = \frac{1}{2}(\partial\psi^a)^2 - \frac{\lambda}{4}(\psi^a\psi^a - \eta^2)^2. \tag{18}$$

Topologically non-trivial self-supporting solutions of this system can be found. The ansatz describing a monopole is

$$\psi^a(\mathbf{x}) = \eta f(r)\frac{x^a}{|\mathbf{x}|} \tag{19}$$

where

$$x^a x^a = r^2 \tag{20}$$

with x^a meaning the corresponding cartesian component of $\mathbf{x}$. Here η is a constant representing the energy scale of the symmetry breaking. Using Lagrangian (18) and the metric (12) the components of energy momentum tensor can be obtained from

$$T_{\mu\nu} = 2\frac{\partial L}{\partial g^{\mu\nu}} - Lg_{\mu\nu}.$$

The stress tensor of the system outside the monopole core (where $f = 1$) for larger can be approximated as [7]

$$T_1^1 = T_4^4 = \frac{\eta^2}{r^2} e^{-2\beta} \tag{21}$$

which means T_k^i (monopole) = diag $\{(\eta^2/r^2)e^{-2\beta}, 0, 0, (\eta^2/r^2)e^{-2\beta}\}$. This is the same as that given in (17) for $\beta = 0$. This shows that the solution dual to the Schwarzschild solution generates stresses of global monopole. That means the Schwarzschild solution with and without global monopole charge are dual of each-other relative to the duality transformation (7).

The equation of motion for ψ^a given in (19) will read as

$$e^{-2\lambda}\left[f'' + \left(\nu' - \lambda' + 2\beta' + \frac{2}{r}\right) f'\right] - \frac{2f}{r^2} e^{-2\beta} - \lambda\eta^2(f^2 - 1)f = 0. \tag{22}$$

It has the approximate solution $f = 1$ when $0(r^{-2}e^{-2\beta})$ is ignorable.

5. General prescription

Let the metric (12) be a solution of the Einstein equation

$$G_k^i = -(T_k^i + T_k^i(10)). \tag{23}$$

The equation dual to it would read as

$$R_k^i = -(T_k^i + T_k^i(10)) \tag{24}$$

because under the duality transformation the Ricci goes to the Einstein. It would be equivalent to

$$G_k^i = -(T_k^i - \frac{1}{2} T g_k^i + T_k^i(11)). \tag{25}$$

In view of the structure of G_k^i in (13)–(16), note that the stresses $T_k^i(21)$ can be incorporated simply by adding a constant in front of the angular part of the metric (12). This will amount to introducing deficit in solid angle without affecting the original spacetime. Equation (25) is dual of (23). It can be seen that solution of (23) for a reasonable T_k^i, would always imply $K = 0$. The solution dual to the metric (12) (i.e. solution of (25)) would hence be given by the metric

$$ds^2 = e^{2\alpha} dt^2 - e^{2\lambda} dr^2 - r^2 e^{2\beta}(1 - \eta^2)(d\theta^2 + \sin^2\theta d\varphi^2) \tag{26}$$

with $T_k^i \to (T_k^i - (1/2)T g_k^i)$ which for a fluid distribution means $\rho \to (\rho + 3p)/2$ and $p \to (\rho - p)/2$ [2].

We thus have the general prescription: If the metric (12) represents a spherically symmetric solution of Einstein's equation for stress tensor T_k^i, the metric (25) will represent a dual spacetime for stress tensor $T_k^i - (1/2)T g_k^i + T_k^i$ (monopole (21)). If we do not apply the duality transformation; i.e. $R^i_k \leftrightarrow G^i_k$, the above metric would have $T^i_k + T^i_k$ (monopole (21)) as source. The duality transformation implies in addition to global monopole also the transformation of T^i_k.

Let us apply this prescription to write the solution dual to the McVittie solution [8] describing a Schwarzschild particle in the FRW model. The solution is given by the metric

$$ds^2 = \left(\frac{1 - M/2w}{1 + M/2w}\right)^2 dt^2 - e^{\gamma}(1 + \frac{M}{2w})^4[dr^2 + h^2(r)(d\theta^2 + \sin^2\theta d\varphi^2)] \tag{27}$$

where

$$M = M(t),\ \gamma = \gamma(t),\ \dot{\gamma} = -\frac{2\dot{M}}{M},\ w = w(r). \tag{28}$$

The functions $h(r)$ and $w(r)$ depend upon the choice of $k(= -1, 0, +1)$, the Riemannian curvature of the surface of homogeneity t = constant in the background FRW universe:

$$h(r) = \begin{cases} \sinh r, & k = -1 \\ r, & k = 0 \\ \sin r, & k = +1 \end{cases} \tag{29}$$

and

$$w(r) = \begin{cases} 2\sinh \frac{r}{2}, & k = -1 \\ r, & k = 0 \\ 2\sin \frac{r}{2}, & k = +1. \end{cases} \tag{30}$$

The metric (27) satisfies the Einstein field equation for a perfect fluid distribution. The pressure p and the density ρ of the distribution are given by

$$p = -\frac{3}{4}\dot{\gamma}^2 - \ddot{\gamma}\left(\frac{1 + M/2w}{1 - M/2w}\right) - \frac{ke^{-\gamma}}{(1 - M/2w)(1 + M/2w)^5} \tag{31}$$

and

$$\rho = \frac{3}{4}\dot{\gamma}^2 + \frac{3ke^{-\gamma}}{(1 + M/2w)^5}. \tag{32}$$

Raychaudhuri [9] has shown that "The McVittie solution is unique under the following assumptions:

(i) The metric is spherically symmetric with a singularity at the centre.
(ii) The matter distribution is a perfect fluid.
(iii) The metric must asymptotically go over to the isotropic cosmological form.
(iv) The fluid flow is shear-free".

Application of the above prescription would give the dual-metric,

$$\mathrm{d}s^2 = \left(\frac{1 - M/2w}{1 + M/2w}\right)^2 \mathrm{d}t^2$$
$$-e^{\gamma}\left(1 + \frac{M}{2w}\right)^4 [\mathrm{d}r^2 + h^2(r)(1-\eta^2)(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2)] \tag{33}$$

where $h(r)$ and $w(r)$ are as before while ρ, p transforming as stated earlier. It can be verified that the above metric is the unique solution of (25) under the above stated assumptions. This describes a Schwarzschild particle with global monopole charge in an FRW expanding model with ρ, p being appropriately transformed.

When $k = 0$ and $\gamma = 0$, the above metric becomes

$$\mathrm{d}s^2 = \left(\frac{1 - M/2r}{1 + M/2r}\right)^2 \mathrm{d}t^2$$
$$-(1 + M/2r)^4[\mathrm{d}r^2 + r^2(1-\eta^2)(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)]. \tag{34}$$

This is the Schwarzschild particle with global monopole in isotropic coordinates. It can be transformed to the standard curvature form

$$\mathrm{d}s^2 = (1 - \eta^2 - \frac{2M}{r})\mathrm{d}t^2 - (1 - \eta^2 - \frac{2M}{r})^{-1}\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2 \mathrm{d}\varphi^2) \tag{35}$$

where $M \to M(1-\eta^2)$. This is the metric associated with the spacetime of a static black hole with a global monopole discussed by Barriola and Vilenkin [3]. Application of the prescription to the Schwarzschild solution would put $1 - \eta^2$ in front of the angular part of the metric, which could always be transformed to the above form. Similarly application to the flat spacetime will lead to the above metric with $M = 0$, the massless pure global monopole. Note that it is a spacetime dual to flat spacetime [1].

Next we consider Vaidya's radiating star solution [10]. Write the solution in the diagonal form as given by [11],

$$\mathrm{d}s^2 = \frac{\dot{x}^2}{f^2}(1 - \frac{2M}{r})\mathrm{d}t^2 - (1 - \frac{2M}{r})^{-1}\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2) \tag{36}$$

where $M = M(x), x = x(r,t)$ and $f(x) = x'(1 - 2M/r)$. Applying the prescription, we write

$$\mathrm{d}s^2 = \frac{\dot{x}^2}{f^2}(1 - \frac{2M}{r})\mathrm{d}t^2 - (1 - \frac{2M}{r})^{-1}\mathrm{d}r^2 - r^2(1-\eta^2)(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2) \tag{37}$$

with M, x and f as defined earlier. It can be easily verified that this is the solution obtained by Yu [12] for a radiating particle with global monopole charge in the diagonal form.

We have thus given a general prescription for writing straightaway solutions dual to spherically symmetric Einstein solutions without solving any differential equations. The procedure automatically includes the monopole or string dust stresses (21), which could be treated independent of the duality. In that case we shall have the original distribution with global monopole or string dust. Note that it is quite an interesting property of spherically symmetric solutions. What does a dual solution signify, is however not very clear except its association with global monopole for particle solutions. The prescription will trivially work for writing solution dual to a charged black hole. The solutions dual to the Kerr and the NUT solutions have also been found [4–5]. That means the solutions (i.e. of (9)) dual to axially and spherically symmetric vacuum solutions have been found. Further it can be shown that spacetime of global texture is dual-flat like the massless global monopole spacetime (35) with $M = 0$. This indicates that the duality transformation automatically incorporates the topological defects of global monopole and texture into the original spacetime [1].

Acknowledgement

L K Patel thanks IUCAA for hospitality.

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