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Exact solutions of the field equations for Charap's chiral invariant model of the pion dynamics

SUSANTO CHAKRABORTY and PRANAB KRISHNA CHANDA*
Central Drugs Laboratory, 3 Kyd Street, Calcutta 700 016, India
*Government Teacher's Training College, Malda, West Bengal 732 101, India

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Abstract. The field equations for the chiral invariant model of pion dynamics developed by Charap have been revisited. Two new types of solutions of these equations have been obtained. Each type allows infinite number of solutions. It has also been shown that the chiral invariant field equations admit invariance for a transformation of the dependent variables.

Keywords. Exact solutions; chiral invariance; pion dynamics; tangential parametrization; Charap.

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1. Introduction

Under tangential parametrization [1] the field equations for the chiral invariant model of the pion dynamics take the form [2]

$$\Box\phi = \eta^{\mu\gamma}\frac{\partial\phi}{\partial x^\mu}\cdot\frac{\partial\beta}{\partial x^\gamma} \tag{1.1a}$$

$$\Box\psi = \eta^{\mu\gamma}\frac{\partial\psi}{\partial x^\mu}\cdot\frac{\partial\beta}{\partial x^\gamma} \tag{1.1b}$$

$$\Box\chi = \eta^{\mu\gamma}\frac{\partial\chi}{\partial x^\mu}\cdot\frac{\partial\beta}{\partial x^\gamma} \tag{1.1c}$$

where,

$$\begin{aligned}\eta^{\mu\gamma} &= 0 \text{ for } \mu\neq\gamma\\ &= 1 \text{ for } \mu=\gamma\neq 4\\ &= -1 \text{ for } \mu=\gamma=4\end{aligned} \tag{1.1d}$$

$$\beta = \ln(f_\pi^2+\phi^2+\psi^2+\chi^2) \tag{1.1e}$$

$$f_\pi = \text{constant.} \tag{1.1f}$$

Charap [2] obtained solutions for (1.1) under the assumption that ϕ, ψ and χ are all functions of $(k_1x^1 + k_2x^2 + k_3x^3 + k_4x^4)$ where k_4 is any four vector. Ray [3] presented two types of solutions for (1.1). For obtaining the first type of solution Ray [3] used the ansatz,

$$\phi = \phi(u),\ \psi = \psi(u),\ \chi = \chi(u) \tag{1.2}$$

This type of solution is a generalization of the solutions obtained by Charap mentioned above and includes a soliton solution as a special case. For obtaining the second type of solutions, Ray [3] used the ansatz:

$$\phi = \phi(x^1, x^2, x^3 - x^4), \tag{1.3a}$$

$$\psi = \psi(x^1, x^2, x^3 - x^4), \tag{1.3b}$$

$$\chi = \chi(x^1, x^2, x^3 - x^4). \tag{1.3c}$$

When (1.1) is reduced to

$$\phi_{11} + \phi_{22} = \beta_1\phi_1 + \beta_2\phi_2, \tag{1.4a}$$

$$\psi_{11} + \psi_{22} = \beta_1\psi_1 + \beta_2\psi_2, \tag{1.4b}$$

$$\chi_{11} + \chi_{22} = \beta_1\chi_1 + \beta_2\chi_2, \tag{1.4c}$$

where,

$$\beta = \ln(f_\pi^2 + \phi^2 + \psi^2 + \chi^2). \tag{1.4d}$$

$$f_\pi = \text{constant}, \tag{1.4e}$$

where $\phi_1 = \partial\phi/\partial x^1, \phi_{11} = \partial^2\phi/\partial(x^1)^2$ etc.

Ray [3] obtained some particular solutions of (1.3) and (1.4). Chanda, De and Ray [4] further generalized them considerably.

The equations of (1.4) are conformally invariant i.e. the form (1.4) remains invariant under transformation $(x^1, x^2) \to (y, z)$ where y and z are two mutually conjugate solutions of Laplace's equations in x^1 and x^2. Hence from any solution of (1.4) one can immediately generate infinitely many other solutions of (1.4) simply by replacing (x^1, x^2) by (y, z) where y and z are two mutually conjugate solutions of Laplace's equations. Owing to the absence of x^3 and x^4 in (1.4) all the arbitrary constants of integration present in the solutions obtained by Ray [3] and Chanda, De and Ray [4] are arbitrary functions of $(x^3 - x^4)$. In other words they obtained infinite number of solutions of (1.1) where the dependence on x^3 and x^4 appears in terms of $(x^3 - x^4)$ only.

In this paper we present two other types of exact solutions each allowing infinite number of solutions where the dependence on x^3 and x^4 is more generalized than in the case mentioned above.

Here it may be mentioned that another class of rather generalized solutions for the nonlinear sigma model of chiral theories has been found by Enikova, Karloukovoski and Valchev [5] and more recently been rediscovered by Anslem [6].

In this article we have also shown that the equations of (1.1) admit invariance for a transformation of the dependent variables.

2. Exact solutions (Class I)

The ansatz which has been used here is given by:

$$\phi = \phi(\tau, \sigma),\ \psi = \psi(\tau, \sigma),\ \chi = \chi(\tau, \sigma), \tag{2.1a,b,c}$$

where,

$$\tau = \tau(x^1, x^2), \sigma = \sigma(x^3, x^4). \tag{2.2a,b}$$

Using the procedure which is exactly similar to that followed by Chakraborty, Chanda and Ray [7] and using relations (2.1 a, b, c) and (2.2 a,b) one can rewrite (1.1) as (Appendix A)

$$(\phi_{\gamma\gamma} - \phi_\gamma \beta_\gamma)R' + (\phi_{\delta\delta} - \phi_\delta \beta_\delta)S' = 0, \tag{2.3a}$$

$$(\psi_{\gamma\gamma} - \psi_\gamma \beta_\gamma)R' + (\psi_{\delta\delta} - \psi_\delta \beta_\delta)S' = 0, \tag{2.3b}$$

$$(\chi_{\gamma\gamma} - \chi_\gamma \beta_\gamma)R' + (\chi_{\delta\delta} - \chi_\delta \beta_\delta)S' = 0, \tag{2.3c}$$

$$\beta = \ln(f_\pi^2 + \phi^2 + \psi^2 + \chi^2),\ f_\pi = \text{ constant}, \tag{2.3d}$$

where γ is given by

$$\gamma = k_2 x^1 + k_3 x^2 + k_1, \tag{2.3e}$$

$$R' = k_2^2 + k_3^2, \tag{2.3f}$$

or

$$\gamma = \left(\frac{1}{2}k_4\right) \ln[(k_4 x^1 + k_5)^2 + (k_4 x^2 + k_6)^2] + [(\ln k_7)/(2k_4)] \tag{2.3g}$$

$$R' = \frac{1}{[(k_4 x^1 + k_5)^2 + (k_4 x^2 + k_6)^2]}, \tag{2.3h}$$

and δ is given by

$$\delta = k_9 x^3 + k_{10} x^4 + k_8, \tag{2.3i}$$

$$S' = k_9^2 - k_{10}^2, \tag{2.3j}$$

or

$$\delta = \left\{\frac{1}{(2k_{11})}\right\} \ln[(k_{11} x^3 + k_{12})^2 - (k_{11} x^4 - k_{13})^2] + \left[\frac{(\ln k_{14})}{(2k_{11})}\right], \tag{2.3k}$$

$$S' = \frac{1}{[(k_{11} x^3 + k_{12})^2 - (k_{11} x^4 - k_{13})^2]}, \tag{2.3l}$$

where $k_i,\ i = 1$ to 13, are arbitrary constants.

It may be noted that here $S' \neq 0$ and hence the solutions of (1.1) via (2.3) are not particular solutions of (1.1) via (1.4). The equations (2.3) reduce to an interesting form when R' = constant and S' = constant. After a transformation $(\gamma, \delta) \to (\gamma', \delta')$ where $\gamma' = \gamma/\sqrt{R}'$ and $\delta' = \delta/\sqrt{S}'$ one gets from (2.3)

$$\phi_{\gamma'\gamma'} + \phi_{\delta'\delta'} = \phi_{\gamma'}\beta_{\gamma'} + \phi_{\delta'}\beta_{\delta'}, \tag{2.4a}$$

$$\psi_{\gamma'\gamma'} + \psi_{\delta'\delta'} = \psi_{\gamma'}\beta_{\gamma'} + \psi_{\delta'}\beta_{\delta'}, \tag{2.4b}$$

$$\chi_{\gamma'\gamma'} + \chi_{\delta'\delta'} = \chi_{\gamma'}\beta_{\gamma'} + \chi_{\delta'}\beta_{\delta'}, \tag{2.4c}$$

$$\beta = \ln(f_\pi^2 + \phi^2 + \psi^2 + \chi^2), f_\pi = \text{ constant}, \tag{2.4d}$$

where

$$\gamma' = \frac{\gamma}{\sqrt{R}'} = \frac{(k_2 x^1 + k_3 x^2 + k_1)}{\sqrt{(k_2^2 + k_3^2)}}, \tag{2.4e}$$

$$\delta' = \frac{\gamma}{\sqrt{S}'} = \frac{(k_9 x^3 + k_{10} x^4 + k_8)}{\sqrt{(k_9^2 - k_{10}^2)}}. \tag{2.4f}$$

The relations (2.4e,f) follow from the fact that for R' = constant, S' = constant, γ and δ are given by (2.3e,f) and (2.3i,j) respectively.

That the equations (2.4) are consistent with the constraint R' = constant and S' = constant can be checked by putting in (1.1)

$$\phi = \phi(\gamma', \delta'),$$
$$\psi = \psi(\gamma', \delta'),$$
$$\chi = \chi(\gamma', \delta'),$$

where γ' and δ' are given by (2.4e,f). This reduces (1.1) exactly to (1.4).

The set of equations (2.4) is conformally invariant, i.e., the form of these equations is retained under any transformation

$$(\gamma', \delta') \to (f, q), \tag{2.5}$$

where f and q are functions of γ' and δ' such that $f_{\nu'} = q_{\delta'}$ and $f_{\delta'} = -q_{\gamma'}$, i.e., f and q are mutually conjugate solutions of the Laplace equation in γ' and δ'.

Hence from any solution of (2.4) one can immediately generate infinitely many other solutions of (2.4) simply by replacing (γ', δ') by (f, q). It is further interesting to note that equations of (2.4) are of the same form as one of the two generalised Lund–Regge equations [8], [9] namely (2.6b).

The generalized Lund–Regge equations are:

$$\theta_{11} + \theta_{22} - 4g(\theta) + h(\theta)(\lambda_1^2 + \lambda_2^2) = 0, \tag{2.6a}$$

$$\left[\lambda_1 \exp\left\{-\int p(\theta)\mathrm{d}\theta\right\}\right]_1 + \left[\lambda_2 \exp\left\{-\int p(\theta)\mathrm{d}\theta\right\}\right]_2 = 0, \tag{2.6b}$$

where

$$\theta = \theta(x^1, x^2), \lambda = \lambda(x^3, x^4).$$

With $g = 0$, the equations in (2.6) reduce to a conformally invariant set of equations, a particular example of which is the physically interesting equations of two dimensional Heisenberg ferromagnets [10], [11].

A direct expansion of (2.6b) leads to

$$\lambda_{11} + \lambda_{22} = p(\theta)[\lambda_1\theta_1 + \lambda_2\theta_2],$$

which is a generalised form of each of the equations in (2,4a,b,c).

Since (1.4) and (2.4) are exactly same in form one can use the solutions of (1.4) presented by Ray [3] and Chanda, De and Ray [4] for obtaining the solutions for (2.4).

However, the following points should be kept in mind. First, the independent variables for (1.4) are x^1 and x^2, whereas the independent variables for (2.4) are γ' and δ'. Second, the arbitrary constants of integration in the solutions for (1.4) are functions of $(x^3 - x^4)$. But for (2.4) they are pure constants. Third, for the solutions of (1.1) via the solutions of (2.4) one has R' = constant and S' = constant. Hence such solutions for (1.1) are found not for all x^1, x^2, x^3, x^4 but only on the two dimensional surfaces R' = constant and S' = constant.

3. Exact solutions (Class II)

Here we have sought a class of solutions by changing variables to functions of spacetime coordinates which are restricted in the following way:

$$(x^1, x^2, x^3, x^4) \to (X, Y, Z, W) \tag{3.1a}$$

such that

$$X_1 = Y_2, X_2 = -Y_1 \tag{3.1b}$$

and

$$Z_3 = W_4, Z_4 = W_3 \tag{3.1c}$$

where

$$X = X(x^1, x^2), \quad Y = Y(x^1, x^2) \tag{3.1d}$$

$$Z = Z(x^3, x^4), \quad W = W(x^3, x^4). \tag{3.1e}$$

Some examples of X and Y satisfying (3.1b) are as follows:

$$X = x^1, Y = x^2, \tag{3.2a}$$

$$X = (x^1)^2 - (x^2)^2, Y = 2(x^1)(x^2), \tag{3.2b}$$

$$X = (x^1)^3 - 3(x^1)(x^2)^2, Y = 3(x^1)^2(x^2) - (x^2)^3. \tag{3.2c}$$

And, some examples of Z and W satisfying (3.1c) are as follows:

$$Z = x^3, W = x^4 \tag{3.2d}$$

$$Z = (\sin x^3)(\sin x^4),\ W = -(\cos x^3)(\cos x^4) \tag{3.2e}$$

$$Z = (\sinh x^3)(\sinh x^4),\ W = (\cosh x^3)(\cosh x^4) \tag{3.2f}$$

As a consequence of the transformation (3.1), (1.1) reduces to:

$$(\phi_{XX} + \phi_{YY} - \phi_X\beta_X - \phi_Y\beta_Y)(X_1^2 + X_2^2) \\ +(\phi_{ZZ} - \phi_{WW} - \phi_Z\beta_Z + \phi_W\beta_W)(Z_3^2 - Z_4^2) = 0, \tag{3.3a}$$

$$(\psi_{XX} + \psi_{YY} - \psi_X\beta_X - \psi_Y\beta_Y)(X_1^2 + X_2^2) \\ +(\psi_{ZZ} - \psi_{WW} - \psi_Z\beta_Z + \psi_W\beta_W)(Z_3^2 - Z_4^2) = 0, \tag{3.3b}$$

$$(\chi_{XX} + \chi_{YY} - \chi_X\beta_X - \chi_Y\beta_Y)(X_1^2 + X_2^2) \\ +(\chi_{ZZ} - \chi_{WW} - \chi_Z\beta_Z + \chi_W\beta_W)(Z_3^2 - Z_4^2) = 0, \tag{3.3c}$$

$$\beta = \ln(f_\pi^2 + \phi^2 + \psi^2 + \chi^2). \tag{3.3d}$$

The first case of (3.3) is given by $Z_3^2 - Z_4^2 = 0$ which has been considered by Ray [3] and Chanda, De and Ray [4].

The second case of (3.3) is given by $Z_3^2 - Z_4^2 \neq 0$.

In the particular situation of this case given by $\phi = \phi(X, Z)$, $\psi = \psi(X, Z)$, and $\chi = \chi(X, Z)$ (3.3) reduces to (2.3) (Appendix B).

Here we have considered another particular situation of this case which is given by the simultaneous satisfaction of the following sets of equations:

$$\phi_{XX} + \phi_{YY} = \phi_X\beta_X + \phi_Y\beta_Y \tag{3.4a}$$

$$\psi_{XX} + \psi_{YY} = \psi_X\beta_X + \psi_Y\beta_Y \tag{3.4b}$$

$$\chi_{XX} + \chi_{YY} = \chi_X\beta_X + \chi_Y\beta_Y \tag{3.4c}$$

$$\beta = \ln(f_\pi^2 + \phi^2 + \psi^2 + \chi^2), \tag{3.4d}$$

and

$$\phi_{ZZ} - \phi_{WW} = \phi_Z\beta_Z - \phi_W\beta_W \tag{3.5a}$$

$$\psi_{ZZ} - \psi_{WW} = \psi_Z\beta_Z - \psi_W\beta_W \tag{3.5b}$$

$$\chi_{ZZ} - \chi_{WW} = \chi_Z\beta_Z - \chi_W\beta_W \tag{3.5c}$$

$$\beta = \ln(f_\pi^2 + \phi^2 + \psi^2 + \chi^2),\ \beta = \text{ constant}. \tag{3.5d}$$

Now (3.4) and (3.5) remain invariant in form under the transformation

$$(X, Y, Z, W) \to (X', Y', Z', W') \tag{3.6a}$$

where

$$X'_X = Y'_Y,\ X'_Y = -Y'_X, \tag{3.6b}$$

$$Z'_Z = W'_W,\ Z'_W = W'_Z, \tag{3.6c}$$

$$X' = X'(X, Y),\ Y' = Y'(X, Y), \tag{3.6d}$$

$$Z' = Z'(Z, W),\ W' = W'(Z, W). \tag{3.6e}$$

Examples of such X', Y', Z', W' are same in form to those given for X, Y, Z, W in (3.2).

Hence from any solution for (1.1) which simultaneously satisfies (3.4) and (3.5) one can generate infinitely many other solutions of (1.1) simply replacing (X, Y, Z, W) by (X', Y', Z', W').

The equations (3.4) are of the same form as that of (1.4). And (3.5) reduces to (3.4) under the transformation $W \to iW$ and the replacement of (Z, W) by (X, Y). Thus if the set of equations (3.5) is a set of coupled equations in two-dimensional spacetime continuum where W is time like and Z is space like, then the set of equations (3.4) is its Euclidian counterpart, where both X and Y are space like.

One simple way for obtaining solutions for (1.1) which satisfy simultaneously (3.4) and (3.5) is to follow the algorithm given below:

(1) Obtain any solution for (3.4) using the formalism of Ray [3] or Chanda, De and Ray [4].

(2) Express the arbitrary constants in that solution as functions of Z and W such that role of X and Z or Y and W remain symmetrical and the final solution satisfies (3.4) and (3.5) simultaneously.

It may be checked that the existence of terms like XZ, YW etc. are not permitted. (For an example, see Appendix C).

We use in the following a particular situation of the solutions reported in case I of the work of Chanda, Ray and De [4] for demonstrating the algorithm stated above.

Here the solution for (3.4) can be written as

$$\phi = f_\pi \tan(2f_\pi X - G), \tag{3.7a}$$
$$\psi = f_\pi \sec(2f_\pi X - G)\cos(AX + BY + C), \tag{3.7b}$$
$$\chi = f_\pi \sec(2f_\pi X - G)\sin(AX + BY + C), \tag{3.7c}$$
$$A^2 + B^2 = 4f_\pi^2, \tag{3.7d}$$

where A, B, C, G are arbitrary constants; X, Y satisfy (3.1b), (3.1d), (3.6b) and (3.6d).

Now following the algorithm stated above one can write:

$$\phi = f_\pi \tan[2f_\pi(X + Z)], \tag{3.8a}$$
$$\psi = f_\pi \sec[2f_\pi(X + Z)]\cos(CX + DY + EZ + FW), \tag{3.8b}$$
$$\chi = f_\pi \sec[2f_\pi(X + Z)]\sin(CX + DY + EZ + FW), \tag{3.8c}$$
$$C^2 + D^2 = 4f_\pi^2,\ E^2 - F^2 = 4f_\pi^2, \tag{3.8d,e}$$

where X, Y, Z, W satisfy (3.1) and (3.6). The equations (3.8) identically satisfy the equations (1.1). That is, (3.8) represent an exact solution of (1.1) wherefrom we obtain infinitely many other solutions with the use of (3.1) and (3.6).

Transformation of the dependent variables

The equation (1.1) admits an invariance under transformation of the dependent variables. For example, they remain invariant under the transformation:

$$\psi = \bar{c}(\bar{c}^2 + 1)^{-1/2}\Sigma + (\bar{c}^2 + 1)^{-1/2}\Lambda \tag{3.9a}$$

$$\chi = (\bar{c}^2 + 1)^{-1/2}\Sigma - \bar{c}(\bar{c}^2 + 1)^{-1/2}\Lambda \tag{3.9b}$$

where $\bar{c}$ is an arbitrary constant, Σ and Λ are functions of (x^1, x^2, x^3, x^4).

So, if one can get any set of solutions for the equations (1.1), it is easy to generate new solutions with the help of the simple relations in (3.9) where Σ and Λ may be treated as the old solutions of χ and ψ respectively. In this way we can generate infinite number of solutions for (1.1). Furthermore it may be noted that the equations (1.1) are symmetric in ϕ, ψ and χ. Hence at the time of generating new solutions, any two, out of ϕ, ψ and χ can be chosen.

4. Summary

(i) We have obtained two types of solutions of (1.1). Each type allows infinite number of solutions.

(ii) The first type of solutions are represented by the solutions of (2.4). Form of the equations (2.4) remains, invariant under any transformation (2.5). Therefore the equations (2.4) are conformally invariant and admit infinite number of solutions of (1.1). However, the solutions are found here not for all (x^1, x^2, x^3, x^4) but on two specific surfaces.

(iii) The second type of solutions are represented by the solutions which satisfy (3.4) and (3.5) simultaneously. Form of the equations (3.4) and (3.5) remains invariant under any transformation (3.6). Therefore (3.4) and (3.5) admit infinite number of solutions. And here again we get infinite number of solutions of (1.1).

(iv) We have also shown that (1.1) admits invariance for a transformation of the dependent variables. This again allows generation of infinite number of solutions.

Appendix A

From (2.1a) and (2.2a), $\phi = \phi(\tau, \sigma)$ which gives,

$$\phi_1 = \phi_\tau \tau_1 \quad \text{and}$$

$$\phi_{11} = \phi_{\tau\tau}\tau_1^2 + \phi_\tau \tau_{11}.$$

Similarly, we have, $\phi_2, \phi_3, \phi_4, \phi_{22}, \phi_{33}$ and ϕ_{44}. Again, from (2.1b,c) and (2.2a,b) we have similar equations as ϕ's. Using these equations in (1.1a), keeping in mind the relations (2.2a,b), we have,

$$(\phi_{\tau\tau} - \phi_\tau \beta_\tau)(\tau_1^2 + \tau_2^2) + (\phi_{\sigma\sigma} - \phi_\sigma \beta_\sigma)(\sigma_3^2 - \sigma_4^2)$$
$$+\phi_\tau(\tau_{11} + \tau_{22}) + \phi_\sigma(\sigma_{33} - \sigma_{44}) = 0. \tag{A1a}$$

Similarly, we have from (1.1b),

$$
\begin{aligned}
&(\psi_{\tau\tau} - \psi_\tau\beta_\tau)(\tau_1^2 + \tau_2^2) + (\psi_{\sigma\sigma} - \psi_\sigma\beta_\sigma)(\sigma_3^2 - \sigma_4^2) \\
&\qquad + \psi_\tau(\tau_{11} + \tau_{22}) + \psi_\sigma(\sigma_{33} - \sigma_{44}) = 0,
\end{aligned} \tag{A1b}
$$

and from equation (1.1c),

$$
\begin{aligned}
&(\chi_{\tau\tau} - \chi_\tau\beta_\tau)(\tau_1^2 + \tau_2^2) + (\chi_{\sigma\sigma} - \chi_\sigma\beta_\sigma)(\sigma_3^2 - \sigma_4^2) \\
&\qquad + \chi_\tau(\tau_{11} + \tau_{22}) + \chi_\sigma(\sigma_{33} - \sigma_{44}) = 0.
\end{aligned} \tag{A1c}
$$

Eliminating $(\sigma_{33} - \sigma_{44})$ and $(\sigma_3^2 - \sigma_4^2)$ from (A1a,b,c) one arrives at $(\tau_{11} + \tau_{22})/(\tau_1^2 + \tau_2^2)$ = a function of $\phi, \psi, \chi, \phi_\tau, \phi_\sigma, \psi_\tau, \psi_\sigma, \chi_\tau, \chi_\sigma, \phi_{\tau\tau}, \phi_{\sigma\sigma}, \psi_{\tau\tau}, \psi_{\sigma,\sigma}, \chi_{\tau\tau}, \chi_{\sigma\sigma}$. LHS of above equation is a function of x^1 and x^2. Hence, RHS will also be function of x^1 and x^2. But on the RHS x^1 and x^2 do not appear in the explicit form, rather as a function of $\tau(x^1, x^2)$ only.

Thus we may write,

$$
\frac{(\tau_{11} + \tau_{22})}{(\tau_1^2 + \tau_2^2)} = \text{ an arbitrary function of } \tau = P(\tau) \text{ (say)}. \tag{A2}
$$

Following the same procedure, i.e. eliminating $(\tau_{11} + \tau_{22})$ and $(\tau_1^2 + \tau_2^2)$ from (A1a,b,c) one gets,

$$
\frac{(\sigma_{33} - \sigma_{44})}{(\sigma_3^2 - \sigma_4^2)} = \text{ an arbitrary function of } \sigma = Q(\sigma) \text{ (say)}. \tag{A3}
$$

Putting (A2) and (A3) in (A1a) one can write,

$$
\frac{(\phi_{\tau\tau} - \phi_\tau\beta_\tau + P\phi_\tau)}{(\phi_{\sigma\sigma} - \phi_\sigma\beta_\sigma + Q\phi_\sigma)} = -\frac{(\sigma_3^2 - \sigma_4^2)}{(\tau_1^2 + \tau_2^2)}. \tag{A4}
$$

Now both the LHS and the RHS of (A4) are functions of x^1, x^2, x^3, x^4. But on the LHS x^1, x^2, x^3 and x^4 do not appear in explicit form, rather as function of $\tau(x^1, x^2)$ and $\sigma(x^3, x^4)$ only.

Thus we may write,

$$
\frac{(\sigma_3^2 - \sigma_4^2)}{(\tau_1^2 + \tau_2^2)} = \text{ an arbitrary function of } \tau \text{ and } \sigma. \tag{A5}
$$

The numerator of the LHS of (A5) is a function of x^3, x^4 only which must appear in terms of σ. Similarly the denominator of the LHS of (A5) is a function of x^1, x^2 only which must appear in terms of τ.
Hence,

$$
(\tau_1^2 + \tau_2^2) = R(\tau) \text{ and } (\sigma_3^2 - \sigma_4^2) = S(\sigma). \tag{A6a,b}
$$

Here $R(\tau)$ is an arbitrary function of σ. Then equations (A1a,b,c) reduce to

$$
(\phi_{\tau\tau} - \phi_\tau\beta_\tau + P\phi_\tau)R + (\phi_{\sigma\sigma} - \phi_\sigma\beta_\sigma + Q\phi_\sigma)S = 0, \tag{A7a}
$$

$$
(\psi_{\tau\tau} - \psi_\tau\beta_\tau + P\psi_\tau)R + (\psi_{\sigma\sigma} - \psi_\sigma\beta_\sigma + Q\psi_\sigma)S = 0, \tag{A7b}
$$

$$
(\chi_{\tau\tau} - \chi_\tau\beta_\tau + P\chi_\tau)R + (\chi_{\sigma\sigma} - \chi_\sigma\beta_\sigma + Q\chi_\sigma)S = 0, \tag{A7c}
$$

$$
\exp(\beta) = f_\pi^2 + \phi^2 + \psi^2 + \chi^2, \tag{A7d}
$$

where τ and σ are given by (A2), (A3), (A6a,b).
Now (A2), (A3), (A6a,b) can be rewritten as

$$\gamma_{11} + \gamma_{22} = 0, \tag{A8a}$$
$$\gamma_1^2 + \gamma_2^2 = R', \tag{A8b}$$
$$\delta_{33} - \delta_{44} = 0, \tag{A8c}$$
$$\delta_3^2 - \delta_4^2 = S' \tag{A8d}$$

where

$$\gamma = \int \left\{ \exp \left[- \int P(\tau) \right] \mathrm{d}\tau \right\} \mathrm{d}\tau, \tag{A8e}$$

$$\delta = \int \left\{ \exp \left[- \int Q(\sigma) \right] \mathrm{d}\sigma \right\} \mathrm{d}\sigma, \tag{A8f}$$

$$R' = \left\{ \exp \left[-2 \int P(\tau) \mathrm{d}\tau \right] \right\} R(\tau), \tag{A8g}$$

$$S' = \left\{ \exp \left[-2 \int Q(\sigma) \mathrm{d}\sigma \right] \right\} S(\sigma). \tag{A8h}$$

By virtue of (A8e) and (A8f) one can consider R' and S' as functions of γ and δ respectively.

The solutions for (A8a) and (A8b) are given by (2.3e, f,g,h). And the solutions for (A8c) and (A8d) are given by (2.3i,j,k,l). The procedure for obtaining (2.3e) to (2.3l) is exactly same as that given in the Appendix B of the work of Chakraborty *et al* [7]. Hence the related calculations have been omitted in this paper.

Finally, without any loss of generality one can transform (τ, σ) to (γ, σ) and (A7) leads to (2.3).

Appendix B

With

$$\phi = \phi(X, Z), \tag{B1a}$$
$$\psi = \psi(X, Z), \tag{B1b}$$
$$\chi = \chi(X, Z), \tag{B1c}$$

one obtains from (3.2) the following equation:

$$(\phi_{XX} - \phi_X \beta_X)(X_1^2 + X_2^2) + \phi_{ZZ} - \phi_Z \beta_Z)(Z_3^2 - Z_4^2) = 0, \tag{B2a}$$
$$(\psi_{XX} - \psi_X \beta_X)(X_1^2 + X_2^2) + \psi_{ZZ} - \psi_Z \beta_Z)(Z_3^2 - Z_4^2) = 0, \tag{B2b}$$
$$(\chi_{XX} - \chi_X \beta_X)(X_1^2 + X_2^2) + \chi_{ZZ} - \chi_Z \beta_Z)(Z_3^2 - Z_4^2) = 0, \tag{B2c}$$
$$\beta = \ln(f_\pi^2 + \phi^2 + \psi^2 + \chi^2). \tag{B2d}$$

From (B2a) one gets,

$$\frac{(\phi_{XX} - \phi_X\beta_X)}{(\phi_{ZZ} - \phi_Z\beta_Z)} = -\frac{(Z_3^2 - Z_4^2)}{(X_1^2 + X_2^2)}. \tag{B3}$$

Now both the LHS and the RHS of (B3) are functions of x^1, x^2, x^3 and x^4. But on the LHS x^1, x^2, x^3 and x^4 do not appear in explicit form, rather as function of $X(x^1, x^2)$ and $Z(x^3, x^4)$ only.

Thus we may write

$$\frac{(Z_3^2 - Z_4^2)}{(X_1^2 + X_2^2)} = \text{ an arbitrary function of } X \text{ and } Z. \tag{B4}$$

Now, the numerator of the LHS of (B4) is a function of x^3 and x^4 only which must appear in terms of Z. Similarly the denominator of the LHS (B4) is a function of x^1 and x^2 only, which must appear in terms of X.

Hence,

$$X_1^2 + X_2^2 = \text{ an arbitrary function of } X = R''(X), \text{ (say)}, \tag{B5a}$$

$$Z_3^2 - Z_4^2 = \text{ an arbitrary function of } Z = S''(Z), \text{ (say)}. \tag{B5b}$$

Putting (B5) in (B2) one can write

$$(\phi_{XX} - \phi_X\beta_X)R''(X) + (\phi_{ZZ} - \phi_Z\beta_Z)S''(Z) = 0, \tag{B6a}$$

$$(\psi_{XX} - \psi_X\beta_X)R''(X) + (\psi_{ZZ} - \psi_Z\beta_Z)S''(Z) = 0, \tag{B6b}$$

$$(\chi_{XX} - \chi_X\beta_X)R''(X) + (\chi_{ZZ} - \chi_Z\beta_Z)S''(Z) = 0, \tag{B6c}$$

$$\beta = \ln(f_\pi^2 + \phi^2 + \psi^2 + \chi^2), f_\pi = \text{ constant}. \tag{B6d}$$

with,

$$X_{11} + X_{22} = 0, \tag{B6e}$$

$$X_1^2 + X_2^2 = R''(X), \tag{B6f}$$

$$Z_{33} - Z_{44} = 0, \tag{B6g}$$

$$Z_3^2 - Z_4^2 = S''(Z). \tag{B6h}$$

The equations (B6e,f,g,h) are exactly same as (A8a,b,c,d). Only the notations are different. Hence (B6) as a whole are exactly same as (2.3).

Appendix C

Instead of (3.8) let us write

$$\begin{aligned}
\phi &= f_\pi \tan[2f_\pi(X + Z)], \\
\psi &= f_\pi \sec[2f_\pi(X + Z)] \cos(AZX + BWY + CX + DY + EZ + FW), \\
\chi &= f_\pi \sec[2f_\pi(X + Z)] \sin(AZX + BWY + CX + DY + EZ + FW),
\end{aligned}$$

This when put in (1.1) leads to

$$(AZ + C)^2 + (BW + D)^2 = 4f_\pi^2,$$

$$(AX + E)^2 - (BY + F)^2 = 4f_\pi^2,$$

which is not permitted.

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